Valuing Capacity Investment Decisions: Binomial vs. Markov Models *

Dalila B. M. M. Fontes[1] and Fernando A. C. C. Fontes[2]

[1] LIACC, Faculdade de Economia da Universidade do Porto
Rua Dr. Roberto Frias, 4200-464 Porto, Portugal.
fontes@fep.up.pt

[2] Officina Mathematica, Departamento de Matemática para a
Ciência e Tecnologia, Universidade do Minho
4800-058 Guimarães, Portugal.
ffontes@mct.uminho.pt

Abstract

In this work, we present a model to value capacity investment decisions based on real options. In the problem considered we incorporate partial reversibility by letting the firm reverse its capital investment at a cost, both fully or partially. The standard RO approach considers the stochastic variable to be normally distributed and then approximated by a binomial distribution, resulting in a binomial lattice. In this work, we investigate the use of a sparse Markov chain, which is derived from demand data previously collected. The main advantages of this approach are: i) the Markov chain does not assume any type of distribution for the stochastic variable, ii) the probability of a variation is not constant, actually it depends on the current value, and iii) it generalizes current literature using binomial distributions since this type of distribution can be modelled by a Markov chain.

Key words. Real Options, Dynamic Programming, Markov Chains.

1 Introduction

Manufacturing flexibility has become a very important competitive aspect for production oriented companies. Several types of flexibility can be valued. Here, we are concerned with the some times called "volume" flexibility, which can be defined as the ability to operate with profit at different output levels. Managerial flexibility has been valued by option pricing for almost

*The financial support from FCT Project POCTI/MAT/61842/2004 is gratefully acknowledged.

two decades and during this time different kinds of real options have been treated. Kulatilaka (1988) uses a stochastic dynamic programming model to value the options in a flexible production process and incorporates the effects of switching costs. He & Pindyck (1992) examine investments in flexible production capacity. As in (Tannous 1996) demand is uncertain but in this case differs, via a demand shift parameter depending on whether market is perfectly competitive or not. Tannous (1996) carries out capital budgeting for volume flexible equipment and compares a non-flexible to a flexible system in a case based on a real company. In his model, demand is uncertain and dependent on price and a stochastic factor. The effect of having inventory available is also considered. Bollen (1999) values the option to switch between production capacities. The demand stochastic process is governed by a stochastic product life cycle which is modelled by using a regime switching process. In his study a comparison between flexible and fixed capacity projects is made.

2 Problem Description and Formulation

As in (Pindyck 1988), we consider a monopolist that faces a demand function that shifts stochastically, towards and away from the origin, over time as given by

$$Q = \theta - \lambda P, \tag{1}$$

where Q is the industry output and θ models the dynamics of demand. Of course, for the case of monopoly (1) is also the demand curve faced by the firm. (In financial options it is standard to assume that the underlying security is traded in a perfectly competitive market. However, many real asset markets are monopolistic or oligopolistic, rather than perfectly competitive.)

The total variable production costs are assumed to be a quadratic function of quantity produced, which is a standard assumption, see for example (Pindyck 1988, Trigeorgis 1996, Bollen 1999). Thus, the total production costs are

$$C(Q, m) = c_1 Q + \frac{c_2}{2m} Q^2 + c_3 m, \tag{2}$$

where the fixed and variable coefficients of the marginal cost function are c_1 and $\frac{c_2}{2m}Q$, m is the installed production capacity, and the fixed component $c_3 m$ represents the overhead costs.

The operating profit of period t, given the demand and production capacity installed, is then computed as

$$\pi\left(\theta_t, m_{t-1}\right) P(Q_t) - C(Q_t, m_{t-1}) = \left(\frac{\theta_t}{\lambda} - c_1\right) Q_t - \left(\frac{1}{\lambda} + \frac{c_2}{2m_{t-1}}\right) Q_t^2 - c_3 m_{t-1}. \tag{3}$$

The firm maximizes operating profit over produced quantity and hence, the optimal operating profit is given by

$$\pi^*\left(\theta_t, m_{t-1}\right) = P\left(Q^{**}(\theta_t)\right) - C\left(Q^{**}(\theta_t), m_{t-1}\right), \tag{4}$$

where $Q^{**}(\theta_t) = max\left(0, min\left(Q^*(\theta_t), m_{t-1}\right)\right)$ and Q^*, which is obtained by solving $\frac{\partial \pi}{\partial Q} = 0$.

The ability to partially reverse investment is modelled through capacity sell out, more specifically, following the work by Bollen (1999), we use $S\left(m_1, m_2\right)$ to represent additional investment or recovered investment associated with changing capacity level from m_1 to m_2:

$$\begin{aligned} S\left(m_1, m_2\right) &= s_1 c_4 \left(m_2 - m_1\right) + s_3, \quad \text{if } m_2 > m_1, \\ S\left(m_1, m_2\right) &= s_2 c_4 \left(m_1 - m_2\right) + s_3, \quad \text{otherwise,} \end{aligned} \tag{5}$$

where s_1 and s_2 are percentages of the initial capacity cost c_4 and s_3 is a fixed switching capacity cost.

3 Solution Methodology

To solve our problem it is necessary to find the optimal sequence of capacity choices, namely: invest in additional capacity, sell out excessive capacity, keep exactly the same capacity; and the optimal production in each period given the capacity decision previously made. These two types of decision must be addressed simultaneously since the existence of switching costs implies that a capacity decision made in a period alters future switching costs and future profits and thus, future switching decisions. Therefore, the project value must be determined simultaneously with the optimal production capacity policy.

As said before, we propose to discretize the problem in two different ways: through the use of a binomial lattice, the standard RO approach and through the use of a Markov chain, a sparse one-step transition probabilities matrix. For each of these approaches, a dynamic programming model is derived and solved by backward induction.

A Markov chain is defined by a one step transition probability matrix and can be obtained from the problem data as follows. The probability of reaching state j at some period of time being in state i at the previous time period is given by the ratio between the number of transitions from demand value θ_i to demand value θ_j in consecutive periods and the total number of transitions out of demand value θ_i to all other possible demand values in consecutive periods.

In each period the firm must make two decisions, one regarding the quantity to produce and another regarding the production capacity that is to be in place for the following period. The decision on the quantity to produce is given by maximizing the operating profit. The decision about the production capacity is related to the future periods profits since the chosen capacity will be available from next period. Therefore, at each period the project value is dependent on the level of demand and production capacity and is obtained by maximizing the sum of the optimal current period's profit with the optimal continuation value for each possible capacity value. The latter value is given by the discounted expected future profits net of switching costs.

The optimal project value at period t given the demand θ_t and available production capacity m_{t-1} is then given by

$$f(\theta_t, m_{t-1}, t) = \pi^*(\theta_t, m_{t-1}) + \max_m \left\{ \frac{E[f(\theta_{t+1}, m_t, t+1)]}{1+r_f} + S(m_{t-1}, m_t) \right\}. \tag{6}$$

As said before, and in order to allow for earlier exercise, the valuation procedure begins at the last stage and works backwards to initial time. At the final period $t = T$, for each demand value and capacity available, the project value is computed as

$$f(\theta_T, m_{T-1}, T) = \pi^*(\theta_T, m_{T-1}) + S(m_{T-1}, 0). \tag{7}$$

The implementation of the Dynamic Programming recursion, given by equation (6), on a standard binomial lattice computes expected value of future profits as

$$E[f(\theta_t, m_{t-1}, t)] = q_u f(u\theta_t, m_t, t+1) + q_d f(d\theta_t, m_t, t+1), \tag{8}$$

where q_u and q_d are the probabilities of an up or down move, respectively, and u and d are the associated up and down rates, while the implementation on a Markov grid is computed as

$$E[f(\theta_t, m_{t-1}, t)] = \sum_{i=1}^{n} P_{\theta_t, \theta_i} f(\theta_i, m_t, t+1), \tag{9}$$

where P_{θ_i, θ_j} is the transition probability from demand value θ_i to demand value θ_j in consecutive periods.

4 Results

In order to test our methodologies we have implemented, in MATLAB, the dynamic programming model on the binomial lattice and on the Markov grid. As we have considered the initial production capacity also to be decided we have to solve

$$Project\ Value = \max_{m_0} \{f(\theta_1, m_0, 1)\} / (1 + r_f) - c_4 m_0. \tag{10}$$

Both the Binomial and Markov models have been used to find out an optimal capacity investment policy, which we call *a priori* solution. The quality of these models is then tested by evaluating the policy performance on specific data realization sets, which we call *a posteriori* solution.

We have collected monthly sales data for 48 months. The first 24 months of data are used to set up the Binomial and the Markov models. The values for the parameters associated with selling price, production and switching costs, and production capacity have been taken from Bollen (1999). The demand data collected has been scaled in order to be of the same magnitude of demand values used in (Bollen 1999). The initial demand was set to the average demand over the first 24 months period. The production capacity values range from 0 up to 2.5 with capacity step values varying between 0.05 and 0.5. The other parameters are as follows: $c_1 = 0.1$, $c_2 = 0.5$, $c_3 = 0.1$, $c_4 = 2$, $s_1 = -1$, $s_2 = 0.85$, $s_3 = 0.05$, $\lambda = 1$, and$r_f = 10\%$.

To value project value accuracy, we compare the predicted project value (or model value) to the value obtained by applying the policy found to the data used to derive the model (months 1 to 24), see Table 1. For each possible value of capacity changing step, we report the model value, i.e. the predicted project value which is computed as given in equation (10), and the corresponding initial capacity. We also give the data value, which is the value obtained by applying the optimal capacity changing policy to the data set used to set up the model.

	Binomial Lattice				Markov Grid				Data
Cap. Step	Model Value	Init. Cap.	Data Value	Mod/Data Ratio (%)	Model Value	Init. Cap.	Data Value	Mod/Data Ratio (%)	Mark/Bin Ratio (%)
0.5	7.91	0.5	4.80	165.03	5.63	1.0	5.28	106.63	140.48
0.4	7.93	0.8	5.13	154.93	5.78	0.8	5.45	106.11	137.12
0.3	7.94	0.6	5.10	155.94	5.74	0.6	5.44	105.67	138.16
0.2	7.94	0.6	4.97	160.16	5.78	0.8	5.45	106.11	137.30
0.1	7.94	0.7	5.13	155.20	5.79	0.7	5.47	105.87	137.12
0.05	7.95	0.65	4.90	162.58	5.79	0.75	5.47	106.00	137.15

Table 1: Predicted project value for binomial and Markov models (months 1-24).

From the results reported it can be concluded that the strategies proposed by the two models are different since the initial capacity values are different. As expected, the better values for the predicted project value are obtained for smaller capacity steps, in both models. Furthermore, the predicted project value is larger for the Binomial model, which although might seem to be an advantage is actually a drawback since in both cases the project value tends to be an overestimation. This can be observed in the columns giving the model to data project value ratio. The project value obtained for the first 24 months period data, is better if the capacity changing policy used is the one provided by the Markov model. The Markov model provides values between 37% and 40% better than the Binomial model, as can be seen in the Mark/Bin Ratio column.

To test the efficiency of the models we have used the capacity policies of each model on 7 different sets of data as given in Tables 2 and 3. Data sets 1 and 2 correspond to, respectively,

	Binomial - Project Value						
Step	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
0.5	4.804	6.729	5.573	5.835	7.357	7.852	9.078
0.4	5.130	7.304	6.075	6.168	8.296	8.600	10.353
0.3	5.101	7.075	5.895	6.011	7.881	8.416	9.744
0.2	4.969	7.075	5.895	6.011	7.881	8.416	9.744
0.1	5.131	7.246	6.052	6.147	8.182	8.734	10.153
0.05	4.900	7.178	5.992	6.097	8.058	8.602	9.978

Table 2: Average project values for specific data realizations using the Binomial model policy.

	Markov - Project Value						
Step	Set 1	Set 2	Set 3	Set 4	Set 5	Set 6	Set 7
0.5	5.280	7.230	5.939	6.037	8.271	8.866	10.435
0.4	5.449	7.304	6.075	6.168	8.296	8.862	10.353
0.3	5.435	7.075	5.895	6.011	7.881	8.416	9.744
0.2	5.449	7.304	6.075	6.168	8.296	8.862	10.353
0.1	5.473	7.246	6.052	6.147	8.181	8.734	10.153
0.05	5.467	7.287	6.077	6.168	8.257	8.816	10.275

Table 3: Average project values for specific data realizations using the Markov model policy.

the first and the last 24 months of the collected data. The remaining data sets were randomly generated between the minimum and maximum values of the data collected having increasing demand averages.

As it can be seen, the *real* project values are higher when the Markov policies are used. Only in 1 out of the 42 values computed the Binomial model performs better. Furthermore, the project values obtained by using the Markov model vary from 99.99% to 114.95% of the project values obtained by using the Binomial model.

5 Conclusions

In this work, we address the problem of making investment decisions on a flexible production capacity firm. We consider the investments to be, at least, partially reversible since capacity sell out allows for partial investment recovering.

We propose to address this problem by using dynamic programming implemented on a Markov grid rather than on the standard binomial lattice. An example using *real* data for the stochastic variable (demand) has been solved, using both discretization approaches. It has been shown that the Markov approach is more reliable and leads to a better decision policy. The computational tests performed, also allowed for the conclusion that the Markov model is less sensitive to capacity changing steps.

References

Bollen, N. P. B. (1999), 'Real options and product lyfe cycles', *Management Science* **45**, 670–684.

He, H. & Pindyck, R. S. (1992), 'Investment in flexible production capacity', *Journal of Dynamics and Control* **16**, 575–599.

Kulatilaka, N. (1988), 'Valuing the flexibility of flexible manufacturing system', *IEEE Transaction in Engineering Management* **35**, 250–257.

Pindyck, R. (1988), 'Irreversible investment, capacity choice and the value of the firm', *American Economic Review* **78**, 969.

Tannous, G. F. (1996), 'Capital budgeting for volume flexible equipment', *Decision Sciences* **27**, 157–184.

Trigeorgis, L. (1996), *Real Options: Managerial Flexibility and Strategy in Resource Allocation*, MIT Press.