IMPROVING REDEVELOPMENT OF REAL ASSETS

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Abstract

Property asset management is keenly influenced by the market rental price for the services and asset quality. Renovation is conceived as a form of redevelopment that returns the property back to its pristine state after experiencing quality deterioration. We formulate a two-factor renovation model under rental price and quality uncertainty and a fixed investment cost. The renovation policy is determined from the economic boundary conditions as an analytical solution in an implicit form. From this, we establish that the incremental value from renovating the asset quality has to exceed the investment cost by a mark-up factor that exceeds one. Further, the most the property owner is willing spend on renovating the property is positively influenced by the prevailing rental price but negatively by the underlying volatilities. Owners renovate their properties whenever rental prices are propitious and volatility is falling.

Introduction

Commercial property assets, such as hotels, apartment blocks and retail parks, experience deterioration in their quality level due to wear and tear. The effects of any decline in quality loss are to make the property less attractive to both current and prospective users and to engender a loss in prestige. This has the possible adverse consequence of a fall in rental income, which is the product of the prevailing market rental price for services and its idiosyncratic quality. Renovation is conceived as the process of repairing the sustained damage and restoring the asset to its pristine state. But, remedying the quality deficit necessitates a capital investment injection. This raises the question of identifying the viable timing for renovating the asset when both the rental price and the asset quality are treated as stochastic. This occurs when the expected improvement in future rental income due to the quality renovation are sufficient to compensate the investment outlay. Conceptually, the opportunity to renovate an asset represents a real option that can be exercised at the discretion of the asset-owner and on a repeated number of occasions. The decision on asset renovation is resolved on the basis of the prevailing rental price for services of the asset and its quality level. Our aim is to determine in an analytical form the timing boundary for the two-factor renovation model.

Real option analysis has evolved from the assertion of Myers and Turnbull (1977) that the DCF analysis of investment opportunities is distorted in the presence of optionality and the derivation by Samuelson (1965) of the American perpetuity option. The original analytical applications include Tourinho (1979) on resource extraction, Brennan and Schwartz (1985) on mining activities with possible suspension, and McDonald and Siegel (1986) on investment opportunities.

In the context of property development, applications of real option analysis can be classified according to its stage in the property life-cycle. Construction opportunities on vacant land is considered through a binomial lattice framework by Titman (1985) and through an analytical two-factor model by Williams (1991). Redevelopment opportunities of an existing structure are investigated by Williams (1997) who determines analytically the optimal inferior and superior quality levels of a property for a variable redevelopment cost, while Childs, Riddiough and Triantis (1996) use a numerical solution technique to find the optimal combination for a mixed use property. The possibility of mothballing the property by suspending its rental services in a market of falling rental prices, with the eventuality of demolition and reconstruction, is investigated analytically by Paxson (2007). There are also other studies related to redevelopment; these include Mauer and Ott (1995) on equipment replacement and Malchow-Møller and Thorsen (2005) on technology replacement.

The methods for solving the timing decision associated with these models of property development can be categorised as either analytical or numerical. An analytical solution is obtainable for a model that can be expressed as a one-factor formulation. In the Paxson (2007) one-factor model, the only source of uncertainty is the profit accruing to the property. Since the homogeneity degree-one property is satisfied by the boundary conditions in the Williams (1991) two-factor model, his formulation is expressible by a single variable. In a similar way, Williams (1997) transforms the boundary condition to produce a model in a tractable form. Whenever dimensional reducing transformations are unavailable, the conventional approach for solving the timing boundary is to adopt a numerical technique such as finite difference, Childs, Riddiough and Triantis (1996). However, these methods are computationally onerous and lack the intrinsic appeal of an analytical solution.

The aim of this paper is to develop an analytical solution of the timing boundary for the two-factor renovation model. In this formulation, the investment cost for renovating the property asset is treated as a fixed quantity, which means that the boundary conditions infringe the homogeneity degree-one property and dimension reducing transformations are unavailable. Despite this, the timing boundary is derived analytically in an implicit form and we demonstrate that the condition for renovating the property adopts a form that is similar to the expression obtained by McDonald and Siegel (1986) and Dixit and Pindyck (1994). Secondly, renovation is conceived as the process of raising the inferior quality of the property back to its pristine state. In this conception, quality upgrades can range from refurbishment to structural renovation and we determine the most the property owner is willing to spend on renovation for given levels of the rental price and asset quality.

The paper is organised in the following way. The second section develops the method for deriving the analytical timing boundary and determines the renovation condition. The third section explores the model behaviour through a sensitivity analysis. The final section is a conclusion.

Valuing the Renovation Opportunity

Two Factor Model

To solve the asset renovation problem, we maximise the expected present value of the stream of net cash flows rendered by the asset over all possible renovation schemes. A commercial property company owns a deteriorating real asset that represents a significant portion of its portfolio. It is seeking to identify the threshold conditions signalling a renovation investment, which will raise the prevailing inferior quality level to a certain superior level. The threshold conditions are characterised by a boundary that discriminates between the two decisions of asset renovation and asset continuance. In contrast to the one-factor model where the threshold is specified by a single level, the discriminatory boundary for the two-factor renovation model is specified by a locus function of the two variables that are uncertain. At any time, the property asset renders a cash flow revenue that depends on the asset quality and the market rental price relevant for the type of property under study. The rental price per unit of quality and the asset quality are denoted by p and q respectively, and the rental income is given by pq. These two variables are taken to evolve stochastically according to the geometric Brownian motion process. For $x \in \{p, q\}$:

$$dx = \alpha_x x dt + \sigma_x x dz_x , \tag{1}$$

Where α_x is the instantaneous drift rate, σ_x is the instantaneous volatility, and dz_x is the increment of a standard Wiener process. Dependence between the two uncertain variables is described by the instantaneous covariance term $\rho\sigma_p\sigma_q$ where $\mathrm{Cov}[dp, dq] = \rho\sigma_p\sigma_q$ with $|\rho| \le 1$. Williams (1997) assumes that the rental price follows a geometric Brownian process but that quality deterioration is deterministic. We diverge from his formulation by allowing the asset quality to be stochastic. If the quality volatility σ_q is set to equal zero, then the quality process becomes deterministic and the model reverts to the Williams (1997) formulation. The quality drift rate α_q represents the mean deterioration rate for the asset, it incorporates both physical depreciation and functional obsolescence, and it is expected to be negative.

Renovating the asset incurs an investment cost that instantaneously raises asset quality from the prevailing inferior level to a certain superior level. This investment cost is denoted by K and is treated as non-stochastic. We assume, initially, that the continuous property operating and maintenance cost is constant throughout the asset life, or that any variability can be subsumed within the rental income. Subsequently, this continuous cost is permitted to be variable.

Valuation Function

The function F is defined as the value of the property asset including its embedded renovation option. All renovation decisions are treated as being made in isolation to any other enacted policies, so temporary suspension and abandonment opportunities are assumed to be unavailable. The value of F depends on the prevailing relevant rental price p and the asset quality level q, so $F = F(p, q)$. By assuming complete markets, standard contingent claims can be applied to the asset with value F to determine its risk neutral valuation relationship, Constantinides (1978) and Mason and Merton (1985). This is expressed by the partial differential equation:

$$\begin{aligned} &\tfrac{1}{2}\sigma_p^2 p^2 \frac{\partial^2 F}{\partial p^2} + \tfrac{1}{2}\sigma_q^2 q^2 \frac{\partial^2 F}{\partial q^2} + \rho\sigma_p\sigma_q pq \frac{\partial^2 F}{\partial p \partial q} \\ &\qquad + \theta_p p \frac{\partial F}{\partial p} + \theta_q q \frac{\partial F}{\partial q} - rF + pq = 0. \end{aligned} \tag{2}$$

Where r denotes the risk-free rate of interest, and θ_p and θ_q are the risk-adjusted drift rates respectively for the rental price and quality.

The simplest form of generic function satisfying (2) takes the form:

$$F = Ap^{\beta}q^{\eta} + \frac{pq}{r-\phi} \tag{3}$$

where A denotes a generic parameter and $\phi = \rho\sigma_p\sigma_q + \theta_p + \theta_q$. This generic functional form can be justified in two ways. In their analysis of an investment opportunity, McDonald and Siegel (1986) propose that the homogenous element of the functional solution to their two-factor model adopts a form similar to that expressed in (3). In their formulation, the boundary conditions support homogeneity degree-one, so they impose the condition $\beta + \eta = 1$. However, as we will explain later, the property homogeneity degree-one is not satisfied by the boundary conditions for the two-factor renovation model, and so we do not impose any conditions on the values of β and η. Other authors, such as Paxson and Pinto (2005), use the similarity principle to convert their two-factor into a one-factor model, but again, the boundary conditions of the renovation model do not support the similarity principle and this dimension reducing method cannot be applied.

Secondly, the functional form (3) satisfies the valuation relationship (2) with characteristic root equation:

$$Q(\beta,\eta) = \tfrac{1}{2}\sigma_p^2\beta(\beta-1) + \tfrac{1}{2}\sigma_q^2\eta(\eta-1) + \rho\sigma_p\sigma_q\beta\eta + \theta_p\beta + \theta_q\eta - r = 0. \tag{4}$$

This is the two-factor equivalent of the Q function for the one-factor model discussed by Dixit and Pindyck (1994). The function $Q = 0$ defines an ellipse that passes through all four quadrants and therefore the roots β and η may take on any value in the real two-dimensional plane:

$$\begin{aligned}
&\mathrm{I}: && \{\beta_1, \eta_1\}\ \beta_1 \geq 0, \eta_1 \geq 0;\\
&\mathrm{II}: && \{\beta_2, \eta_2\}\ \beta_2 \geq 0, \eta_2 \leq 0;\\
&\mathrm{III}: && \{\beta_3, \eta_3\}\ \beta_3 \leq 0, \eta_3 \leq 0;\\
&\mathrm{IV}: && \{\beta_4, \eta_4\}\ \beta_4 \leq 0, \eta_4 \geq 0.
\end{aligned}$$

This suggests that the generic function (3), which is the solution to (2), takes the specific form:

$$F = A_1 p^{\beta_1} q^{\eta_1} + A_2 p^{\beta_2} q^{\eta_2} + A_3 p^{\beta_3} q^{\eta_3} + A_4 p^{\beta_4} q^{\eta_4} + \frac{pq}{r - \phi}. \quad (5)$$

We now invoke the limiting boundary conditions to constrain the form of (5). The effect of a renovation investment is to improve the asset quality from its prevailing inferior level to a certain superior level in order to benefit from an ameliorated cash flow stream. For a high quality level close to its initial level, there is no economic incentive to make the renovation investment, and so the renovation option value under these conditions has to be close to zero. It follows that η has to be negative and cannot belong to quadrants I or IV, and so A_1 and A_4 have to be zero. In contrast, if the prevailing quality level is low and close to zero, there is a strong economic justification for making the renovation investment. It follows that η can only belong to quadrants II or III. Although property renovation improves the asset quality level by a certain degree, renovation has no effect on the rental price since this is a market figure. Even so, the renovation is only affordable for a sufficiently high rental price. If the rental price is low and close to zero, renovation is unlikely to occur because it is not economically viable, so the renovation option value is close to zero. It follows that β cannot be negative and the solution quadrant is not III or IV. In contrast, if the prevailing rental price is high, there is an incentive to renovate and to take advantage of the propitious conditions, so the renovation option value is high. This implies that β is positive and the solution quadrant is I or II.

Collectively, the relevant solution quadrant for the renovation model is II. It follows from (5) that the specific solution becomes:

$$F = A_2 p^{\beta_2} q^{\eta_2} + \frac{pq}{r - \phi}. \quad (6)$$

Economic Boundary Conditions

The value matching boundary condition identifies the renovation event when it is economically justified to make an investment in quality improvement. Renovation is assumed to occur when the asset quality has deteriorated to the inferior threshold level $\underline{q}$.

At the prevailing market rental price p, the value of the asset in this state is $F(p,\underline{q})$. The act of renovating the asset raises the quality to the superior threshold level $\overline{q}$ and the renovated asset value becomes $F(p,\overline{q})$ at the prevailing market rental price p. The effect of making the renovation investment cost K is to improve the asset quality level from $\underline{q}$ to $\overline{q}$ where $\overline{q} > \underline{q}$. The two threshold levels, $\underline{q}$ and $\overline{q}$, are selected to maximise the expected benefits altered by the renovation. The optimal choices for p, $\underline{q}$ and $\overline{q}$ are the solutions to:

$$0 = \max_{p,\overline{q},\underline{q}} \left\{ F(p,\overline{q}) - K - F(p,\underline{q}) \right\}. \tag{7}$$

subject to $\overline{q} > \underline{q}$. The expression being maximized represents the renovated asset value less the renovation cost and the sacrificed asset value prior to renovation, all defined at the renovation event. The value matching condition requires an overall zero gain at renovation because of the indifference between the asset with a superior threshold quality after expending the renovating cost and the same asset prior to renovation. The optimal values determined from (7) are denoted by $\hat{p}$, $\underline{\hat{q}}$ and $\hat{\hat{q}}$ respectively. The explicit value matching relationship for the renovation model is found by substituting (6) in (7):

$$A_2 \hat{p}^{\beta_2} \underline{\hat{q}}^{\eta_2} + \frac{\hat{p}\underline{\hat{q}}}{r - \phi} = A_2 \hat{p}^{\beta_2} \hat{\hat{q}}^{\eta_2} + \frac{\hat{p}\hat{\hat{q}}}{r - \phi} - K. \tag{8}$$

The smooth pasting condition for p can be expressed as:

$$A_2 = \frac{\hat{p}\left(\hat{\hat{q}} - \underline{\hat{q}}\right)}{\beta_2 (r - \phi)} \frac{1}{\hat{p}^{\beta_2} \left(\underline{\hat{q}}^{\eta_2} - \hat{\hat{q}}^{\eta_2}\right)}, \tag{9}$$

which is positive since $\beta_2 > 0$ and $\eta_2 < 0$. Replacing A_2 in (8) yields the reduced form value matching relationship:

$$\frac{\hat{p}\hat{\hat{q}} - \hat{p}\underline{\hat{q}}}{(r - \phi)} = \left(\frac{\beta_2}{\beta_2 - 1} \right) K > 0. \tag{10}$$

Since both sides of (10) have to be positive, otherwise no renovation would take place, then $\beta_2 > 1$. It follows that the mark-up factor $\beta_2 / (\beta_2 - 1)$ is at least one.

Renovating a deteriorating asset is viable when the incremental rental income rendered by the renovation, evaluated as a perpetuity, equals or exceeds the renovation investment cost adjusted by the markup factor. This finding is universally valid for all renovation investment costs provided that K is not dependent on the rental price p. The viability condition (10) represents the asset renovation version of the equivalent statement for an investment opportunity, McDonald and Siegel (1986) and Dixit and Pindyck (1994). Renovating a deteriorating asset in an uncertain environment is economically justified only if the incremental rental revenue yielded by the renovation significantly exceeds the investment cost. This result contrasts with the net present value rule that justifies asset renovation provided that the investment cost is exceeded by the incremental rental revenue.

The smooth pasting condition for $\underline{q}$ can be expressed as:

$$A_2 = -\frac{\hat{p}\hat{\underline{q}}}{\eta_2 (r - \phi)} \frac{1}{\hat{p}^{\beta_2} \hat{\underline{q}}^{\eta_2}} \tag{11}$$

which is positive since $\eta_2 < 0$. Substituting for A_2 in (7) yields:

$$\frac{\hat{p}\hat{\overline{q}} - \hat{p}\hat{\underline{q}}}{r - \phi} = K - \frac{\hat{p}\hat{\underline{q}}}{\eta_2 (r - \phi)} \left(1 - \frac{\hat{\overline{q}}^{\eta_2}}{\hat{\underline{q}}^{\eta_2}} \right), \tag{12}$$

where $\frac{-1}{\eta_2} \left(1 - \frac{\hat{\overline{q}}^{\eta_2}}{\hat{\underline{q}}^{\eta_2}} \right) > 0$. The viability condition (12), similarly, demonstrates that the incremental rental income rendered by the renovation has to exceed the investment cost by a positive amount.

The smooth pasting condition for $\overline{q}$ can be expressed as:

$$A_2 = -\frac{\hat{p}\hat{\overline{q}}}{\eta_2 (r - \phi)} \frac{1}{\hat{p}^{\beta_2} \hat{\overline{q}}^{\eta_2}} . \tag{13}$$

A comparison of (11) and (13) reveals an obvious contradiction that infringes the value matching relationship and makes the model indeterminate. The indeterminacy of the model arises because for any given values of $\hat{p}$ and K, there are an infinite number of combinations of $\underline{q}$ and $\overline{q}$ that satisfy the value matching relationship. There are three alternative methods for resolving the indeterminacy: (i) the renovation investment cost can be expressed as a function of the quality levels, (ii) the ratio of quality levels can be treated as fixed, or (iii) the upper quality level can be set to equal a known, fixed upper limit.

If the superior quality level $\hat{\hat{q}}$ is fixed at the norm level of one, the reduced value matching relationships (10) and (12) become respectively:

$$\frac{\hat{p}\left(1-\hat{\underline{q}}\right)}{\left(r-\phi\right)}=\left(\frac{\beta_2}{\beta_2-1}\right)K, \tag{14}$$

$$\frac{\hat{p}\left(1-\hat{\underline{q}}\right)}{r-\phi}=K-\frac{\hat{p}\hat{\underline{q}}}{\eta_2\left(r-\phi\right)}\left(1-\hat{\underline{q}}^{-\eta_2}\right). \tag{15}$$

The two factor model is constituted by the two expressions of the value matching relationships (14) and (15), and the characteristic root equation (4).

Single Renovation Opportunity

When only a single renovation opportunity exists for the asset under study, the valuation of the renovated asset depends on only the present value of the revenue at the superior quality level of one, but excludes the option element. Under these circumstances, the amended value matching relationship from (7) can be expressed as:

$$A_{22}\hat{p}^{\beta_{22}}\hat{\underline{q}}^{\eta_{22}}+\frac{\hat{p}\hat{\underline{q}}}{r-\phi}=\frac{\hat{p}}{r-\phi}-K. \tag{16}$$

By substituting for A_{22} in (16) by using the smooth pasting condition for p and $\underline{q}$, the reduced forms of the value matching relationship become respectively:

$$\frac{\hat{p}\left(1-\hat{\underline{q}}\right)}{\left(r-\phi\right)}=\left(\frac{\beta_{22}}{\beta_{22}-1}\right)K, \tag{17}$$

$$\frac{\hat{p}\left(1-\underline{\hat{q}}\right)}{r-\phi} = K - \frac{1}{\eta_2}\frac{\hat{p}\underline{\hat{q}}}{r-\phi}. \tag{18}$$

Renovation Opportunities with Costs

We now extend the renovation model and investigate the impact of an explicit cost structure on the renovation policy. Our aim is to formulate, within the confines of a two-factor model, a variable that represents the continuous operating and maintenance cost for the incumbent. The process of asset deterioration is anticipated to result in an increase in operating and maintenance cost as well as a relative fall in rental revenue due to the loss in quality. The extent of the deterioration experienced by the asset since its last renovation is the decline in quality level, or $q_0 - q$, where q_0 denotes the quality level immediately following the renovation. If the variable cost element and deterioration are related such that the cost depends linearly on the quality loss, this cost element can be specified by $k_q\left(q_0 - q\right)$, where the constant of proportionality k_q measures the increase in operating and maintenance cost due to a unit quality loss. Since any constant term appears identically in the valuations of the incumbent value and the renovated asset value, it can be safely ignored. Therefore, the valuation relationship (2) has to be amended by incorporating the cost variable, which is represented by $-k_q q$. The relationship becomes:

$$\begin{aligned}\tfrac{1}{2}\sigma_p^2 p^2\frac{\partial^2 F}{\partial p^2} + \tfrac{1}{2}\sigma_q^2 q^2\frac{\partial^2 F}{\partial q^2} + \rho\sigma_p\sigma_q pq\frac{\partial^2 F}{\partial p\partial q}\\ + \theta_p p\frac{\partial F}{\partial p} + \theta_q q\frac{\partial F}{\partial q} - rF + pq + k_q q = 0.\end{aligned} \tag{19}$$

The policy for renovation with costs is derived by applying the line of argument similar to that used to develop the original renovation model. The valuation function, which is the solution to (19), is given by:

$$F = A_{3,2}p^{\beta_2}q^{\eta_2} + \frac{pq}{r-\phi} + \frac{k_q q}{r-\theta_q}. \tag{20}$$

The value matching relationship becomes:

$$A_{3,2}\hat{p}^{\beta_2}\underline{\hat{q}}^{\eta_2} + \frac{\hat{p}\underline{\hat{q}}}{r-\phi} + \frac{k_q\underline{\hat{q}}}{r-\theta_q} = A_{3,2}\hat{p}^{\beta_2}q_0^{\eta_2} + \frac{\hat{p}q_0}{r-\phi} + \frac{k_q q_0}{r-\theta_q} - K. \tag{21}$$

When the expression for $A_{3,2}$, which is determined from the smooth pasting condition with respect to p and is similar to (9), is substituted into (21), the reduced form becomes:

$$\frac{\hat{p}\left(q_0-\underline{\hat{q}}\right)}{r-\phi} = \left(\frac{\beta_2}{\beta_2-1}\right)\left[K - \frac{k_q\left(q_0-\underline{\hat{q}}\right)}{r-\theta_q}\right]. \tag{22}$$

By comparing (17) with (22), the effect of including a operating and maintenance cost term in the valuation relationship is to effectively reduce the renovation cost K by the amount $\hat{p}\left(q_0-\underline{\hat{q}}\right)/\left(r-\theta_q\right)$, which denotes the cost saving expressed as a perpetuity. The reduced form (22) states that the incremental revenue rendered by the renovation has to equal the renovation cost less the cost saving adjusted by the mark-up factor $\beta_2/\left(\beta_2-1\right)$.

The smooth pasting condition for $\underline{q}$ can be expressed as:

$$A_{3,2} = \frac{-1}{\eta_2}\left[\frac{\hat{p}\underline{\hat{q}}}{r-\phi} + \frac{k_q\underline{\hat{q}}}{r-\theta_q}\right]\frac{1}{\hat{p}^{\beta_2}\underline{\hat{q}}^{\eta_2}}. \tag{23}$$

Replacing $A_{3,2}$ from (23) in (21) yields the second reduced form:

$$\frac{\hat{p}\left(q_0-\underline{\hat{q}}\right)}{r-\phi} + \frac{k_q\left(q_0-\underline{\hat{q}}\right)}{r-\theta_q} = K - \frac{1}{\eta_2}\left[\frac{\hat{p}\underline{\hat{q}}}{r-\phi} + \frac{k_q\underline{\hat{q}}}{r-\theta_q}\right]\left[1-\left(\frac{q_0}{\underline{\hat{q}}}\right)^{\eta_2}\right]. \tag{24}$$

This states that by renovating the asset, the sum of the incremental rental revenue and the incremental cost saving, both evaluated as perpetuities, exceeds the renovation investment cost by the positive amount $-\frac{1}{\eta_2}\left[\frac{\hat{p}\underline{\hat{q}}}{r-\phi} + \frac{k_q\underline{\hat{q}}}{r-\theta_q}\right]\left[1-\left(q_0/\underline{\hat{q}}\right)^{\eta_2}\right]$. The renovation model with costs is constituted by the two reduced form equations, (22) and (24), and the Q function (4).

Numerical Solution and Sensitivity Analysis

Renovation Boundary

The renovation models that are developed in the last section are characterised by two reduced form equations, which are derived from the value matching relationship, and the characteristic root Q function. Unfortunately, none of these models is sufficiently tractable to yield an explicit solution for the renovation policy. It is, therefore, necessary to determine the solution numerically from the three simultaneous equations. Even so, this is considerably less onerous than resorting to the finite difference method that is used by Childs, Riddiough and Triantis (1996) and Childs, Mauer and Ott (2005). Although only three equations constitute the model, there are four unknowns, which are the rental price level $\hat{p}$, the quality threshold $\underline{\hat{q}}$, and the two characteristic roots β_2 and η_2. But, model indeterminacy is not relevant here since we are seeking to determine the renovation boundary that relates $\hat{p}$ with $\underline{\hat{q}}$.

Normally, the discriminatory boundary dividing the continuance and the investment decisions is determined from pairs of values for the two stochastic factors. However, the circumstances surrounding the phenomenon of renovation are different and suggest that an alternative representation would be more informative. At any time during the asset lifetime, the owner observes the market rental price and then deliberates on whether or not renovation is viable at the prevailing inferior quality level for a certain investment cost. This view of the renovation phenomenon recommends that the discriminatory boundary should be formed as pairs of values for the prevailing quality level and the renovation investment cost for a specified rental price. Accordingly, the determination of the discriminatory boundary first requires setting the rental price $\hat{p}$ at a specific level, and then proceeds to sequentially evaluate from the model equations the numerical solutions of the quality threshold $\underline{\hat{q}}$ and the two characteristic roots, β_2 and η_2, for a set of values for the investment cost K. The base case data for the renovation model are presented in Table 1.

TABLE 1
Base Case Data for the Renovation Model

Rental Price Level	$\hat{p}$	50.0
Superior Quality Level	q_0	1.0
Rental Price Risk Neutral Drift Rate	θ_p	0%
Rental Price Volatility	σ_p	20%
Quality Risk Neutral Drift Rate	θ_q	-4%
Quality Volatility	σ_q	10%
Rental Price Quality Correlation	ρ	0%
Risk-free Interest Rate	r	7%

Figure 1 illustrates the discriminatory boundary labeled AB, which relates $\underline{\hat{q}}$ with K, for the original renovation model characterized by (14), (15) and (4), and using the data exhibited in Table 1. This figure also reveals the regions of continuance and renovation that are separated by the boundary. For a specific rental price and quality level, renovating a property remains viable for low investment costs, but as soon as the investment cost exceeds the level identified by the boundary, renovation is no longer viable and the optimal strategy is not to renovate. Further, the boundary is downward sloping so falling quality levels are associated with rising investment cost levels. At the ruling market rental price, property assets having a lower prevailing quality level have to incur a greater renovation investment cost in order to bring the quality up to the norm level. Greater differences between the norm and the prevailing quality levels demand higher investment cost levels for renovation. From Figure 1, the most an owner should be willing to commit to renovating their asset for a prevailing rental price of 50 and an observed quality level of 50% is 105.33[1].

At the point A on the discriminatory boundary, a zero investment cost is insufficient to raise the quality level by any positive amount. Figure 1 reveals that at A, $\beta_2 = 1$ and $\eta_2 = -1.352$, which is the negative characteristic root η_{2-} of the equation $Q(1, \eta_2) = 0$. As we move along the boundary from A towards B, both β_2 and η_2 increase. Although

[1] The real option rule for an investment opportunity with value V, investment cost K and mark-up factor $\beta/(\beta-1)$ is $V = K\beta/(\beta-1)$. This can be interpreted as the greatest investment cost for acquiring the asset is no more than $K = V(\beta-1)/\beta$.

β_2 can in principle increase without limit, the value of η_2 is only permitted to increase to zero. If η_2 exceeds zero and becomes positive, then A_2 whether evaluated from (9) or (11) is negative. This implies that the option element of the asset valuation is negative, which is a result that contradicts option pricing theory. Clearly, the value of η_2 is restricted to the range from η_{2-} to zero. Moreover, this restriction places a limit on the inferior quality threshold. By L'Hospital's rule:

$$\lim_{\eta_2 \to 0^-} \left(1 - \underline{q}^{-\eta_2}\right) / \eta_2 = \ln \underline{q}_{\min} ,$$

where $\underline{q}_{\min}$ denotes the minimum quality threshold. It follows from (14) and (15) that:

$$\underline{q}_{\min} \left(1 - \beta_{2+} \ln \underline{q}_{\min}\right) = 1 ,$$

where β_{2+} is the positive characteristic root of the equation $Q\left(\beta_2, 0\right) = 0$. Since the minimum quality level $\underline{q}_{\min}$ depends on β_{2+}, its value is determined by only the stochastic properties for the rental price. Surprisingly, the minimum quality level is independent of the stochastic properties for the quality level, and is not affected by either the prevailing rental price or the renovation investment cost. It is only possible for $\underline{q}_{\min}$ to approach zero provided that β_{2+} tends to infinity, which only occurs when the rental price volatility σ_p tends to zero. In a world of rental price certainty, a minimum quality threshold level of zero exists. The consequence of rental price uncertainty is to give rise to a minimum quality threshold level such that renovations become infeasible whenever the prevailing quality level falls beneath that minimum. The existence of a minimum quality threshold implies that there is a maximum investment cost above which renovation is not feasible. Using the data exhibited in Table 1, then $\beta_{2+} = 2.4365$ and $\underline{q}_{\min} = 0.2064$, so the maximum renovation investment cost is 212.67.

Redevelopment in the Williams (1997) model is signaled when the asset state variable attains the threshold level. Like other one-factor models, his model supplies a threshold condition based on a single variable since the transformation on the value matching relationship reduces the dimensionality to one. The discriminatory boundary is represented by a single point. In contrast, the two-factor renovation model yields a two-dimensional discriminatory boundary that is composed of a countless set of paired values. The renovation investment cost that is required to remedy the quality deficit is evaluated for all possible quality levels. In essence, the renovation cost is not treated as fixed but is allowed to vary. This means that an asset-owner can, at any time during the renovation cycle, apply the renovation discriminatory boundary to investigate whether or not the required renovation cost is sufficiently small to make renovation economically acceptable. Accordingly, the renovation model interprets renovation as an act that can possibly happen throughout the lifetime of the incumbent, rather than at some predetermined point.

Variations in Rental Price

Figure 2 illustrates the effects of variations in the rental price p on the renovation boundary. Three different rental prices are presented, namely 40, 50 and 60. All three renovation boundaries start at the identical renovation investment cost of zero for a quality level of 100%, and then the boundaries slope downwards as a declining quality level is associated with a greater investment cost for remedying the deficit. Further, all three boundaries terminate at the identical minimum quality level since this level only depends on β_{2+}. However, at the minimum quality level, the cost of renovating the asset is different for each boundary and the largest renovation cost is associated with the greatest rental price. This is explain by their positive association as expressed in (14).

In Figure 2, the renovation for greater rental prices always lies to the right. For any specified quality level, a rise in rental price leads to an increase in the renovation cost. Higher rental prices are able to support a greater cost incurred in renovating the asset, so when the rental price increases, the asset-owner is more agreeable to spending more on asset renovation for remedying the quality deficit because of the greater improved future benefits from the renovation. In fact, a proportionate increase in rental price corresponds to an identical proportionate change in the renovation cost. Asset-owners are more predisposed to renovating their properties when rental prices are observed to be increasing.

Variations in Volatility

Figure 3 illustrates the effects of variations in the rental price volatility σ_p on the renovation boundary. Four different rental price volatilities are presented, namely 0%, 10%, 20% and 30%. The renovation boundaries, which all start at the identical point at a quality level of 100% and a renovation cost of zero, are all downward sloping. All four renovation boundaries terminate at different minimum quality levels, which increase for increases in volatility since β_{2+} varies inversely with σ_p. The renovation boundaries for greater rental price volatilities always lie to the left. For a specific quality level, the most the asset-owner is willing to spend on renovating the asset and remedying the quality deficit is always less for increases in the rental price volatility. If the rental price volatility rises, there is greater uncertainty concerning the improved future benefits produced by the renovation and consequently, this increases the sacrificial value of the renovation cost. Asset-owners are more reluctant to renovate their properties in a climate of increasing rental price volatility.

Figure 4 illustrates the effect of variations in the asset quality volatility on the renovation boundary. Two boundaries are presented for $\sigma_q = 0\%$ and $\sigma_q = 100\%$. Both curves are downward sloping. They both start at the same point where $\underline{\hat{q}} = 100\%$ and $K = 0$, and terminate at the same point where $\underline{\hat{q}}$ equals the minimum quality level. The two curves form an envelope, within which lies the boundaries for other volatility levels that fall within the range. Increases in the quality volatility bends the boundary towards the left, but leaves the two end points unaffected.

Conclusion

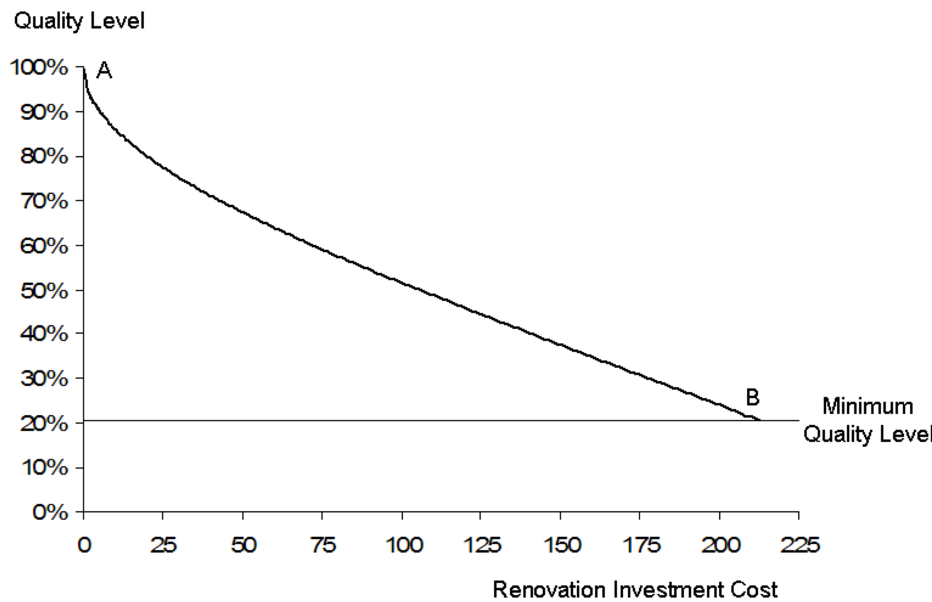
We determine the optimal discriminatory boundary for renovating a property asset that is subject to stochastic rental prices and quality levels, and a fixed investment cost. The renovation policy is found analytically as the implicit solution to a set of simultaneous equations derived from the economic boundary conditions. This quasi-analytical method has several advantages compared with the alternatives. An implicit solution to the boundary is obtainable even though dimension reducing transformations are unavailable. Further, numerical methods such as finite difference are computationally onerous and lack the transparency of an analytical solution. In contrast, the quasi-analytical method delivers a relatively effortless solution, albeit in an implicit form, on which various sensitivity experiments are facilitated.

The economic boundary conditions for the renovation model yield a rule on the viability of renovating. It states that renovation is economically prudent whenever the incremental value from upgrading the property quality exceeds by a mark-up factor the investment cost of renovation. This rule on renovation investment is the two-factor equivalent of the one-factor rule formulated by McDonald and Siegel (1986) and Dixit and Pindyck (1994) and demonstrates that their fundamental conclusion is extendable to other contexts. The renovation policy is characterised by the discriminatory boundary that is expressed as the trade-off between inferior quality levels and the renovation cost required to bring the property back to its pristine state. The shape of the discriminatory boundary establishes that the level of renovation investment required for a specific market rental price increases with the quality deficit and that progressively inferior quality levels demand greater amounts of investment to remedy the deterioration. Moreover, the most the property owner is willing to invest on property renovation depends positively on the market rental price and for a specific level of quality deterioration, higher rental prices can absorb greater amounts of renovation investment. Finally, the renovation policy is affected by the degree of uncertainty. The most the property owner is willing to invest on renovation is adversely affected by the extent of the volatility for specific levels of rental price and quality. A reduction in volatility for either the rental price or quality increases the amount the property owner is prepared to invest viably on renovation. Since volatility increases reduces the viable amount of renovation investment, property owners have to exercise greater caution whenever there is a rise in volatility.

The combination of market rental price and property quality in a renovation model produces a richer conceptualization than the existing one-factor representations and the quasi-analytical approach delivers the renovation policy with greater facility than the finite difference method. However, the representation ignores the existence of other options available to the property owner such as the temporary suspension of rental services or abandonment.

Figure 1

Discriminatory Boundary for the Multiple Opportunity Renovation Model



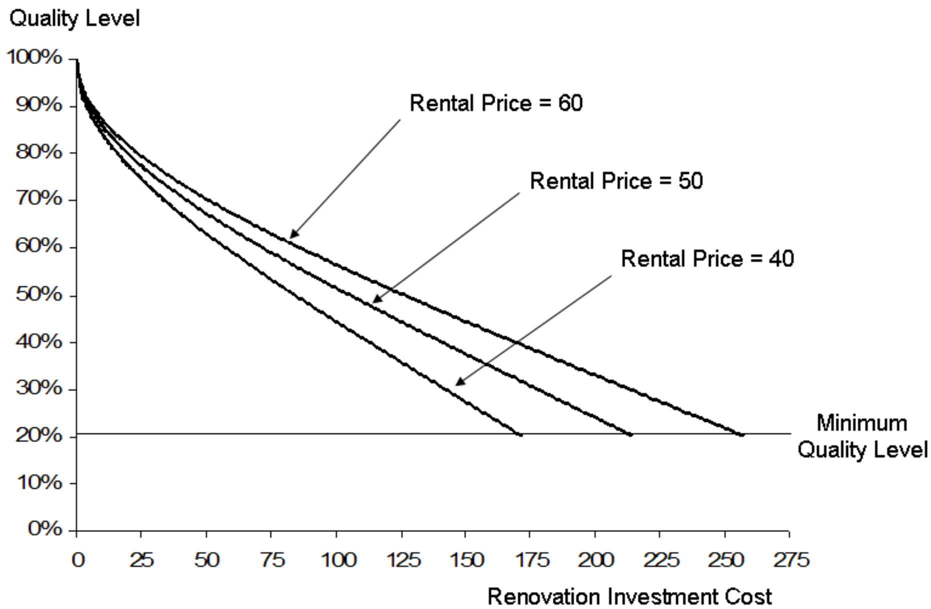
Typical values are

$\underline{\hat{q}}$	K	β_2	η_2
100.0%	0.000	1.00000	-1.35235
90.0%	5.179	1.12860	-1.30254
80.0%	19.935	1.28087	-1.22809
70.0%	42.802	1.45748	-1.11987
60.0%	71.946	1.65482	-0.96952
50.0%	105.335	1.86384	-0.77350
40.0%	141.107	2.07207	-0.53630
30.0%	177.916	2.26842	-0.26873
20.6%	212.686	2.43649	0.00000

Figure 2

Discriminatory Boundary for the Multiple Opportunity Renovation Model

for Three Different Rental Prices



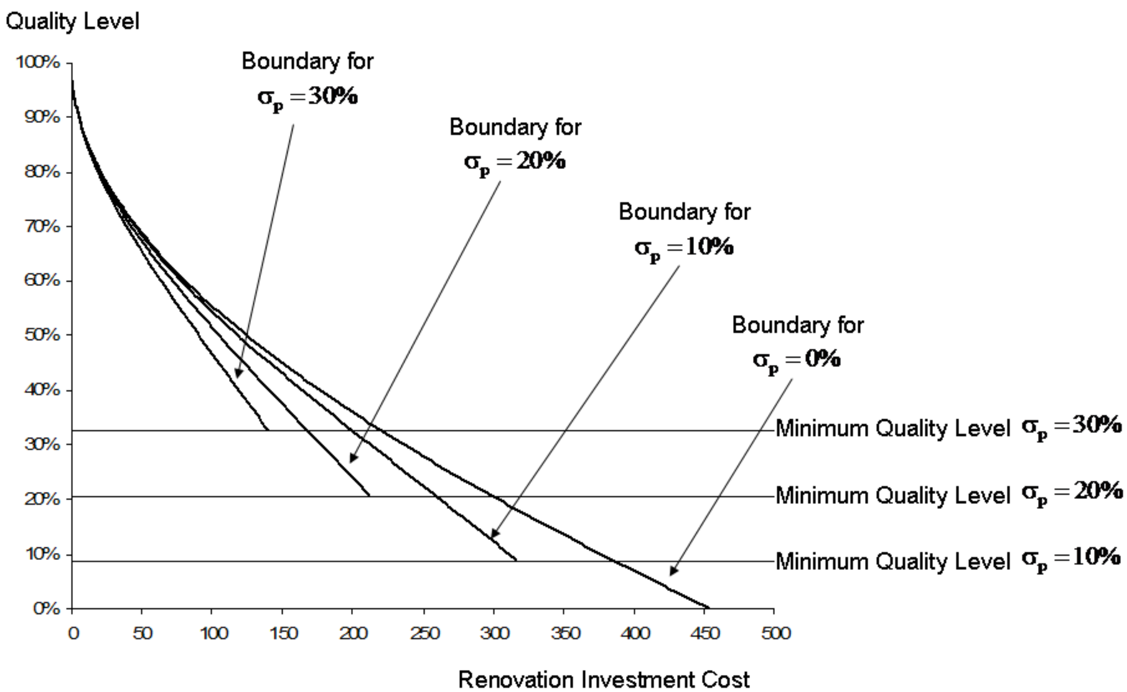
Typical values are:

			Rental Price p		
			40	50	60
$\underline{\hat{q}}$	β_2	η_2	K	K	K
100.0%	1.00000	-1.35235	0.000	0.000	0.000
90.0%	1.12860	-1.30254	4.143	5.179	6.215
80.0%	1.28087	-1.22809	15.948	19.935	23.921
70.0%	1.45748	-1.11987	34.242	42.802	51.363
60.0%	1.65482	-0.96952	57.557	71.946	86.335
50.0%	1.86384	-0.77350	84.268	105.335	126.402
40.0%	2.07207	-0.53630	112.885	141.107	169.328
30.0%	2.26842	-0.26873	142.333	177.916	213.499
20.6%	2.43649	0.00000	170.149	212.686	255.224

Figure 3

Renovation Boundary for Multiple Opportunities

for Three Different Rental Price Volatilities



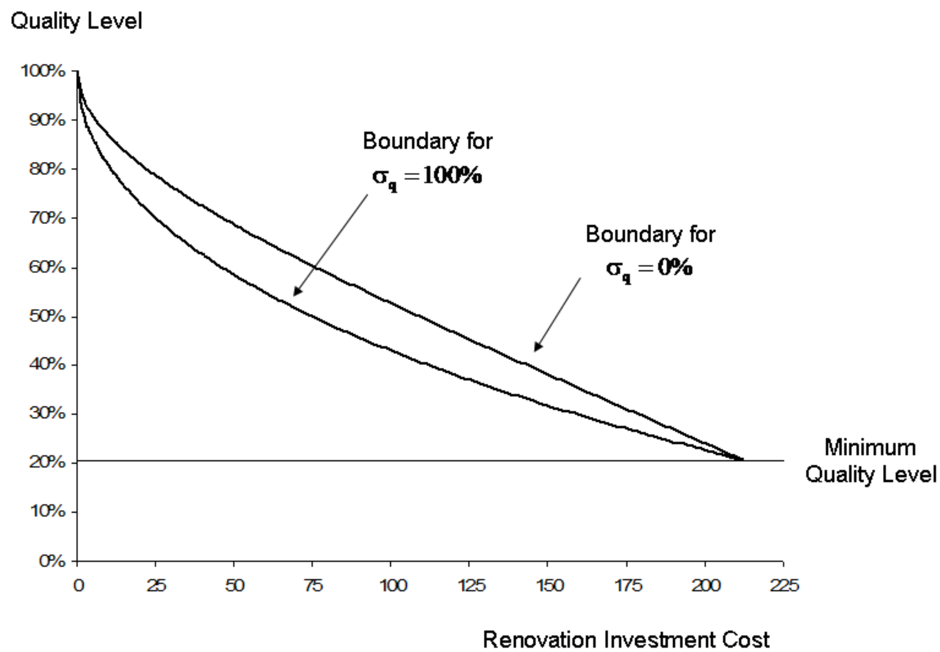
Typical values are:

	$\sigma_p = 0\%$	$\sigma_p = 10\%$	$\sigma_p = 20\%$	$\sigma_p = 30\%$
$\underline{\hat{q}}$	K	K	K	K
100.0%	0.000	0.000	0.000	0.000
90.0%	5.282	5.256	5.179	5.056
80.0%	20.865	20.623	19.935	18.895
70.0%	46.329	45.374	42.802	39.259
60.0%	81.219	78.541	71.946	63.866
50.0%	125.035	118.796	105.335	90.858
45.0%	150.120	141.040	123.022	104.843
40.0%	177.221	164.344	141.107	119.018
35.0%	206.257	188.419	159.441	133.319
30.0%	237.139	212.975	177.916	
25.0%	269.770	237.747	196.466	
20.0%	304.039	262.520		
15.0%	339.822	287.162		
10.0%	376.973	311.656		
5.0%	415.305			

Figure 4

Renovation Boundary for Multiple Opportunities

for Two Different Quality Volatilities



Typical values are

$\hat{q}$	$\sigma_q = 0\%$	$\sigma_q = 20\%$	$\sigma_q = 40\%$	$\sigma_q = 60\%$	$\sigma_q = 80\%$	$\sigma_q = 100\%$
100.00%	0.000	0.000	0.000	0.000	0.000	0.000
90.00%	5.919	4.277	3.336	2.932	2.728	2.613
80.00%	22.211	16.906	13.515	11.995	11.216	10.774
70.00%	46.423	37.408	30.767	27.627	25.986	25.044
60.00%	76.031	64.974	55.243	50.299	47.645	46.102
50.00%	108.876	98.345	86.901	80.480	76.898	74.779
40.00%	143.459	135.792	125.273	118.510	114.510	112.078
30.00%	178.942	175.329	169.067	164.242	161.116	159.124
20.64%	212.686	212.686	212.686	212.686	212.686	212.686

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