The Learning Curve and the Optimal Investment Under Uncertainty

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Abstract

We investigate the effects of the learning curve on the timing and intensity of investment. The learning curve generates signficant scale effects: if the efficiency gains from learning are large and the learning speed fast, marginal capital profits are increasing. In a capital expansion model this implies that the firm undertakes an initial lumpy investment followed by further marginal adjustments. However, we investigate a model in which investment is lumpy by assumption and the firm chooses its capacity once and for all. We find that the effects of the learning curve are ambiguous: if the speed of learning is high investment occurs earlier and on a smaller scale. If the learning speed is low the scale of investment increases while the effect on timing is ambiguous.

1 Introduction

The learning curve hypothesis states that unit costs decrease with cumulative production. While producing, a firm exploits a process of learning-by-doing that will lead to increased efficiency and lower production costs in the future. There is ample empirical evidence documenting the presence of the learning effects in various industries (Wright (1936), Hirsh (1952), Webbink (1977), Zimmerman (1982), Lieberman (1984), Argote, Beckman and Epple (1990), Gruber (1992) and Bahk and Gort (1993) among others).

The learning curve has been recognized as a key factor behind the firms' production and competitive policies (see Spence (1981), Fudenburg and Tirole (1983), Dasgupta and Stiglitz (1988), Majd and Pindyck (1989), Cabral and Riordan (1994), Cabral and Riordan (1997)). Majd and Pindyck (1989) determine the optimal production rate under the learning curve and uncertain demand. They show that the conditions that make the firm willing to produce are less stringent. Even when marginal revenue is lower than marginal cost it may be optimal for the firm to produce.

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This is because, in addition to the value of generating profits, production has the additional value of reducing future marginal costs. However, Majd and Pindyck (1989) only focus on the production choice while the capital stock is fixed, implying that production capacity is given. It follows that the speed of learning is bounded by the fixed capacity.

We extend their investigation by analyzing how the learning curve affects investment. To do so we exploit the real options framework which allows us to analyze optimal capacity choice in continuous time, with flexible timing, irreversibility and an uncertain demand.

Our first result is that the learning curve generates a significant scale effect. If the learning efficiency gains are sufficiently large and the learning process relatively fast, the profit function is convex-concave, that is, for low levels of capacity, returns to capital are increasing. For a linear demand and a constant returns to capital production function we can derive a simple condition under which increasing returns are generated. However, this result is robust also to different specifications.

A question that immediately arises is how increasing returns affects the firm's investment strategy. Before answering this question, however, we must define what the firm can actually do when choosing its capacity. In other words, we have first to specify our assumptions about which investment option(s) the firm holds. In the real options literature a large strand of research (see, Pindyck (1988), He and Pindyck (1992), Abel and Eberly (1994), Aguerrevere (2003), Guo, Miao and Morellec (2005)) investigates capacity expansion models. In those models the firm, under the standard assumption of decreasing marginal returns, expands its capacity sequentially and can choose in any point in time to increases its capital stock on the basis of the evolution of the stochastic selling price. In this context, the optimal strategy of the firm is to adjust its capacity to prevent the marginal productivity to cross a barrier that defines the investment threshold. It is shown that this threshold equals the sunk marginal cost multiplied by a factor larger than one which gives account the option to wait generated by uncertainty (see Dixit and Pindyck (1994), Chapter 11 p.362).

What would the effect of increasing returns be in a capital expansion model? More specifically, how would the expansion path of a learning firm differ from that one of a firm producing at a constant marginal cost equal? Two facts must be taken into account. First, as we will show, the value of a marginal unit for the learning firm is larger than that one of the constant marginal cost firm. This implies that the investment threshold for the learning firm will be lower. Second, when the conditions for the initial investment are met, the constant marginal cost firm will invest only in the first marginal unit. On the contrary, when the profit function is convex-concave, the learning firm will invest in a "discrete" amount of capital. Given increasing marginal profits, the selling price that justify the investment in the first unit of capital triggers the investment also in some following units. Dixit (1995) investigates the expansion model in a real option context with a convex-concave production function. He shows how the firm's investment path is indeed defined by an initial lumpy investment followed by further marginal adjustments. On the basis of these considerations, we can unambiguously conclude that in a capacity expansion model the effect of the learning curve is twofold: investment occurs earlier and on a larger scale.

To enrich the analysis, however, we depart from the framework described above and investigate the consequences of the presence of the learning curve in a model where investment is lumpy by assumption. In other words while in a capacity expansion model lumpiness is an endogenous effect of the presence of the learning curve (if the learning curve generates increasing marginal profits the investment is lumpy, but it is marginal otherwise) we consider a model in which investment is lumpy both for a constant marginal cost firm and for a learning firm, independently of the presence of increasing returns. In order to do so we build on the framework proposed by Bar Ilan and Strange (1999) (BI&S, henceforth) and consider a strategy of a firm that has the option to choose once and for all its productive capacity. Investigating a lumpy investment model, allows us also to give account of the well acknowledged fact that investment occurs in spikes not only at aggregate but also at micro level (see, e.g. Caballero et al. (1995), Cooper and Haltiwanger (1999), Doms and Dunne (1998))

A direct consequence of our modelling choice is that the implications of increasing returns are less striking. As already mentioned, in the expansion models lumpiness is an endogenous effect of the presence of the learning curve, while in our specification this is an exogenously imposed assumption. But, on the other hand, in our model the effects of learning are more difficult to pin down ex ante.

We find that the influence of the learning curve on timing and intensity of investment is ambiguous. To explain the ambiguity we need to remind, first, a result of BI&S (1999): in general, factors that delay the investment increases its intensity when it occurs. In the remainder we will call this result bad news effect. When a bad news arrives, the option value to delay the investment increases so that investment occurs when the value of the project, determined by the output selling price, is higher. Therefore, at the time of investment the marginal productivity of capital is higher and the scale of the optimal capacity also grows. For the same principle, factors that accelerate the investment usually decrease its intensity. Given that the learning curve increases the value of capital, investment tends to occurs earlier and, according the above described mechanism, on a smaller scale.

However, larger capacity, increasing the per-period prodution rate, guarantees that the benefits of the learning curve are obtained more rapidly. For this reason, the firm has an incentive to increase its scale, in contrast to the bad news effect. When the learning curve increases the scale of the investment, given that larger capacity implies a higher sunk costs, the firm may prefer to postpone it. Therefore, the effect of the learning curve on timing and intensity of investment is ambiguous. Compared to a firm that produces at constant marginal cost, a learning firm invests earlier and less if the learning speed is high. High learning speed implies that the firm does not need a large per-period production rate, i.e. large capacity, to move down along the learning curve. On the contrary, when the learning speed is low, it invests on a larger scale, while the effect on timing is ambiguous.

This article is organized as follows. The next section presents our modelling assumptions. Section 3 shows the optimal policy of a firm that produces at constant marginal cost. Section 4 is the bulk of our work and investigates the choice of timing and intensity of investment for a firm facing the learning curve, while Section 5 concludes.

2 The learning curve

Consider a firm that at any time t sells its product at a price determined by the inverse demand function

$$X = P - \varphi q, \tag{1}$$

where q is the quantity of output produced in each period, φ is a strictly positive and constant parameter, and P is a demand shift parameter that fluctuates over time according to

$$\frac{dP}{P} = \mu dt + \sigma dZ_t, \tag{2}$$

where dZ_t is the increment of a standard Wiener process. Each unit of the capital stock produces one unit of output, and we assume that the firm always produces up to its capacity, which means that

$$q = K \tag{3}$$

at all times without the possibility of temporary suspension of the production. As we we clarify more in detail in Section 4, the firm holds an option to choose once and for all its productive capacity at per unit cost of capital equal to k. We further suppose that capital does not depreciate and investment is fully irreversible.

To model the learning curve we follow Majd and Pindyck (1989). Starting from an initial level c, marginal cost asymptotically declines to zero with cumulative output Q_t.[1] Given that in our model firm's per-period production rate is constant and equal to the chosen capacity K, it holds $Q_t = tK$. Marginal cost equals

$$c(Q_t) = ce^{-\gamma Q_t}. \tag{4}$$

It follows that, the instantaneous profit at time t is

$$\pi(P, K, Q_t) = \left(P - \varphi K - ce^{-\gamma Q_t}\right) K \tag{5}$$

[1] While we consider an infinite learning, Majd and Pindyck (1989) assume that the learning process as soon as marginal cost reaches a strictly positive lower bound. Using Majd and Pindyck's specification our qualitative results wouldn't change.

The value of the capital in place and the value of the marginal unit of capital stock are equal to their expected discounted stream of profits:

$$V(P,K,0) = E\left[\int_0^\infty \pi(P;Q_t;K)e^{-\rho s}ds \middle| P_0 = P, Q_0 = 0\right],$$

where ρ is the time discount rate. The integral can be directly evaluated.

$$V(P,K,0) = K\left(\frac{P}{\delta} - \frac{\varphi K}{\rho} - \frac{c}{\gamma K + \rho}\right), \tag{6}$$

in which $\delta = \rho - \mu > 0$[2]. Marginal capital profis equals

$$V_K(P,K,0) = \frac{P}{\delta} - \frac{2\varphi K}{\rho} - \frac{c}{\gamma K + \rho} + \frac{\gamma K c}{(\gamma K + \rho)^2}. \tag{7}$$

where the last term represents the benefit that a larger capacity gives to the firm in terms of an increased speed of learning. Note that, for a firm that produces at constant marginal cost, i.e. $\gamma = 0$, value and marginal value of capital are given by $V(P,K) = K\left(\frac{P}{\delta} - \frac{\varphi K}{\rho} - \frac{c}{+\rho}\right)$ and $V_K(P,K,0) = \frac{P}{\delta} - \frac{2\varphi K}{\rho} - \frac{c}{\rho}$, and are lower than (6) and (7).

The presence of the learning curve has an important implication for the scale effect which we establish the in following proposition.

Proposition 1 *The profit function (6) is convex-concave in K if*

$$\gamma > \frac{\varphi\rho}{c}. \tag{8}$$

The proof is relegated in Appendix A.1. The result in Proposition 1 means that, when γ is large enough, the learning curve gives rise to an initial region of increasing marginal profits. In that case marginal and average profit curves have inverse U-shapes.

For given φ and ρ marginal profits are increasing when the initial marginal cost c is large and the learning speed parameter γ is high. The underlying logic is straightforward. The speed at which the firm moves down along the learning curve depends on the learning rate, defined as γK. Therefore, larger capacity guarantees faster learning and greater profits. But on the other hand, it implies also a cost because it reduces the selling price proportional to the parameter φ. When the efficiency gains are substantial and (c is high) and can be reached quickly (γ is also high), marginal productivity of capital is maximal for a sufficiently large K, that is the profit function is convex-concave. Figure 1 shows the shape of the marginal (V_K) and average ($AV = \frac{V}{K}$) profit curves for the parameters values specified in the figure's caption.

[2] If $\rho - \mu \leq 0$, it would be optimal to indefinetely postpone the investment, i.e an investment would never occur.

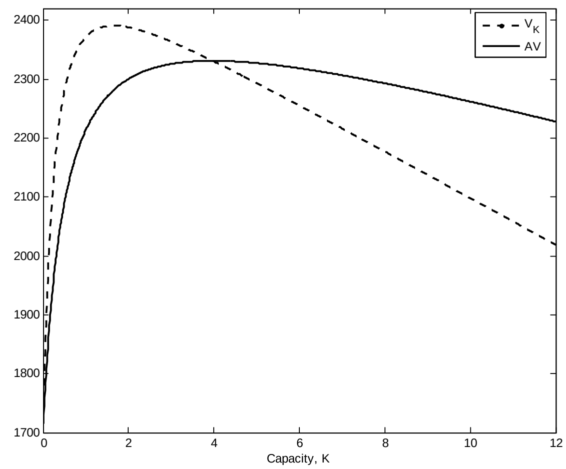

Figure 1: Average (AV) and marginal (V_K) revenue curves. Parameters values are: $P = 100$, $\bar{c} = 40$, $\gamma = 0.1$, $\rho = 0.05$, $\mu = 0.01$, $\varphi = 1$.

3 Timing and intensity of investment

3.1 Constant marginal cost

In this section we investigate the investment strategy of a firm that produces at a constant marginal cost c. Here, our analysis is analogous to Bar Ilan and Strange (1999) with the sole difference that while they employ a Cobb-Douglas production function, $q = K^{\alpha}$ with $\alpha \in (0, 1)$, we use the linear demand with constant returns production specified in equations (1) and (3).

Given our assumptions, the per-period profit is

$$\pi(P, K,) = (P - \varphi K - c)\, K$$

and the value of capital

$$V(P, K) = E\left[\int_t^{\infty} \pi(P, K,) e^{-\rho(s-t)} ds \,\middle|\, P_0 = P, \right],$$

Solving the integral yields

$$V(P, K) = K\left(\frac{P}{\delta} - \frac{\varphi K}{\rho} - \frac{c}{\rho} \right) \tag{9}$$

The firm's problem is to choose once and for all the optimal size and timing of investment. Denote by $F(P, K)$ the option value associated with this investment opportunity. A standard analysis shows that this option satisfies the ordinary differential equation

$$\frac{1}{2}\sigma^2 P F_{PP} + \mu P F_P - \rho F = 0. \tag{10}$$

with general solution $A(K)P^{\beta_1} + B(K)P^{\beta_2}$, where

$$\beta_1 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} + \sqrt{\left[\frac{(\rho-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} > 1$$

$$\beta_2 = \frac{1}{2} - \frac{(\rho-\delta)}{\sigma^2} - \sqrt{\left[\frac{(\rho-\delta)}{\sigma^2} - \frac{1}{2}\right]^2 + \frac{2\rho}{\sigma^2}} < 0$$

are the roots of the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta-1) + \mu\beta - \rho = 0$. The boundary condition

$$\lim_{P\to 0} F(P,K) = 0, \tag{11}$$

implies $F(P,K) = A(K)P$. The coefficient $A(K)$ and the investment trigger $\overline{P}(K)$ are obtained from the value matching and smooth pasting conditions

$$F(\overline{P}(K),K) = V(\overline{P}(K),K) - kK, \tag{12}$$

$$F_P(\overline{P}(K),K) = V_K(\overline{P}(K),K), \tag{13}$$

Substituting the expression for (9) and using the solution of $F(P,K)$, (12) and (13) can be rewritten as

$$A(K)\overline{P}^{\beta_1} = K\left(\frac{\overline{P}}{\delta} - \frac{\varphi K}{\rho} - \frac{c}{\rho} - k\right), \tag{14}$$

$$\beta_1 A(K)\overline{P}^{\beta_1 - 1} = \frac{K}{\delta}. \tag{15}$$

Simple algebra yields

$$\overline{P}(K) = \frac{\beta_1\delta}{\beta_1 - 1}\left(\frac{\varphi K}{\rho} + \frac{c}{\rho} + k\right), \tag{16}$$

and

$$F(P,K) = \frac{K}{\beta_1 - 1}(\frac{\varphi K}{\rho} + \frac{c}{\rho} + k)\left(\frac{\rho\,(\beta_1 - 1)}{\beta_1\delta}\frac{P}{(\varphi K + c + \rho k)}\right)^{\beta_1}. \tag{17}$$

Given that the firm has only the possibility to undertake one investment, it chooses its capacity to maximize $F(P,K)$. Differentiating (17) with respect to K and rearranging yields

$$\overline{K} = \frac{1}{\varphi\,(\beta_1 - 2)}(c + \rho k) \tag{18}$$

Subtituting this expression in (16) gives us the solution for the trigger price

$$\overline{P} = \frac{\beta_1 \delta}{(\beta_1 - 2)} (\frac{c}{\rho} + k), \tag{19}$$

Note that we require $\beta_1 > 2$. If this condition is not satisfied (it happens when volatility is high and the discount factor ρ is low) the firm's investment problem has no finite solution.

In Table 1 we report comparative statics results of our model compared with those of BI&S (1999). All the derivations are in the Appendix. As shown, the effects in the two models are in general consistent and, as stressed by BI&S, display surprising patterns.

For example consider a rise in σ and c. When volatility and marginal production cost rise both in the optimal capacity $\overline{K}$ and in the investment trigger $\overline{P}$ increase. While the effect on investment timing is expected, (the investment is delayed) the effect of intensity is a first sight surprising: the firm invests more. One would expect that the "bad news" of rising uncertainty and production cost should lead to lower investment. However, this is not the case. BI&S (1999) clarify the reason why this happens. When a bad news arrives, the option value to delay the investment increases so that the value matching condition (12) is satisfied for a larger $\overline{P}$. For this reason, at the time of investment the marginal productivity of capital is higher and the scale of the optimal capacity $\overline{K}$ grows. This principle implies that, in general (but not always), the sign of the derivatives of $\overline{K}$ and $\overline{P}$ goes in the same direction. Factors that delay the investment increase its intensity when it occurs. This property will be called in the remainder "bad news effect". To clarify the point and ease the understanding of the learning curve case, tackled in the next section, we further briefly investigate the mechanism underlying the bad news effect.

Our model implies a simultaneous solution for timing and intensity of investment, but for the sake of clarity we consider them separately. The timing is determined by (12) and (13). The intensity is determined by the condition

$$F_K(P, K) = \frac{1}{\beta_1 \delta} \left(\frac{P}{\overline{P}(K)} \right)^{\beta_1} \left[\overline{P}(K) + (1 - \beta_1) \overline{P}_K(K) K \right] = 0 \tag{20}$$

When $F_K(P, K)$ is positive it is optimal for the firm to increase the investment size, and the other way around.[3] The sign of (20) depends on the expression in the square brackets. The first term, $\overline{P}(K)$, is always positive. Therefore, $\overline{P}_K(K)K$ must be also positive.[4] Now, consider how a bad news affects this optimality condition. When a bad news arrives investment is delayed, i.e. $\overline{P}(K)$ increases, and this tends to rise $F_K(P, K)$. In general, $(1 - \beta_1)\overline{P}_K(K)K$ may decrease but not sufficiently to offset

[3] Indeed, $\beta_1 > 2$ guarantees that $F(P, K)$ is a concave function, thus $F_{KK}(P, K) < 0$.

[4] It is never optimal to invest in a capacity level lying in the increasing part of the marginal profits curve. Therefore, it always holds $\overline{P}_K(K) > 0$.

the increase in $\overline{P}(K)$. Therefore, the net effect is to rise $F_K(P,K)$ and this implies that the firm must invest in a larger capacity to restore optimality.

For example, consider an increase in the marginal production cost c, starting from optimality (i.e. (20) is satisfied). From (16) it is immediate to see that, other things being equal, $\overline{P}(K)$ rises. In addition, $\overline{P}_K(K)$ does not depend on c, leaving the second term in (20) unaffected. This implies that the effect on $F_K(P,K)$ is unambiguously positive and that K^* must increase.

For the interpretation of the effect of other parameters we remand the reader to BI&S (1999). Notice how, contrary to BI&S's model, with a linear demand and constant returns to capital the bad news effect applies also for the sunk investment cost k.

3.2 Learning curve

The solution procedure for the learning firm's investment problem follows the same steps outilined in the previous section and eventually yields

$$P^*(K) = \frac{\beta_1\delta}{\beta_1 - 1}\left(\frac{\varphi K}{\rho} + \frac{c}{\gamma K + \rho} + k\right), \tag{21}$$

and

$$F(P,K) = \frac{K}{\beta_1 - 1}\left(\frac{\varphi K}{\rho} + \frac{c}{\gamma K + \rho} + k\right)\left(\frac{\beta_1 - 1}{\beta_1\delta}\frac{P}{\left(\frac{\varphi K^2}{\rho} + \frac{Kc}{\gamma K+\rho} + k\right)}\right)^{\beta_1}, \tag{22}$$

The optimal capacity K^* is implicitly defined by the condition

$$F_K(P,K) = \frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left[P^*(K) + (1-\beta_1)P^*_K(K)K\right] = 0 \tag{23}$$

Condition (23) is satisfied when

$$\frac{\varphi K}{\rho} + \frac{c}{\gamma K + \rho} + k + (1-\beta_1)\left[\frac{\varphi K}{\rho} - \frac{\gamma K c}{(\gamma K + \rho)^2}\right] = 0 \tag{24}$$

or

$$(\beta_1 - 2) = \frac{\rho}{\varphi K}\left[\frac{c\,(\beta_1\gamma K + \rho)}{(\gamma K + \rho)^2} + k\right] > 0. \tag{25}$$

Note that K^* is finite if $(\beta_1 - 2) > 0$, the same condition that we found in the constant marginal cost case.[5] The optimal capacity K^* is the real solution of a cubic equation, but the explicit expression that defines it results too complex to be informative. Yet,

[5] As shown in the Appendix $(\beta_1 - 2)$ guarantees that $F(P,K)$ as defined in (23) is concave.

	Our Model	BI&S (1999)
$V(P,K)$	$K\left(\frac{P}{\delta}-\frac{\varphi K}{\rho}-\frac{c}{\rho}\right)$	$K^{\alpha}\left(\frac{P}{\delta}-\frac{c}{\rho}\right)$
$\overline{K}_{\rho};\,\overline{P}_{\rho}$	$\pm;\pm$	$-;\pm$
$\overline{K}_{k};\overline{P}_{k}$	$+;+$	$-;=$
$\overline{K}_{c};\,\overline{P}_{c}$	$+;+$	$+;+$
$\overline{K}_{\mu};\,\overline{P}_{\mu}$	$+;\pm$	$+;\pm$
$\overline{K}_{\sigma};\overline{P}_{\sigma}$	$+;+$	$+;+$

Table 1: Comparative statics results with constant marginal production costs for our model (linear demand with constant returns to capital production function) and BIS (1999) (Cobb-Douglas production function).

	constant c	learning
$K^*_{\rho};\,P^*_{\rho}$	$\pm;\pm$	$\pm;\pm$
$K^*_{k};P^*_{k}$	$+;+$	$+;+$
$K^*_{\mu};\,P^*_{\mu}$	$+;\pm$	$+;\pm$
$K^*_{\sigma};P^*_{\sigma}$	$+;+$	$+;+$
$K^*_{c};\,P^*_{c}$	$+;+$	$+;+$
$K^*_{\gamma};\,P^*_{\gamma}$	$N.A.$	$\pm;\pm.$

Table 2: Comparative statics. Learning curve and constant marginal cost firm

differentiating the above optimality condition we can analytically define how factors affect K^*.

In Appendix we show that

$$K^*_{\rho} \lesseqgtr 0,\ K^*_{\sigma} > 0,\ K^*_{\mu} > 0, K^*_{k} > 0,\ \ K^*_{\gamma} \lesseqgtr 0,\ \ K^*_{c} < 0 \tag{26}$$

where $\frac{\partial K^*}{\partial \rho} \equiv K^*_{\rho}$. Also for K constant,

$$P^*_{\sigma}(K) > 0,\ P^*_{\mu}(K) \lesseqgtr 0,\ P^*_{k}(K) > 0,\ P^*_{\rho}(K) \lesseqgtr 0,\ P^*_{\gamma}(K) < 0,\ P^*_{c}(K) \tag{27}$$

Given that $P^*_{K}(K) > 0$, $K^*_{\sigma} > 0$ and $P^*_{\sigma}(K) > 0$, the trigger price unambiguously increases in σ. The same reasoning holds for P^*_{k} and P^*_{c}. On the contrary, we cannot analytically pin down the sign of P^*_{μ}, P^*_{ρ} and P^*_{γ}. Numerically we show that the effect is ambiguous for all the three variables.

Table 2 summarizes the results and compare them with the constant marginal cost case. As we show, where a direct comparison is possible, the direction of the effects is the same.

The main goal of our analysis is, however, to investigate the effect of the learning curve on timing and intensity of investment. In this perspective, our interest is focused on the comparative statics effect of γ. As γ decreases the learning effect progressively weakens and eventually vanishes. $\gamma = 0$ corrspond to the constant marginal cost case.

Thus, K^*_γ and P^*_γ provide information about what happens to intensity and timing of investment when the influence of the learning curve on production process becomes more powerful.

As indicated in Table 2, the effect of γ is ambiguous.

Proposition 2 *Consider the optimality condition (23). The optimal capacity is increasing in γ ($K^*_\gamma > 0$), if*

$$\gamma K < \frac{(\beta_1 - 2)\,\rho}{\beta_1} \tag{28}$$

and the other way around.

The above proposition states that an increase in γ is more likely to increase the optimal capacity when the learning rate γK is small. To ease the interpretation of Proposition 2 we rewrite the optimality condition (23) as

$$P^*(K) + (\beta_1 - 1)AV_K(K)K = 0, \tag{29}$$

where $AV = \frac{V(P,K,0)}{K}$. Optimality requires $AV_K(K) < 0$, because K^* lies in the decreasing returns region where both the average and marginal profit curve are declining in K.

An increase in γ, affects both tems in condition (29). The effect on $P^*(K)$ in unambiguous. Given that faster learning increases the value of capital the threshold investment curve defined by $P^*(K)$ shifts down. For this reason, $F_K(P,K)$ tends to be lower pushing the firm to invest in a smaller capacity. This is the bad news effect identified by BI&S (1999). The good news of a faster learning tends to anticipate the investment. Therefore. marginal capital profits when the investment occurs are lower and this reduces the investment size.

However, the effect on AV_K is ambiguous. It is easily shown that $AV_{K\gamma}(K) > 0$ if $\gamma K < \rho$, and the other way around. If (28) holds, the increase in AV_K (a reduction of AV_K in absolute value) is sufficiently strong to more than counterbalance the bad news effect. A larger $AV_K(K)$ means that the the rate at which the average productivity of capital (which determines $P^*(K)$) decreases in K is lower. Indeed, larger capacity increases production costs but also increases the speed of learning making a larger capacity relatively more convenient. If this effect is dominant the size of investment rises. The effect on P^* is also ambiguous. With γ larger, $P^*(K)$ declines, and if the firm chose same capacity stock it would invest earlier. However, the optimal capacity may increase, rising the irreversible expenditure wK. If the increase is the sunk cost is substantial, investment occurs later.

Hence, the ambigous effect of the learning curve on the optimal investment size is totally driven by its influence on the shape of the average productivity curve. Figure 2 helps to clarify this point. The figure shows AV curves for different levels of γ. Two fact are immediately clear. First, lower γ shifts the AV curve down because the value of capital is lower. Second, the slopes change, but the effect is non monotonic. For example, start from the capacity level $K_{\max}$, identified by the verical line, that maximizes the average value of capital for $\gamma = 0.1$. In that point the average profit curve is flat, i.e. $AV_K = 0$. As γ decreases AV_K first becomes positive (note how for $\gamma = 0.01$ the the average profit curve at $K_{\max}$ is increasing), then negative (for $\gamma = 0.001$ and $\gamma = 0$ the profit function is concave and the AV curves are monotonically decreasing).

The reason is the following. With high γ marginal returns to capital are increasing (see Proposition 1) but the peak of the average productivity of capital occurs for a relatively small K because the firm does not need a large capacity to exploit the learning curve. Given the high γ the learning process proceeds already fast. With γ smaller, but not too low, marginal returns are still increasing but the average productivity of capital is maximized for a larger K, because the firm needs to compensate the low γ with a larger per-period production rate to optmally exploit the learning curve. This implies that at $K_{\max}$ $AV_K(K) > 0$. If γ further decrease, marginal returns becomes decreasing and $AV_K(K)$ is always negative.

Compare, now, learning and constant marginal cost firms. As suggested by Proposition 2 and confirmed by the numerical results of Table 4 in the Appendix, the learning firm invest on a larger scale if γ is small and on a smaller scale if γ is high. When γ is high, the difference in average profits between learning and costant marginal cost firms is large, because the learning process proceeds fast. The large value of capital is an incentive to anticipate the investment, and the already fast learning induces the firm to install a small capacity. Given that marginal cost will rapidly decline in the future, waiting too long before undertaking the investment is not desirable. The learning firm invest much earlier than the constant marginal cost firm, but this implies that bad news effect is stronger. Investment occurs on a smaller scale.

When γ is small, however, the learning process is slow so that the difference in average profits between learning and costant marginal cost firms is also small. This implies that the bad news effect is weakened, because the downward shift of the threshold curve is less pronounced. Moreover, a small γ also implies that a larger capacity is needed to exploit the learning curve. Therefore, investment occurs on a larger scale. With a larger investment the effect on timing is ambiguous. The average profit per unit of capital is always larger for the learning firm. Thus, for given K investment would still occur earlier. But on the other hand the irreversible sunk expenditure wK, is also larger. Therefore, it might be optimal to undertake it later, i.e. for a larger P^*.

Given that we investigate a real options model, a question that naturally arises is how uncertainty affects the above described mechanism. The answer is, in this case, clear-cut. Uncertainty, increasing β_1, makes condition (28) more stringent, i.e. it reduces the likelihood that the learning curve increases the scale of investment. Indeed, other things being equal, higher volatility itself triggers the bad news effect inducing the firm to invest later and on a larger capacity. This implies that the additional incentive to further increase the capacity due to the learning curve is

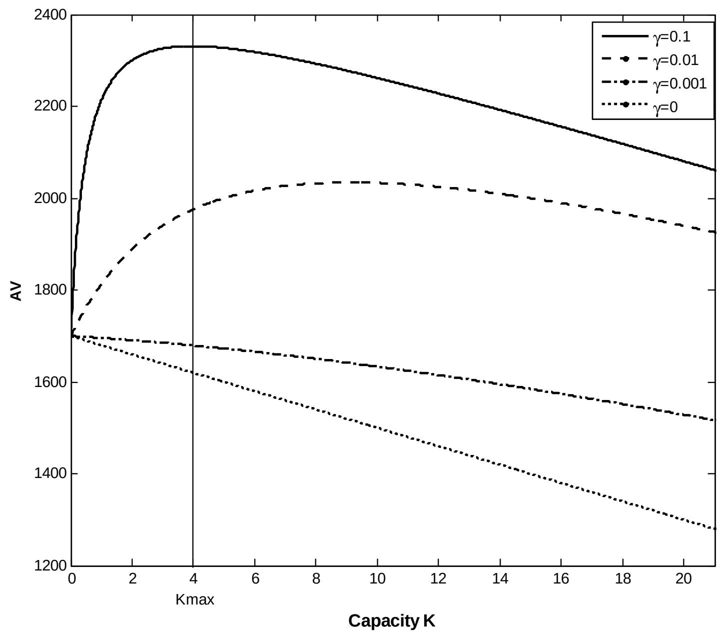

Figure 2: Average (AV) revenue curves for different values of γ. Parameters values are: $P = 100$, $\bar{c} = 40$, $\rho = 0.05$, $\mu = 0.01$, $\varphi = 1$.

weaker, because the firm already invest on a large scale. As also shown by the numerical results of Table 4 in the Appendix, with higher σ the region where $K_\gamma^* > 0$ progressively shrinks.

To summarize, we find that the influence of the learning curve on timing and intensity of investment is ambiguous. On one side, the value of capital increases so that, other things being equal, the investment trigger P^* is lower. A lower trigger implies that, at the moment the investment occurs, also the marginal productivity of capital is also lower. Therefore, the firm prefers to choose a smaller capacity. On the other side, larger capacity, increasing the per-period prodution rate, guarantees that the benefits of the learning curve are obtained more rapidly. Therefore, the firm has an incentive to increase its scale. Given that larger capacity implies a higher sunk costs, the firm may prefer to postpone the investment (higher P^*). The effet on timing and intensity of investment depends on which of these two opposing forces prevail. Typically, when the speed of learning is high, investment occurs earlier and on a smaller scale. When the learning speed is low, the intensity of investment increase, while the effect on timing is ambiguous. Finally, uncertainty decrease the probability that the learning curve will increase the scale of investment.

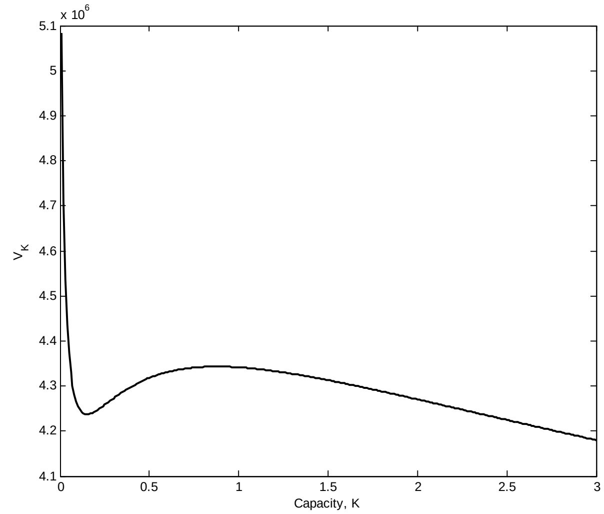

Figure 3: Marginal revenue curve with a production for of the form K^α. Parameters values are: $P = 100$, $\bar{c} = 40$, $\gamma = 0.1$, $\rho = 0.05$, $\mu = 0.01$, $\alpha = 0.8$

3.3 Extension: a Cobb-Douglas production function

In this section we extend the results of Section 3 and investigate the effects of the learning curve using a production function analogous to BI&S (1999), that is $q = K^\alpha$ with $\alpha \in (0, 1)$. This extension represents a robustness check for one of the main results of our model, that is the fact that the learning curve may generate increasing marginal profits. Indeed, for a Cobb-Douglas production function, and more in general for the broader class of functions which satisfy the Inada conditions, marginal productivity of capital at $K = 0$ is infinite. Given that marginal profits must be initially decreasing, a textbook convex-concave profit function, as that we found for the linear case, is immediately ruled out. However, we will show that the marginal profit curve can be still characterized by an inverse U-shaped region, that is the learning may generate increasing marginal profits.

As in Section 2, the firm's cost function is (4). Therefore, profit at time is

$$\pi(P, K, Q_s) = K^\alpha \left(P - ce^{-\gamma Q_t}\right)$$

It follows that value and marginal value of capital are

$$V(P, K, 0) = \frac{PK^\alpha}{\delta} - \frac{K^\alpha c}{\gamma K^\alpha + \rho} \tag{30}$$

$$V_K(P, K, 0) = \frac{\alpha}{\delta} PK^{\alpha-1} - \frac{\alpha c K^{\alpha-1}}{\gamma K^\alpha + \rho} + \frac{\alpha\gamma K^{2\alpha-1} c}{(\gamma K^\alpha + \rho)^2} \tag{31}$$

Our main interest here is to understand whether, for some parameters values, marginal profit function (31) displays an increasing region. Figure 3 helps to clarify this point. Marginal profit is very large for small capacity (it is infinite for $K = 0$) and is initially steeply decreasing in K. For K sufficiently large, however, the curve displays the familiar inverse U-shape. In other words, the profit function (30) is concave-convex-concave and our intuition is robust: the learning curve may generate increasing marginal profits.

The firm investment problem can is solved with procedure outlined in the previous sections and eventually yields

$$P^*(K) = \frac{\beta_1 \delta}{\beta_1 - 1}\left(\frac{c}{\gamma K^\alpha + \rho} + kK^{1-\alpha}\right),$$

and

$$F(P,K) = \frac{K^\alpha}{\beta_1 - 1}\left(\frac{c}{\gamma K^\alpha + \rho} + kK^{1-\alpha}\right)\left(\frac{\beta_1 - 1}{\beta_1 \delta}\frac{P}{\left(\frac{c}{\gamma K^\alpha + \rho} + kK^{1-\alpha}\right)}\right)^{\beta_1}. \quad (32)$$

The optimal capacity K^* maximizes (32) and satisfies

$$\frac{\alpha c}{\gamma K^\alpha + \rho} + \alpha k K^{1-\alpha} + (1-\beta_1)\left(-\frac{\alpha\gamma c K^\alpha}{(\gamma K^\alpha + \rho)^2} + (1-\alpha)\, kK^{1-\alpha}\right). = 0 \quad (33)$$

Equation (33) is satisfied if

$$\beta_1(1-\alpha) = 1 + \frac{\alpha c K^{\alpha-1}}{k\,(\gamma K^\alpha + \rho)^2}\left(\frac{\beta_1 \gamma K^\alpha + \rho}{(\gamma K^\alpha + \rho)}\right) > 1. \quad (34)$$

Note that a finite solution for the optimal capacity can be obtained if $\beta_1(1-\alpha) > 1$ and $k > 0$.[6]

Analogously to the linear demand case the effect of learning on investment intensity is ambiguous, with K^* increasing in γ if $\gamma K^\alpha < \rho\frac{(\beta_1 - 2)}{\beta_1}$, and the other way around. The interpretation follows the same lines of the previous section.

In the Appendix we show that other factors affect K^* in the following directions:

$$K^*_\rho \lesseqgtr 0,\ K^*_\sigma > 0,\ K^*_\mu > 0,\ K^*_k < 0,\ K^*_c > 0 \quad (35)$$

The same argument of the previous section implies that $P^*_\sigma > 0$, $P^*_c > 0$. The effect of other factors on the investment timing is shown numerically.

[6] An analogous condition the same condition found in BI&S (1999). Condition (34) guarantees that the investment option as specified in (32) is a concave function of K.

4 Conclusions

We investigated the optimal investment policy of a firm that faces the learning curve and a stochastic demand. The learning curve generates significant scale effects: if the efficiency gains from learning are large and the speed of learning sufficiently fast, the profit function is convex-concave, i.e the marginal profit function has inverse U-shape. As shown in Dixit (1995) this result implies that in a capacity expansion model, the initial investment would be lumpy followed by additional marginal adjustments. However, following Bar Ilan and Strange (1999), we investigate a model in which the firm chooses once and for all its capital stock and investment is lumpy by assumption. We find that the influence of the learning curve on timing and intensity of investment is ambiguous.When the learning speed is low, the intensity of investment increase, while the effect on timing is ambiguous. Uncertainty decrease the probability that the learning curve increases the scale of investment.

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A Appendix

A.1 Proof of Proposition 1

In order to prove that the profit function is convex-concave in K, we have to show that $V_{KK}(P,K,Q_t) > 0$ holds for $K = 0$, and $V_{KK}(P,K,Q_t) < 0$ for a given $\widehat{K} > 0$. This second part is immediately verified by noticing that, for finite values of φ, c and $\underline{c}$, $\underset{K\to\infty}{lim} V_{KK} = -\frac{2\varphi}{\rho}$. For the second part note that

$$V_{KK}(P,K,0) = -\frac{2\varphi}{\rho} + \frac{2\gamma c}{(\gamma K+\rho)^2} + \frac{2\gamma^2 Kc}{(\gamma K+\rho)^3}.$$

Therefore, $V_{KK}(P,0,0) > 0$ holds if condition (8) is satisfied. ■

A.2 Comparative statics

A.2.1 Constant marginal cost

$$\overline{P}_k = \frac{\beta_1 \delta}{(\beta_1 - 2)\,\rho} > 0$$

$$\overline{K}_k = \frac{\rho}{\varphi\,(\beta_1 - 2)} > 0$$

With an abuse of notation we indicate the partial derivatives of β_1 with respect a generic factor x not with the subscripts but as $\frac{\partial \beta_1}{\partial x}$. Note that

$$\frac{\partial \beta_1}{\partial \mu} < 0 \text{ which implies } \frac{\partial}{\partial \mu}\frac{\beta_1}{(\beta_1 - 2)} > 0$$

$$\frac{\partial \beta_1}{\partial \rho} > 0 \text{ which implies } \frac{\partial}{\partial \rho}\frac{\beta_1}{(\beta_1 - 2)} < 0$$

$$\frac{\partial \beta_1}{\partial \sigma} < 0 \text{ which implies } \frac{\partial}{\partial \sigma}\frac{\beta_1}{(\beta_1 - 2)} > 0$$

It follows that

$$\overline{K}_\sigma = -\frac{\partial \beta_1}{\partial \sigma}\frac{(c+\rho k)}{\varphi\,(\beta_1 - 2)^2} > 0$$

$$\overline{P}_{\sigma} = \frac{\partial}{\partial \sigma} \frac{\beta_1}{(\beta_1 - 2)} \delta(\frac{c}{\rho} + k) > 0$$

$$\overline{K}_{\mu} = -\frac{\partial \beta_1}{\partial \mu} \frac{(c + \rho k)}{\varphi \left(\beta_1 - 2\right)^2} > 0$$

$$\overline{P}_{\mu} = \frac{\partial}{\partial \mu} \frac{\beta_1}{(\beta_1 - 2)} \delta(\frac{c}{\rho} + k) - \frac{\beta_1}{(\beta_1 - 2)} (\frac{c}{\rho} + k) \lesseqgtr 0$$

$$\overline{K}_{\rho} = -\frac{\partial \beta}{\partial \rho} \frac{1}{\varphi \left(\beta_1 - 2\right)^2} (c + \rho k) + \frac{k}{\varphi \left(\beta_1 - 2\right)} \lesseqgtr 0$$

$$\overline{P}_{\rho} = \frac{\partial}{\partial \rho} \frac{\beta_1}{(\beta_1 - 2)} \delta(\frac{c}{\rho} + k) + \frac{\beta_1}{(\beta_1 - 2)} (\frac{c}{\rho} + k) - \frac{\beta_1}{(\beta_1 - 2)} \delta \frac{c}{\rho^2} \lesseqgtr 0$$

$$\overline{K}_{c} = \frac{1}{\varphi \left(\beta_1 - 2\right)} > 0$$

$$\overline{P}_{c} = \frac{\beta_1 \delta}{\left(\beta_1 - 2\right) \rho} > 0 \tag{36}$$

A.2.2 Learning curve: linear demand and CRS

$$F(P, K) = \frac{P^*(K) K}{\beta_1 \delta} \left(\frac{P}{P^*(K)} \right)^{\beta_1}, \tag{37}$$

The firm maximizes this option choosing the optimal capacity. The first order condition with respect to K is

$$F_K(P, K) = \frac{1}{\beta_1 \delta} \left(\frac{P}{P^*(K)} \right)^{\beta_1} \left[P^*(K) + (1 - \beta_1) P_K^*(K) K \right] = 0$$

For K chosen optimally the term inside the square brackets must be equal to zero. Thus,

$$\frac{\varphi K}{\rho} + \frac{c}{\gamma K + \rho} + k + (1 - \beta_1) \left[\frac{\varphi K}{\rho} - \frac{\gamma K c}{\left(\gamma K + \rho\right)^2} \right] = 0$$

Note that in order for the above equation, the term in the square brakets must be positive. This property will result useful to evaluate the sign of some of the partial derivatives below.

To perform a comparative statics analysis we differentiate the equilibrium condition. Hence all the following partial derivatives are evaluated at $K = K^*$, but we write K omitting the star $(*)$ to make the notation less burdensome.

$$F_{KK}(P, K) = \frac{1}{\beta_1 \delta} \left(\frac{P}{P^*(K)} \right)^{\beta_1} \left\{ \frac{\varphi}{\rho} - \frac{\gamma c}{\left(\gamma K + \rho\right)^2} + (1 - \beta_1) \left[\frac{\varphi}{\rho} - \frac{\gamma c}{\left(\gamma K + \rho\right)^2} + \frac{2\gamma^2 K c}{\left(\gamma K + \rho\right)^3} \right] \right\}$$

In order to have a well defined a problem it must be that $F_{KK}(P,K)<0$, which implies

$$(\beta_1-2)>\left[\frac{\varphi}{\rho}-\frac{\gamma c}{(\gamma K+\rho)^2}\right]^{-1}(\beta_1-1)\frac{2\gamma^2K^2c}{(\gamma K+\rho)^3}<0.$$

This condition is immediately satisfied when (25) holds.

Notice that

$$F_{K\sigma}=-\frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\frac{\partial\beta}{\partial\sigma}\left[\frac{\varphi K}{\rho}-\frac{\gamma Kce^{-\gamma Q_t}}{(\gamma K+\rho)^2}\right]>0,$$

$$F_{K\mu}=-\frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\frac{\partial\beta}{\partial\mu}\left[\frac{\varphi K}{\rho}-\frac{\gamma Kce^{-\gamma Q_t}}{(\gamma K+\rho)^2}\right]>0,$$

$$F_{Kk}=\frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}>0.$$

$$F_{K\gamma}=\frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{\frac{cK^2}{(\gamma K+\rho)^3}\left[\rho\left(\beta_1-2\right)-2\beta_1\gamma K\right]\right\}\lesseqgtr 0$$

$$\begin{aligned}F_{K\rho} \;=\;& \frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{(\beta_1-2)\frac{\varphi K}{\rho^2}+\frac{cK}{(\gamma K+\rho)^3}\left[\gamma K\left(1-2\beta_1\right)-\rho\right]\right.\\ &\left.-\frac{\partial\beta}{\partial\rho}\left[\frac{\varphi K}{\rho}-\frac{\gamma Kc}{(\gamma K+\rho)^2}\right]\right\}\lesseqgtr 0\end{aligned}$$

$$F_{Kc}=\frac{1}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left[\frac{1}{\gamma K+\rho}-(1-\beta_1)\frac{\gamma Kc}{(\gamma K+\rho)^2}\right]>0$$

Rearranging the partial derivatives (26) follows.

For given K it holds

$$P^*_\sigma(K)=\frac{\partial}{\partial\sigma}\frac{\beta_1}{\beta_1-1}\delta\left(\frac{\varphi K}{\rho}+\frac{c}{\gamma K+\rho}+k\right)>0, \tag{38}$$

$$P^*_k(K)=\frac{\beta_1\delta}{\beta_1-1}>0$$

$$\begin{aligned}P^*_\rho(K) \;=\;& \left(\frac{\varphi K}{\rho}+\frac{c}{\gamma K+\rho}+k\right)\left[\frac{\partial}{\partial\rho}\frac{\beta_1}{\beta_1-1}\delta+\frac{\beta_1}{\beta_1-1}\right]+\\ &-\frac{\beta_1\delta}{\beta_1-1}\left[\frac{\varphi K}{\rho^2}+\frac{c}{(\gamma K+\rho)^2}\right]\lesseqgtr 0\end{aligned}$$

$$P^*_\mu(K)=\left(\frac{\varphi K}{\rho}+\frac{c}{\gamma K+\rho}+k\right)\left[\frac{\partial}{\partial\mu}\frac{\beta_1}{\beta_1-1}\delta-\frac{\beta_1}{\beta_1-1}\right]\lesseqgtr 0$$

$$P_\gamma^*(K,0) = -\frac{\beta_1\delta}{\beta_1 - 1}\frac{Kc}{(\gamma K+\rho)^2} < 0$$

$$P_c^*(K,0) = \frac{\beta_1\delta}{\beta_1 - 1}\frac{1}{\gamma K+\rho}$$

A.2.3 Learning curve: Cobb Douglas

$$F_K(P,K,0) = \frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{\frac{\alpha c}{\gamma K^\alpha+\rho} + \alpha k K^{1-\alpha} + (1-\beta_1)\left[-\frac{\alpha\gamma c K^\alpha}{(\gamma K^\alpha+\rho)^2} + (1-\alpha)\,kK^{1-\alpha}\right]\right\}$$

$$\begin{aligned} F_{KK}(P,K) &= \frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{-\frac{\alpha^2\gamma c K^{\alpha-1}}{(\gamma K^\alpha+\rho)^2} + (1-\alpha)\,\alpha k K^{-\alpha} + \right. \\ &\left. + (1-\beta_1)\left[-\frac{\alpha^2\gamma c K^{\alpha-1}}{(\gamma K^\alpha+\rho)^2} + 2\frac{\alpha^2\gamma^2 c K^{2\alpha-1}}{(\gamma K^\alpha+\rho)^3} + (1-\alpha)^2 k K^{-\alpha}\right]\right\}\end{aligned}$$

In order for the problem to have a unique solution it must be $F_{KK} < 0$. Therefore, it must be that

$$\beta_1(1-\alpha) > 1 + \frac{\alpha}{1-\alpha}\frac{\gamma c K^{2\alpha-1}}{k\,(\gamma K^\alpha+\rho)^3}\left[\rho\,(\beta_1-2) - \beta_1\gamma K^\alpha\right].$$

This condition is satisfied when (34) holds.

$$F_{K\mu}(P,K) = -\frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\frac{\partial\beta}{\partial\mu}\left[-\frac{\alpha\gamma c K^\alpha}{(\gamma K^\alpha+\rho)^2} + (1-\alpha)\,kK^{1-\alpha}\right] > 0$$

$$F_{K\sigma}(P,K) = -\frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\frac{\partial\beta}{\partial\sigma}\left[-\frac{\alpha\gamma c K^\alpha}{(\gamma K^\alpha+\rho)^2} + (1-\alpha)\,kK^{1-\alpha}\right] > 0$$

$$F_{Kk}(P,K) = \frac{K}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left[1-\beta_1(1-\alpha)\right] < 0$$

$$F_{K\gamma}(P,K) = \frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{\frac{\alpha K^\alpha c}{(\gamma K^\alpha+\rho)^3}\left[(\beta_1-2) - 2\beta_1\gamma K\right]\right\} \lesseqgtr 0$$

$$\begin{aligned} F_{K\rho}(P,K) &= \frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left\{\left[\frac{\alpha c}{(\gamma K^\alpha+\rho)^3}\left(\gamma K^\alpha(1-2\beta_1) - \rho\right)\right] + \right. \\ &\left. -\frac{\partial\beta}{\partial\rho}\left[-\frac{\alpha\gamma c K^\alpha}{(\gamma K^\alpha+\rho)^2} + (1-\alpha)\,kK^{1-\alpha}\right]\right\} < 0\end{aligned}$$

$$F_{Kc}(P,K,0) = \frac{K^{\alpha-1}}{\beta_1\delta}\left(\frac{P}{P^*(K)}\right)^{\beta_1}\left[\frac{\alpha}{\gamma K^\alpha+\rho} - (1-\beta_1)\frac{\alpha\gamma K^\alpha}{(\gamma K^\alpha+\rho)^2}\right] > 0.$$

	constant c	learning
$K^*_\rho;\ P^*_\rho$	$-;\pm$	$-;\pm$
$K^*_k; P^*_k$	$-;=$	$-;+$
$K^*_\mu;\ P^*_\mu$	$+;\pm$	$+;\pm$
$K^*_\sigma; P^*_\sigma$	$+;+$	$+;+$
$K^*_c;\ P^*_c$	$+;+$	$+;+$
$K^*_\gamma;\ P^*_\gamma$	$N.A.$	$\pm;\pm.$

Table 3: Comparative statics. Learning curve and constant marginal cost with a Cobb-Douglas production function

Rearranging the partial derivatives (26) follows.

For given K it holds

$$P^*_\sigma(K) = \frac{\partial}{\partial\sigma}\frac{\beta_1}{\beta_1-1}\delta\left(\frac{c}{\gamma K^\alpha+\rho}+kK^{1-\alpha}\right) > 0,$$

$$P^*_k(K) = \frac{\beta_1\delta}{\beta_1-1}K^{\alpha-1} > 0$$

$$\begin{aligned} P^*_\rho(K) \quad &= \quad \left(\frac{c}{\gamma K^\alpha+\rho}+kK^{1-\alpha}\right)\left[\frac{\partial}{\partial\rho}\frac{\beta_1}{\beta_1-1}\delta-\frac{\beta_1}{\beta_1-1}\right]+ \\ &\quad -\frac{\beta_1\delta}{\beta_1-1}\frac{\alpha c e^{-\gamma Q_t}}{(\gamma K^\alpha+\rho)^2} \lesseqgtr 0 \end{aligned}$$

$$P^*_\mu(K) = \left(\frac{c}{\gamma K^\alpha+\rho}+kK^{1-\alpha}\right)\left[\frac{\partial}{\partial\mu}\frac{\beta_1}{\beta_1-1}\delta-\frac{\beta_1}{\beta_1-1}\right] \lesseqgtr 0$$

$$P^*_\gamma(K,0) = -\frac{\beta_1\delta}{\beta_1-1}\frac{\alpha K^\alpha c}{(\gamma K^\alpha+\rho)^2} < 0$$

$$P^*_\sigma(K) = \frac{\beta_1\delta}{\beta_1-1}\frac{1}{\gamma K^\alpha+\rho} > 0,$$

$$P^*_c(K,0) = \frac{\beta_1\delta}{\beta_1-1}\frac{1}{\gamma K^\alpha+\rho} > 0$$

A.3 Numerical results

	$\sigma = 0$[7]		$\sigma = 0.05$		$\sigma = 0.1$		$\sigma = 0.15$	
γ	K^*	P^*	K^*	P^*	K^*	P^*	K^*	P^*
0	13.50	54	23.14	69.02	57.02	123.23	243.2	421.12
0.0001	14.59	53.95	24.9	68.87	60.19	121.80	227.96	379.83
0.001	20.63	49.45	29.30	59.60	52.12	91.11	126.43	204.86
0.01	14.01	25.03	16.59	28.26	23.81	39.06	47.53	76.36
0.1	5.42	9.30	6.14	10.01	8.37	13.50	15.84	25.37
1	1.87	3.41	2.03	3.27	2.73	4.38	5.09	8.15
10	0.66	1.46	0.65	1.04	0.87	1.40	1.62	2.59

Table 4: The effect of γ on timing and intensity of investment for different values of σ. Parameters values are: $c = 40$, $\rho = 0.05$, $\mu = 0$, $\phi = 1$, $w = 10$.

γ	K^*	P^*
0	23.14	69.02
0.000001	23.16	69.03
0.00001	23.33	69.03
0.0001	24.90	68.87

Table 5: The effect of γ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.05$, $\rho = 0.05$, $\mu = 0.01$, $\phi = 1$ $w = 100$.

ρ	K^*	P^*
0.4	9.78	14.71
0.05	8.37	13.50
0.06	7.99	13.48
0.07	7.89	13.76
0.08	7.91	14.14

Table 6: The effect of ρ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.1$, $\gamma = 0.1$, $\mu = 0.01$, $\phi = 1$, $w = 50$.

μ	K^*	P^*
0	7.28	15.13
0.01	8.08	14.98
0.02	9.45	15.41
0.03	12.18	17.21

Table 7: The effect of μ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.1$, $\gamma = 0.1$, $\rho = 0.1$, $\phi = 1$, $w = 50$.

μ	K^*	P^*
0	7.28	15.13
0.01	8.08	14.98
0.02	9.45	15.41
0.03	12.18	17.21

Table 8: The effect of ρ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.1$, $\gamma = 0.1$, $\mu = 0.01$, $\phi = 1$, $w = 50$.

γ	K^*	P^*
0	315.2	123.22
0.000001	316	123.26
0.00001	323.7	123.61
0.0001	403.2	126.57
0.001	510.8	112.73
0.01	121.2	58.71
0.1	15.8	25.44
1	1.7	10.40

Table 9: Cobb-Douglas production function. The effect of γ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.01$, $\rho = 0.05$, $\mu = 0.0$, $\alpha = 0.6$, $w = 100$.

	$\gamma = 0.001$		$\gamma = 1$	
ρ	K^*	P^*	K^*	P^*
0.4	604.7	69.47	5.10	2.67
0.05	317.2	67.37	4.6	3.16
0.06	182.6	65.24	4.3	3.65
0.07	113.8	63.34	4.1	4.12
0.08	75.6	61.70	3.9	4.60

Table 10: Cobb-Douglas production function. The effect of ρ on timing and intensity of investment. Parameters values are: $c = 40$, $\sigma = 0.1$, $\mu = 0.$, $\alpha = 1$, $w = 10$.