An Optimal Investment Policy in Equity-Debt Financed Firms with Finite and Infinite Maturities

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Abstract

In this paper we examine the effect of maturity for optimal investment policy of the firm that is financed by issuing equity and debt. Specifically, we discuss the investment timing, the firm value, the optimal leverage and coupon payment. Recently, a number of researchers have studied the interaction among firm's investment and financing decisions under uncertainty by means of real option framework. In the literature, investment problems for a firm with growth options, that is financed with equity and debt are investigated. In most studies, in order to simplify the problem, the infinite maturity for the investment and debt is assumed. However, the assumption of the infinite maturity restricts the problem. In this paper, we examine the optimal investment policy of the firm which is financed by issuing equity and debt during a period of time.

Keywords: Real options, investment, finite maturity, debt financing, capital structure

1 Introduction

Recently, a number of studies have investigated the interaction among investment and financing decisions of a firm under uncertainty by means of real option framework. In the literature, investment problems for a firm with growth options, which is financed with equity and debt are investigated (e.g. Lyandres and Zhdanov [2], Mauer and Ott [3], Mauer and Sarkar [4], Sundaresan and Wang [6], Zhdanov [7]). In these studies, in order to simplify the problem the infinite maturity of the investment and debt is assumed. However, the assumption of the infinite maturity restricts the problem. In this paper we examine the optimal investment policy of the firm that is financed by issuing equity and debt during a fixed period.

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The value of American option with finite maturity is higher when the remaining time to maturity is longer. Similarly, the value of investment in equity-debt financed firm with finite maturity is also higher and the investment timing is later when the remaining time to maturity is longer. When coupon payment is low, the firm should invest later because of a limited financing by issuing debt. On the other hand, when coupon payment is high, the firm should also invest late because the possibility of default is high. This implies that the investment timing over coupon payment does not have monotonicity. The optimal coupon payment maximizing the firm value changes over time. When the remaining time to maturity is shorter, the firm should exercise the investment by issuing even low coupon debt. One of interest result is that the optimal leverage does not depend on time under the optimal coupon payment.

2 The Model

Consider a firm with an option to invest at any time by paying a fixed investment cost I. The firm partially finances the cost of investment with straight debt with the instantaneous contractual coupon payment of c and infinite maturity. The coupon payment is tax-deductible at a constant corporate tax rate λ. We suppose that the firm observes the demand shock x_t for its product, where x_t is given by a geometric Brownian motion

$$dx_t = \mu x_t dt + \sigma x_t dW_t, \tag{1}$$

where μ and σ are the risk-adjusted expected growth rate and the volatility of x_t, respectively, and W_t is a standard Brownian motion defined on a probability space $(\Omega, F, \mathbb{P})$. Once the investment option is exercised, we assume that the firm can receive the instantaneous profit

$$\pi(x_t) = (1-\lambda)(Qx_t - c), \tag{2}$$

where $Q > 0$ is the quantity producted from the asset in place.

In order to examine the interaction of debt issuing and the investment maturity, we consider several settings. First, we present two benchmark models in which the investments with the infinite maturity are financed with all-equity and with equity and debt. Second, we examine the cases in which the investments with the finite maturity are financed with all equity and with equity and debt.

2.1 Investment Option with Infinite Maturity

In this section, we consider the investment option with the infinite maturity.

2.1.1 Equity Financing

First, we assume that the investment is financed entirely with equity, i.e. the coupon payment equals zero, $c = 0$. This case has been investigated for real options(e.g., [1, 5]).

The optimal investment rule is to exercise the investment option at the first passage time of the stochastic shock to an upper threshold x^*. Let $\mathcal{T}_{t_1,t_2}$ be the set of stopping times with respect to the filtration as $\{F_s; t_1 \leq s \leq t_2\}$, $\tau \in \mathcal{T}_{0,\infty}$ the investment time (the stopping time). Supposing that the firm can perpetually receive the profit after the investment, the value of an investment option $F(x)$ can be formulated as

$$F(x) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[\int_\tau^\infty e^{-ru}(1-\lambda)Qx_u du - e^{-r\tau} I \right], \tag{3}$$

where $\mathbb{E}_t^x$ the conditional expectation operator upon $x_t = x$, and r is the risk-free interest rate. For convergence, we assume $r > \mu$. Letting x^* be the optimal investment threshold, the optimal investment time is given by

$$\tau^* = \inf\{\tau > 0 \mid x_\tau \geq x^*\}. \tag{4}$$

Since the ordinary differential equation, which is satisfied by the value of investment option in Eq. (3), is derived from Bellman equation[1]:

$$\frac{1}{2}\sigma^2 x^2 \frac{d^2F}{dx^2} + \mu x \frac{dF}{dx} - rF = 0, \quad x < x^* \tag{5}$$

and the general solution of Eq. (5) is given by

$$F(x) = a_1 x^{\beta_1} + a_2 x^{\beta_2}, \quad x < x^*, \tag{6}$$

where $\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} > 1$ and $\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{1}{2} - \frac{\mu}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0$. Using standard arguments, $a_2 = 0$ and the investment threshold x^* is given by

$$x^* = \frac{1}{1-\lambda} \frac{\beta_1}{\beta_1 - 1} \frac{r-\mu}{Q} I. \tag{7}$$

Then, the value of the investment option $F(x)$ is given by

$$F(x) = \left(\frac{x}{x^*}\right)^{\beta_1} \left(\epsilon(x^*) - I\right), \quad x < x^*, \tag{8}$$

where $\epsilon(x)$ is the total post-investment profit in which the investment is financed entirely with equity,

$$\epsilon(x) = \frac{1-\lambda}{r-\mu} Qx. \tag{9}$$

From $\beta_1 > 1$ and $r > \mu$, the investment threshold $x^* > I$. This means that the investment is made when the demand is higher than the investment cost.

2.1.2 Equity and Debt Financing

Next, we consider a firm which has an option of the investment with the infinite maturity, that is financed with equity and debt ($c > 0$), introduced in Mauer and Sarkar [4], and Sundaresan and Wang [6].

[1] See, e.g., Dixit and Pindyck [1].

We model the values of equity and debt with coupon payment c after the exercise of investment option. Once the investment option has been exercised, the optimal default policy is established from the issue of debt. The optimal default strategy of the equity holders maximize the equity value, selecting the default threshold x_d. Let $E(x;c)$ be the total value of equity issued at time t and $\tau_d \in \mathcal{T}_{0,\infty}$ the default time (the stopping time). The value of equity $E(x;c)$ is equal to zero at the default time τ_d. The optimization problem of the equity holders can be formulated by

$$E(x;c) = \sup_{\tau_d \in \mathcal{T}_{t,\infty}} \mathbb{E}_t^x \left[\int_t^{\tau_d} e^{-r(u-t)}(1-\lambda)(Qx_u - c)du \right]. \tag{10}$$

The optimal default time τ_d^* is given by

$$\tau_d^* = \inf\{\tau > 0 \mid x_\tau \le x_d\}, \tag{11}$$

where x_d is the optimal default threshold.

Using standard arguments as in Sec. 2.1.1, the equity value $E(x;c)$ is given by

$$E(x;c) = \begin{cases} \epsilon(x) - \dfrac{(1-\lambda)c}{r} - \left(\epsilon(x_d) - \dfrac{(1-\lambda)c}{r}\right)\left(\dfrac{x}{x_d}\right)^{\beta_2}, & x > x_d \\ 0, & x \le x_d \end{cases} \tag{12}$$

and the default threshold x_d is

$$x_d = \frac{r-\mu}{Q}\frac{\beta_2}{\beta_2 - 1}\frac{c}{r}. \tag{13}$$

Let $D(x;c)$ be the total value of straight debt issued at investment time t. Since the holders of debt can receive the continuous coupon payment of c, the value of debt is given by

$$D(x;c) = \mathbb{E}_t^x \left[\int_t^{\tau_d^*} e^{-r(u-t)}cdu + e^{-r(\tau_d^*-t)}(1-\theta)\epsilon(x_{\tau_d^*}) \right], \tag{14}$$

where θ is the proportional bankruptcy cost, $0 \le \theta \le 1$. The holders of straight debt are entitled to the unlevered value of the firm net of proportional bankruptcy cost, $(1-\theta)\epsilon(x)$. Using the default threshold x_d, the value of straight debt can be represented by

$$D(x;c) = \frac{c}{r} - \left(\frac{c}{r} - (1-\theta)\epsilon(x_d)\right)\left(\frac{x}{x_d}\right)^{\beta_2}, \quad x > x_d. \tag{15}$$

The sum of $E(x;c)$ in Eq. (12) and $D(x;c)$ in Eq. (15) gives the firm value as

$$\begin{aligned} V(x;c) &= E(x;c) + D(x;c) \\ &= \epsilon(x) + \frac{\lambda c}{r}\left\{1 - \left(\frac{x}{x_d}\right)^{\beta_2}\right\} - \theta\epsilon(x_d)\left(\frac{x}{x_d}\right)^{\beta_2}, \quad x > x_d. \end{aligned} \tag{16}$$

We consider the optimal investment policy maximizing the firm value in Eq. (16). The value of the investment option $F(x;c)$ is given by

$$F(x;c) = \sup_{\tau \in \mathcal{T}_{0,\infty}} \mathbb{E}_0^x \left[e^{-r\tau}\left(V(x_\tau;c) - I\right)\right]. \tag{17}$$

Since the value of the investment option satisfies Eq. (5), it is given by

$$F(x;c) = \begin{cases} \left(\dfrac{x}{x^*}\right)^{\beta_1} \{V(x^*;c) - I\}, & x < x^*, \\ V(x;c) - I, & x \geq x^*. \end{cases} \tag{18}$$

Denote c^* as the optimal coupon payment maximizing the firm value at the investment. Then, the optimal capital structure of the firm is determined by

$$F(x^*;c^*) = \sup_{c>0} F(x^*;c). \tag{19}$$

2.2 Investment Option with Finite Maturity

In this section, we consider the firm with an option for investment with the finite maturity $T < \infty$.

2.2.1 Equity Financing

First, we assume the all-equity financing. Let $\tau \in \mathcal{T}_{0,T}$ be the investment time (the stopping time). The value of the investment option at time $t \in [0,T]$ is formulated by

$$F(x,t) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^x \left[e^{-r(\tau - t)} \left(\int_\tau^\infty e^{-r(u-\tau)} (1-\lambda) Q x_u du - I \right)^+ \right], \tag{20}$$

where $(x)^+ = \max(x,0)$. Denoting x_t^* as the optimal investment threshold at time t, the optimal investment time is given by

$$\tau_t^* = \inf\{\tau \in [t,T) \mid x_\tau \geq x_\tau^*\} \wedge T. \tag{21}$$

From Bellman equation, the value of the investment option with finite maturity T satisfies the partial differential equation

$$\frac{1}{2}\sigma^2 x^2 \frac{\partial^2 F}{\partial x^2} + \mu x \frac{\partial F}{\partial x} + \frac{\partial F}{\partial t} - rF = 0, \quad x < x_t^* \tag{22}$$

and the boundary conditions

$$\begin{cases} F(x_T, T) = (\epsilon(x_T) - I)^+, \\ \lim_{x \uparrow x_t^*} F(x,t) = (\epsilon(x_t^*) - I)^+, & t \in [0,T), \\ \lim_{x \uparrow x_t^*} \dfrac{\partial F}{\partial x}(x,t) = \dfrac{1-\lambda}{r-\mu} Q, & t \in [0,T). \end{cases} \tag{23}$$

The first condition is the terminal condition that ensures the investment option of the firm at the maturity. The second condition is the value matching condition requiring that the value of investment option at the investment threshold be the post-investment profit minus the investment cost. The last condition is the smooth-pasting condition that ensures the optimality of the the investment threshold x_t^*.

2.2.2 Equity and Debt Financing

Next, we consider the investment option of the firm, that is financed with equity and debt ($c > 0$). The values of equity and debt admit similar representations to Eqs. (10) and (14). That is, we discuss the optimal investment policy maximizing the firm value in Eq. (16). The value of the investment option $F(x, t; c)$ at time t is given by

$$F(x, t; c) = \sup_{\tau \in \mathcal{T}_{t,T}} \mathbb{E}_t^x \left[e^{-r(\tau - t)} \left(V(x_\tau; c) - I \right)^+ \right]. \tag{24}$$

The option value satisfies the partial differential equation in Eq. (22) as in the case of all-equity financing and the boundary conditions

$$\begin{cases} F(x_T, T; c) = \left(V(x_T; c) - I \right)^+, \\ \lim_{x \uparrow x_t^*} F(x, t; c) = \left(V(x_t^*; c) - I \right)^+, \quad t \in [0, T), \\ \lim_{x \uparrow x_t^*} \dfrac{\partial F}{\partial x}(x, t; c) = \dfrac{dV}{dx}(x_t^*; c), \qquad t \in [0, T). \end{cases} \tag{25}$$

Denote c_t^* as the optimal coupon payment maximizing the firm value at the investment time t. The optimal capital structure of the firm is determined by

$$F(x_t^*, t; c_t^*) = \sup_{c > 0} F(x_t^*, t; c). \tag{26}$$

3 Numerical Analysis

In this section, the calculation results of the value of investment option, the equity value, the debt value, the investment threshold, the optimal coupon payment and optimal leverage are presented in order to examine the effect of the finite maturity for investment. We use the following base case parameters: $Q = 1,\ x = 0.3,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ c = 0.3,\ \theta = 0.3,\ \lambda = 0.3.$

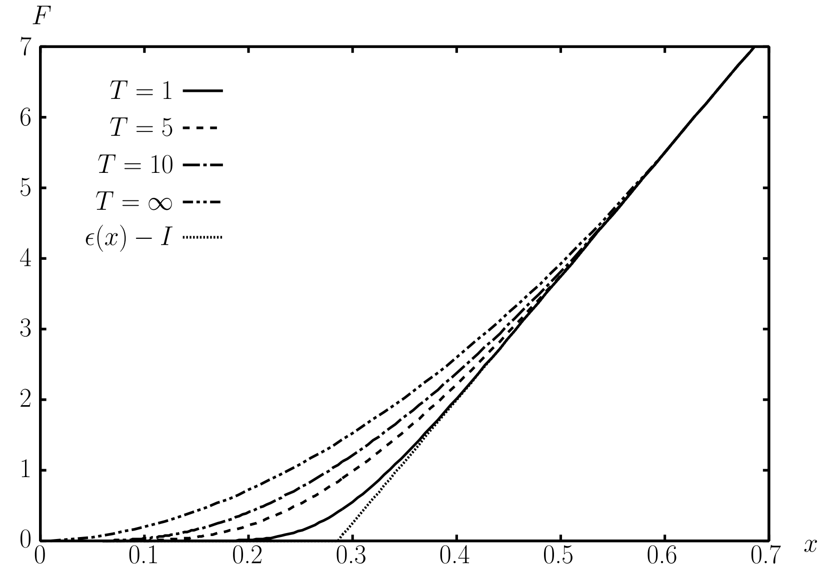

Figure 1: The value of investment option for the firm issuing only equity

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

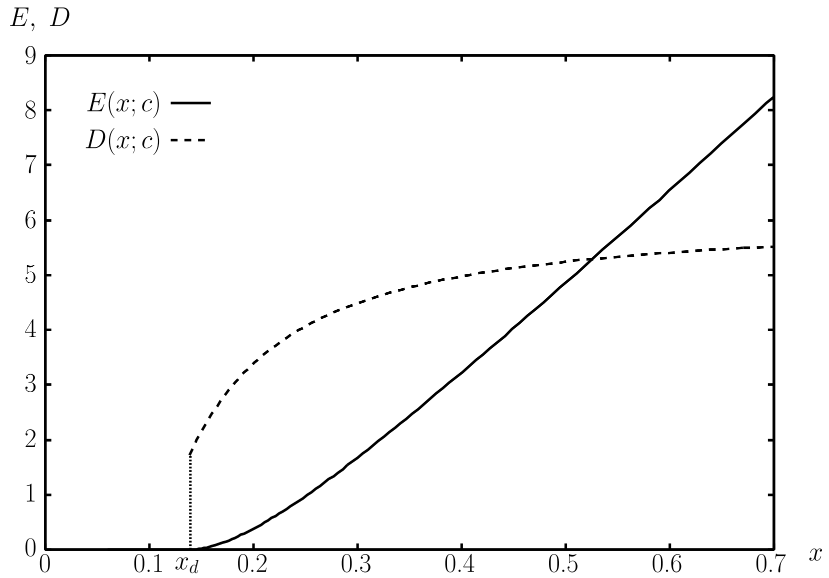

Figure 2: Equity and debt value ($c = 0.3$)

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ \theta = 0.3,\ \lambda = 0.3$

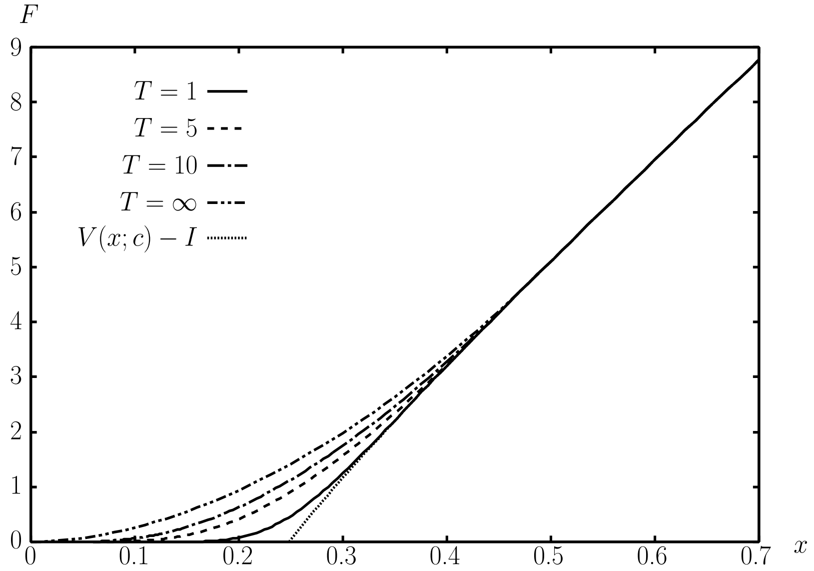

Figure 3: The value of investment option for the firm issuing equity and debt ($c = 0.3$)

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

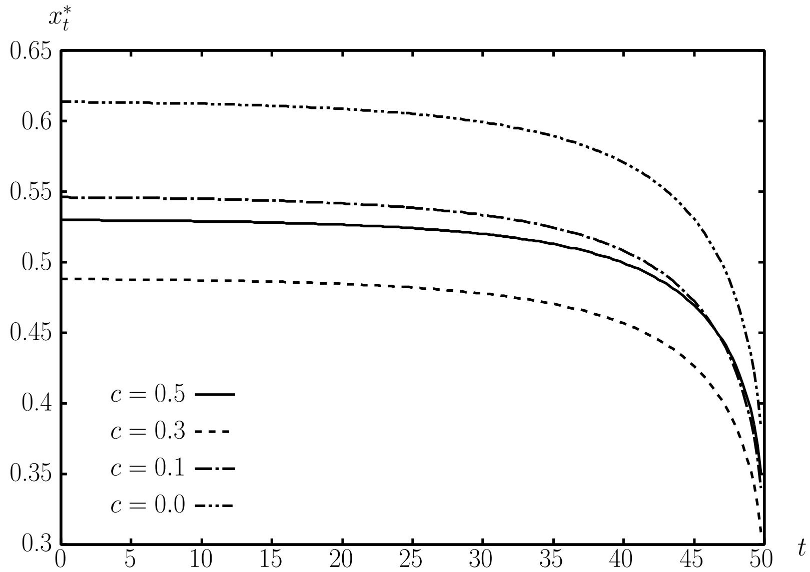

Figure 4: Optimal investment threshold ($T = 50$)

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

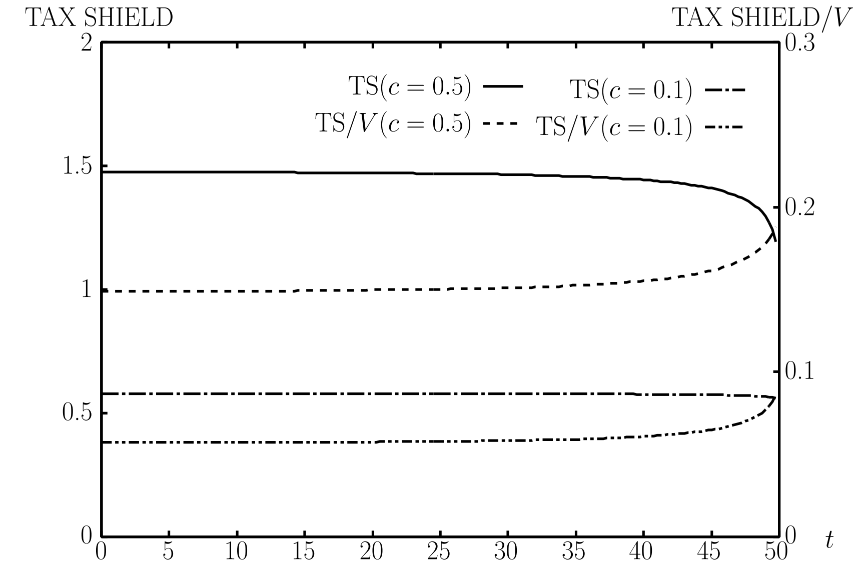

Figure 5: The value and the ratio of tax shield ($T = 50$)

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

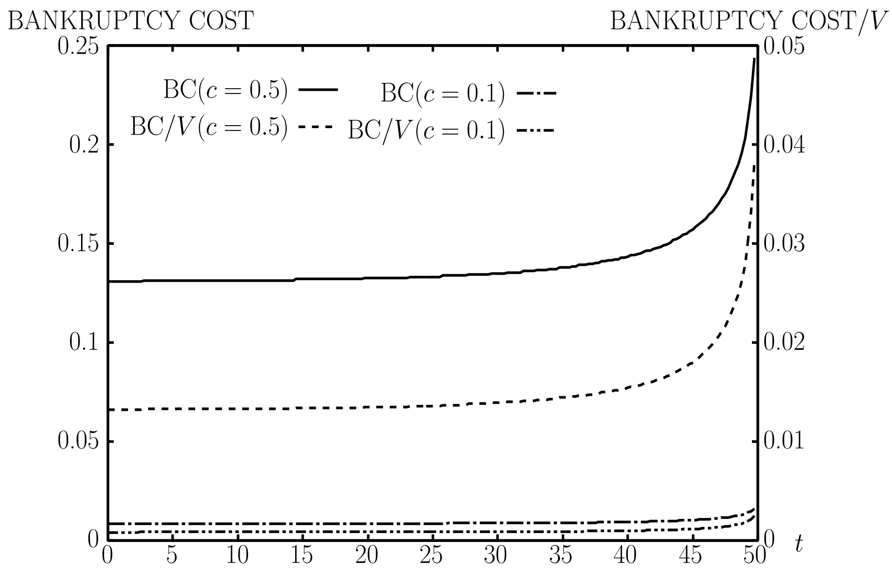

Figure 6: The value and the ratio of bankruptcy cost ($T = 50$)

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

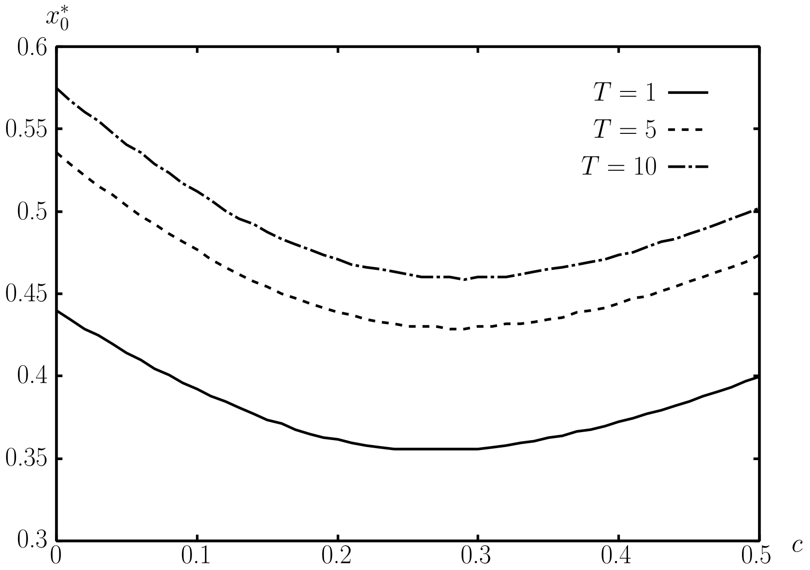

Figure 7: The investment threshold over coupon payment

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

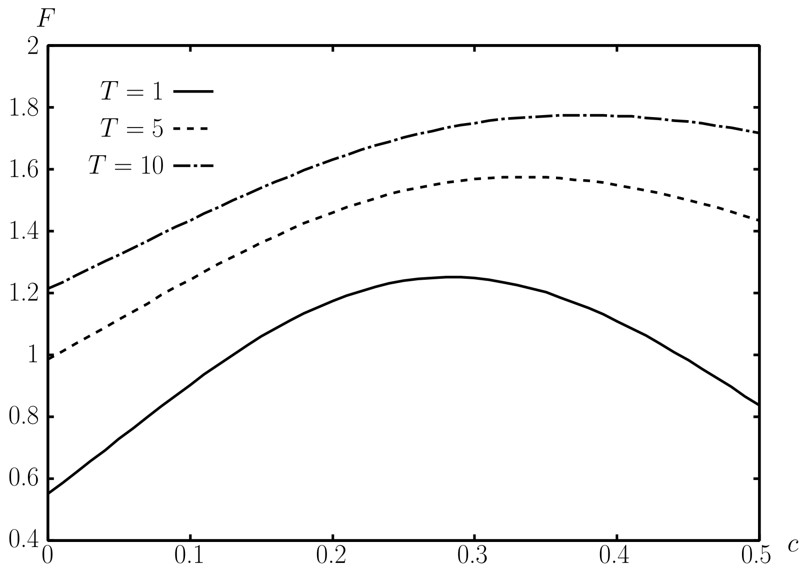

Figure 8: The value of investment option over coupon payment $(x = 0.3)$

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

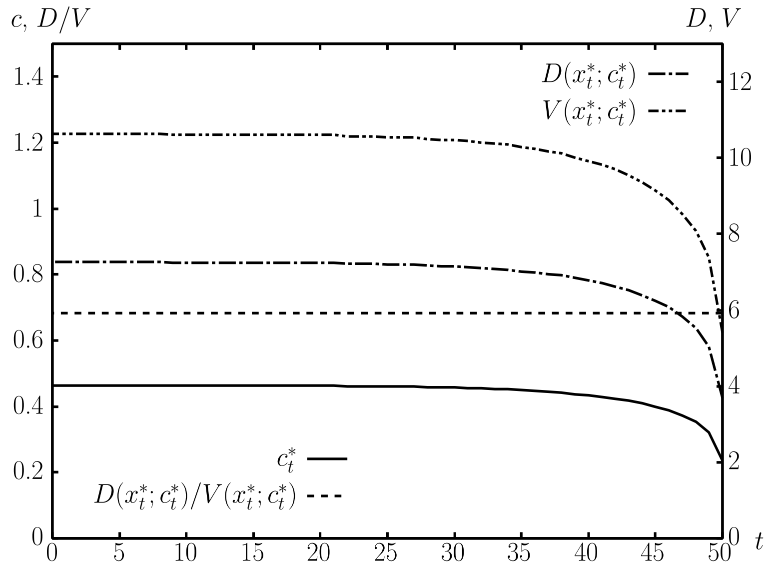

Figure 9: The optimal coupon payment, leverage, debt value and firm value

$Q = 1,\ \mu = 0.01,\ \sigma = 0.2,\ r = 0.05,\ I = 5,\ \theta = 0.3,\ \lambda = 0.3$

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