Strategic R&D Investment Under Uncertainty in Information Technology: Tacit Collusion and Information Time Lag*

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Abstract

This paper develops a stochastic differential game framework for analyzing strategic exercise of options. We focus on research and development (R&D) competition in information technology (IT) investment projects with technical and market uncertainty. According to the theory of real options and game theory, uncertainty generates an option value of delay which can be diminished by the threat of competition. An important feature of the IT projects is that the firms make investment decision on an ongoing basis before the success of the R&D process. Consequently, repeated strategic interactions may facilitate self-enforcing tacit collusion on R&D. We explore the possibility of defining a collusion (cooperative) equilibrium based on the use of a trigger strategy with an information time lag. When the information time lag is long, a preemptive (noncooperative) equilibrium emerges in which the option values of delay are reduced by competition. When the information time lag is sufficiently short, a collusion equilibrium emerges in which investment is delayed more than the single-firm counterpart. An analysis of the equilibrium exercise policies of firms provides a potential explanation for several otherwise puzzling innovation market phenomenons. We also analyze the role of uncertainty on the likelihood of tacit collusion on R&D and provide implications of strategic effects for antitrust and merger control policies.

Keywords: Investment Under Uncertainty, Stochastic Differential Games, Real Options, Information Technology, Trigger Strategy, Tacit Collusion, Information Time Lag

1. Introduction

The valuation of Research and Development or R&D investment projects is an important problem for Information Technology (IT) firms. R&D investments in IT have experienced a rapid growth in the past 20 years, and were at the center of the high-tech boom and bust of the late 1990s. The complexity of R&D projects makes a proper analysis of the associated investments particularly challenging. Much of the difficulty arises from technical and market uncertainty.

Investment under uncertainty problems have been analyzed using the real options approach, which improves upon traditional net present value (NPV) evaluation by recognizing the flexibility of managers to delay, suspend, or abandon a project once it has started. Implementing this approach helps to structure the project as a sequence of managerial decisions over time and clarify the role of uncertainty in project evaluation, which allows us to apply models that have been developed for valuing financial options to project investments (Schwartz and Zozaya-Gorotiza (2003), Berk, Green, and Naik (2002)). A key feature of R&D investments, however, is that they cannot be held independently of strategic considerations. When the options are held by a small number of firms with an advantage to the first mover, each firm's ability to wait is diminished by the threat of preemption. Firms also may have an incentive to delay R&D investments to enhance corporate profits by avoiding an R&D war. The competitive pressure and possibility of tacit collusion [1] of many R&D projects create the motivation for a systematic analysis of the effect of strategic interactions on the firms' optimal exercise strategies. When and how do we need to account competition and tacit collusion in evaluating R&D projects? What is the role of uncertainty on the likelihood of tacit collusion?

In this paper, we develop an game-theoretic framework which include the possibility of tacit collusion and preemption based on a trigger strategy to solve for optimal option exercise

[1] Tacit collusion needs not involve any collusion in legal sense, and needs to involve no communication between the parties. Since explicit collusion is usually banned by antitrust law, we will focus here on the possibility of tacit collusion.

strategies. In order to demonstrate the applicability of such an approach, we focus on a particular real-world example: the behavior of innovation markets [2] in information technology industry. [3] This analysis of the strategic equilibrium exercise policies of firms conducting R&D investments provide a potential explanation for several otherwise puzzling innovation market phenomenons. For example, some strategic R&D investments have been prone to be more delayed than the single form counterpart. Thus, one can use the model to examine the investment thresholds. Firms, fearing to start an R&D war, hold back from investing to proceed a tacit collusion equilibrium, which corresponds to a higher threshold. Similarly, some R&D markets have been prone to overinvestment, where an R&D war may lead to a resumption of a previously discontinued R&D program even when market conditions might still be worse than when they were discontinued. The model provides a potential rational explanation for this phenomenon. Firms react to a deviation from the collusive path with retaliation to follow a preemption equilibrium with lower thresholds. We also analyze the role of uncertainty on the likelihood of tacit collusion on R&D and provide implications of strategic effects for antitrust and merger control policies.

Consideration of strategic exercise of investment projects using the result of a merger between the real options and game theory approaches is an emerging research trend in recent

[2]Innovation markets, sometimes called R&D markets, are markets in which firms compete in research and development. Introduced in the 1995 Antitrust Guidelines for the Licensing of Intellectual Property, innovation markets has quickly become an accepted part of the government's antitrust arsenal. Historically, antitrust focused on price and output effects in markets for goods and services, based on an analysis of historic market shares. In today's dynamic high-tech industries, anticompetitive effects on innovation can have far greater impact than effects on price. Therefore it is not surprising that merger enforcement in these industries often focuses on so-called innovation markets. Two cases are SNIA S.p.A., FTC Dkt. No. C-3889 (July 28, 1999), and Medtronic, Inc., FTC Dkt. No. C-3842 (Dec. 21, 1998). (Morse 2001)

[3]The IT sector spends much more on research and development, relatively speaking, than industry as a whole does. IT companies accounted for a disproportionate share of company-funded R&D (31 percent). Its R&D intensity (i.e., R&D spending divided by industry sales) is three times the national average (U.S. Department of Commerce 2003). IT R&D investments have a high-upside potential, high uncertainty, and indirect returns, and face intensive competitive pressure and propensity to collusion. Thus they are good candidates for being evaluated with a strategic R&D investment framework. For example, effective use of economics is critical in defining the relevant market of high-end ERP research and development and explaining competitive effects and coordinated effects in the DOJ v. Oracle/PeopleSoft case. SAP, Oracle and PeopleSoft are three big players in the ERP industry. The merger of Oracle and PeopleSoft may lead to anticompetitive concern. Another example is the EDA duopoly market of Cadence and the Synopsys, which came after Synopsys' acquisition of Avant! in 2001.

years. For example, Smets (1991) considers irreversible entry for a duopoly facing stochastic demands. Grenadier (1996) uses the strategic exercise of options games to provide a rational explanation for development cascades and recession-induced construction booms in real estate markets. Huisman (2001) studies option games in a technology adoption context. Besides combining irreversible investment under uncertainty with strategic interactions, Weeds (2002) examines R&D by taking technical uncertainty into account. She identifies a preempted leader-follower solution and a joint-investment outcome as two forms of noncooperative equilibrium. The joint-investment outcome leads to greater delay than the single-firm counterpart.

However, these papers typically assume a one-shot investment cost within a stopping time game formulation. The value of active ongoing management of R&D investment projects [4] is not captured in the standard one-shot model. In that type of model, a firm can neither stop the project once it starts, nor resume investment once it terminates the project. In reality, when a firm has an opportunity to invest in an R&D project, it owns an option to invest. After the R&D project begins, a firm maintains the R&D process by making continuous expenditures and receives no income until successful completion of the project. Thus, during the active investment period, it has an option to suspend the R&D project. In fact, firms face the investment decision of whether to invest or suspend R&D in each time period until the project is completed. Under such circumstance, the strategic delay outcome mentioned by Weeds (2002) is no longer a noncooperative equilibrium as each firm can manage its investment actively over time. The ongoing (continuous) nature of many R&D projects creates the motivation for a systematic analysis of R&D investment decision with ongoing (continuous) resource requirements.

In this paper, we analyze a duopoly case in which an R&D project requires ongoing (continuing) costs by developing a stochastic differential game model that allows for consideration of technical and market uncertainty and strategic interactions among firms. There is one R&D

[4] Cooper, Edgett, and Kleinschmidt (1998) have an intensive study of portfolio management as currently practiced in industry and define decision-making process on individual projects on an ongoing basis. The Real Options Group has applied an option-based strategic planning and control framework of Trigeogis (1996) to active management of investment projects over time.

investment opportunity in a new product or technology. The firms compete by their choice of individual investment strategies. The potential future market cash flow uncertainty is taken as an exogenous state of the system, represented by a controlled stochastic process. The technical uncertainty of the R&D process is modeled as a Poisson jump process. [5] We formulate a stochastic optimal control problem which is governed by stochastic differential equations (SDEs) of a type known as Ito equations. Our goal is to synthesize optimal feedback controls for systems subject to SDEs in a way that maximizes the expected value of a given objective function. This one-player stochastic optimal control problem is then expanded to a two-player stochastic differential game.

The games considered here are non zero-sum, in that the sum of the payoffs achieved by the firms is not a constant, and cooperation between the two firms may lead to their mutual advantage. In a Nash equilibrium, no firm can improve its payoff by a unilateral deviation from the equilibrium strategy. However, joint deviations by more than one firm could lead to such improvements. In particular, Nash equilibria are usually not Pareto efficient, that is, maximizing the sum of the payoffs to the two firms. This deviation raises the question of whether there exists efficient Nash equilibria at all and whether there are any general methods to construct such equilibria. This paper presents one such method which is based on the use of trigger strategies. These trigger strategies monitor an implicit Pareto optimal cooperative solution and implement a punitive plan when there is an indication that at least one firm is departing from the cooperative solution.

Evidence of the existence of trigger strategy [6] equilibria in a discrete-time stochastic games was first given by Green and Porter (1984). They view the oligopolistic interaction as a repeated game with imperfect public information and propose a monitoring scheme where a change in the mood of play from cooperation to retaliation would occur when the observed price falls below a triggering level. The application of game theory to continuous-time mod-

[5] Technical uncertainty is similarly modeled as a Poisson arrival in Weeds (2002), Dixit (1988), Reinganum (1983), Lee and Wilde (1980), Dasgupta and Stiglitz (1980), Loury (1979).

[6] Trigger strategies have been mainly discussed in the framework of infinitely repeated games in discrete time (supergames); see e.g. Friedman (1986), Friedman (1991).

els [7] is not well developed and can be quite challenging. Stochastic nonzero-sum differential games [8] have not been extensively used in modeling economic competition as the mathematical apparatus is quite complicated, the Hamilton-Jacobi-Bellmen (HJB) equations do not lead easily to a qualitative analysis of their solutions, and only the noncooperative feedback Nash solution has been characterized for this class of games. Haurie, Krawczyk, and Roche (1994) is an exception. They formulate a stochastic differential game for fisheries management and identify a collusion equilibrium based on a memory strategy with an extended observable state. Usually the implementation of a numerical approximation technique adapted from stochastic control problems is necessary to circumvent the difficulties that arrive in trying to solve this problem directly. Our approach allows us to obtain a close form solution or a sufficiently tractable nonlinear approximation solution and to provide qualitative and quantitative analysis of the solution. We construct a dominating collusion equilibria by implementing monitoring with trigger strategies, which is related to the results of Dockner, Jorgensen, Van Long, and Sorger (2000). They define trigger strategy equilibria by assuming that players observe a defection by any opponent immediately and react to it with a fixed positive time delay $\delta > 0$. Our approach assumes an observation delay with an information time lag, which allows us to remains the subgame perfectness of the trigger strategy equilibrium.

We explore the possibility of defining a so-called collusion equilibrium based on the use of a trigger strategy with an information time lag. When the information lag is long, a preemptive equilibrium emerges in which the option values of delay are reduced by competition. When the information lag is sufficiently short, a collusion equilibrium emerges in which investment is delayed more than the single-firm counterpart and delayed less than that from a one-shot investment cost formulation like that in Weeds (2002).

[7] The value of continuous-time method lies in the clarity, with which optimal strategy or equilibria can be characterized using HJB equations. The continuous-time approach also significantly simplifies the computation of values and risk premium.

[8] References of stochastic differential game see Fleming (1969), Fleming and Rishel (1975), Uchida (1978), Uchida (1979), Basar and Olsder (1995), pierre Cardaliaguet and Plaskacz (2003), Buckdahn, Cardaliaguet, and Rainer (2003)

While economic theory provide many insights on the nature of tacit collusive conducts, it says little on how R&D in a particular industry will or will not coordinate on a collusion equilibrium. Collusion on R&D has been considered as very unlikely, though still possible (Ivaldi, Jullien, Rey, Seabright, and Tirole 2003). The situation may have changed with the rapid growth of R&D expenditures and recent consolidation trend in R&D intensive industry. US government began to respond with the new approach based on the analysis of innovation market introduced in the 1995 Antitrust Guidelines. This generates the need to study the likelihood of tacit collusion on R&D.

The evaluation of tacit collusion calls for a structural quantitative approach, rather than a pure "check list" factors method, to incorporate the various effects. [9] The main problem is that models incorporating all the relevant dimensions would in most cases be unmanageable and unlikely to yield clear-cut predications. (Ivaldi, Jullien, Rey, Seabright, and Tirole 2003). We analyze the characteristics that can affect the sustainability of collusion with a structural quantitative approach from the application of our framework. The goal of this paper is to analyze the role of technical uncertainty and market uncertainty on the likelihood of tacit collusion in innovation marketers of IT industry. In particular, we determine the impact of probability of successful innovation, market growth drift and market volatility on the degree of market transparency that is necessary to sustain the collusion.

This paper, inspired by the single decision maker analysis in Berk, Green, and Naik (2002), develops a stochastic differential game approach to consider strategic interactions in the duopoly case. A similar attempt can be found in Garlappi (2003) to analyze the impact of competition on the risk premium of R&D projects. By developing a more general stochastic differential game framework, we are able to study response maps and introduce a noncooperative collusion equilibrium with a trigger strategy. Miltersen and Schwartz (2003) develop

[9] Another reason for little evidence of tacit collusion on R&D may be that in the past there lacked a structural quantitative approach to assess the likelihood of tacit collusion. The qualitative analysis by government in the DOJ v. Oracle/PeopleSoft case was not accepted by the district court. Anticipating such result, government may take less blocking actions as really needed. So it might not be the small likelihood of tacit collusion, but the difficulty to implement and win in court makes it seemed unlikely.

a model to analyze patent-protected R&D investment projects when there is competition in the development phase and marketing phase of the resulting product. Numerical methods to deal with optimal stopping time problems (Longstaff and Schwartz (2001)) make it possible to analyze their complex model. Their focus is on the impact of R&D competition on production markets and prices instead of the nonprice competition on innovation markets, focused on in this paper.

In summary, the main contributions of this paper are: (1) the development of a stochastic differential game model that allows consideration of technical uncertainty, market uncertainty, strategic interactions and ongoing decision making, (2) the derivation a tacit collusion equilibrium and a preemption equilibrium based on a trigger strategy of nonprice competition in innovation markets, (3) the construction of a structural quantitative approach to evaluate the likelihood of tacit collusion by analyzing the information time lag, (4) the application of the proposed model to provide a potential explanation of several innovation market phenomena, and (5)the analysis of the role of uncertainty on the likelihood of tacit collusion to provide implications for antitrust and merge control in IT industry. These contributions, however, are not limited to IT R&D investments because the basic framework developed in this paper can be applied to other types of highly uncertain R&D investments in which flexibility and strategic interaction among competing firms plays a major role. [10]

An outline of the remainder of the paper is as follows. Section 2 describes the formal model structure and various equilibria and solutions. Section 3 presents analytical results by solving the coupled Hamilton-Jacobi-Bellman equations. Section 4 applies the framework from the model to the case of duopoly competition for one R&D product within a winner-take-all market environment. section 5 examines the role of uncertainty on the likelihood of tacit collusion. Finally, section 6 presents our conclusions and discusses potential future research.

[10] For example, new drugs development in biotech/pharmaceutical industry , and alternative technology fuel cell vehicles or hybrid electric cars in automotive industry.

2. Model

2.1. Setup

Figure 1 illustrates the flow chart of the model.

We are given a standard Brownian motion B in $\mathbb{R}$ on a probability space $(\Omega, \mathcal{F}, P)$. We fix the standard filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ of $\mathcal{F}$ and the time horizon $[0, T]$. [11]

The potential cash flow stream of a project when the R&D is completed is modeled as a process: $\{X_t, t \geq 0\}$ valued in the state space $\mathbb{X} \subset \mathbb{R}$. We will assume that the process follows a geometric Brownian motion, i.e., X_t satisfies the stochastic differential equation

$$dX_t = \mu X_t dt + \sigma X_t dB_t^Q, X_0 = x_0, \tag{1}$$

where dB_t^Q is the increment of B under a risk-neutral Q-martingale. [12]

Now we consider two firms A and B that are competing for one product. Denote the index set $\{A, B\}$. Suppose there are N stages for the R&D process. Then firm i's stage $n_t^i \in \{0, 1, ..., N\}$, for all $i \in \{A, B\}$. Denote by $n_t^{-i} \in \{0, 1, ..., N\}$ the stage of the other firm. We denote the system stage $n_t = (n_t^i, n_t^{-i})$ for all $i \in \{A, B\}$.

[11] When $T = \infty$, the horizon is denoted $[0, \infty)$.

[12] In traditional financial option pricing models, the approach to valuation is based on no-arbitrage augments, where one can trade the underlying asset and a riskless asset so that the option is replicated. However, the assumptions about the liquidity of the underlying assets for such approach is questionable when the applications of this model involve real assets like technology projects. An equilibrium approach relaxes the tradability assumptions needed for arbitrage pricing, although an appropriate equilibrium model must be chosen. This equilibrium model will be used to derive the corresponding Bellman equation. For example, Grenadier (1995) uses the continuous-time version of the capital asset pricing model of Merton (1973).

We model the success of an active R&D process as a Poisson process with parameter π_i, i.e.,

$$dN_t^i = \begin{cases} 1 & \text{with probability } \pi_i dt \\ 0 & \text{with probability } 1-\pi_i dt \end{cases}$$

Each firm's decision at each point of time, given that it has not yet completed the R&D process, is whether to invest in R&D, i.e. to choose a control variable u^i from its set of feasible controls $\mathbb{U}^i : [0,T] \to \mathbb{A}$, where the actions set $\mathbb{A} = \{0,1\}$. Denote u^{-i} as the control of the other firm beside firm i.

The R&D cost process is defined as $u^i I$, where I is the intensity level of the R&D investment, $u^i \in \mathbb{U}^i$ is an admissible control for firm i.

Let $\theta^i \cdot X_t$ denote firm i's cash flow from the completed R&D project at time t, where θ^i can have two possible values: θ_p and θ_f, respectively, corresponding to the market pioneer and the follower. The so called market pioneer is the firm that completes the R&D process first while the market follower is the firm that finishes the R&D process second. Here we refer the market pioneer and market follower as lower case p and f, respectively. [13]

The payoff functionals

$$L^i(t,x_t,n_t^i,n_t^{-i},u^i,u^{-i}) = \int_t^T e^{-r(s-t)}(\zeta_s^i \theta^i X_s - u_s^i I)ds + e^{-r(T-t)}F(X_T,n_T); i \in \{A,B\} \quad (2)$$

where $F(X_T,n_T)$ is the terminal payoff, r is the discount factor, and

$$\zeta_t^i = \begin{cases} 1 & \text{if } n_t^i = N \\ 0 & \text{otherwise} \end{cases}$$

[13] Later, we will refer the R&D Leader and R&D Follower as upper case letter L and F, where R&D Leader is the firm that choose a lower investment threshold and R&D Follower is the firm that choose a higher investment threshold.

is the complete characteristic function for firm i.

Firm i's expected payoff functional is defined as

$$V^i(t,x_t,n_t,u^i,u^{-i}) = E_t^Q\left[L^i(t,x_t,n_t,u^i,u^{-i})|j \in \{A,B\}\right] \tag{3}$$

2.2. Definition of Solutions

In this section, we describe different solution concepts for this differential game.

2.2.1. Noncooperative Equilibrium

The information structure with feedback control is defined by the function

$\eta(t) = \{x_t,n_t\}, t \in [0,\infty]$, where $n_t = (n_t^A, n_t^B)$. The information space for firm i, N_η^i, is induced by its information η^i.

A 2-tuple differential game is formulated as (1),(2) with the admissible control $u^i \in \mathbb{U}^i, i \in \{A,B\}$ and the information structure η. We shall use the notation $\mathbb{P}_\eta(x_0,0)$ for this game.

A strategy for firm i in $\mathbb{P}_\eta(x_0,0)$ is a mapping $\phi^i : [0,T] \times N_\eta^i \to \mathbb{U}^i$. Formally $u^i(t) = \phi^i(t,x_t,n_t)$, for $i \in \{A,B\}$ and all $t \in [0,T]$.

A 2-tuple $\phi = (\phi^A,\phi^B)$ of strategies is called a strategy profile. Denote the set of all feasible strategy profiles for $\mathbb{P}_\eta(x_0,0)$ by S_η. The set of ϕ^i is denoted by S_η^i. The set of $\phi^{-i} = \phi^j$, $i,j \in \{A,B\}, i \neq j$ for which there exist a strategy ϕ^i such that $(\phi^i,\phi^{-i}) \in S_\eta$ is denoted by S_η^{-i}. Finally, the set of feasible responses by firm i to a given $\phi^{-i} \in S_\eta^{-i}$ is denoted by $S_\eta^i(\phi^{-i})$.

Firm i's expected payoff functional can be denoted as

$$V^i(t,x_t,n_t,\phi^i,\phi^{-i}) = E_t^Q\left[L^i(t,x_t,n_t,u^i,u^{-i})|u^j(t) = \phi^j(t,x_t,n_t), j \in \{A,B\}\right] \tag{4}$$

Definition 1. A Nash equilibrium for the differential game $\mathbb{P}_\eta(x_t, n_t, t)$ is a strategy profile $\phi^* = (\phi^{i*}, \phi^{-i*})$ such that for all $i \in \{A, B\}$ and all strategies $\phi^i \in S_\eta^{-i}(\phi^{-i*})$ it holds that $V^i(t, x_t, n_t, \phi^{i*}, \phi^{-i*}) \geq V^i(t, x_t, n_t, \phi^i, \phi^{-i*})$

2.2.2. Cooperative Optimum

The cooperative solution is usually required to be Pareto optimal.

Definition 2. A cooperative (Pareto) solution for the differential game $\mathbb{P}_\eta(x_0, 0)$ is a strategy profile $\phi^{C*} = (\phi^{Ci*}, \phi^{-Ci*})$ such that for all strategy profile $\phi = (\phi^i, \phi^{-i})$ it holds that
$\sum_{i=\{A,B\}} V^i(t, x_t, n_t, \phi^{Ci*}, \phi^{-Ci*}) \geq \sum_{i=\{A,B\}} V^i(t, x_t, n_t, \phi^i, \phi^{-i})$

2.2.3. Response Solution

Definition 3. A response solution by firm i to a given $\phi^{-i} \in S^{-i}$ is a strategy $\phi^{ir} \in S^i(\phi^{-i})$ such that for all strategies $\phi^i \in S^i(\phi^{-i})$ it holds that $V^i(t, x_t, n_t, \phi^{ir}, \phi^{-i}) \geq V^i(t, x_t, n_t, \phi^i, \phi^{-i})$.

2.2.4. Trigger Strategy Equilibria

Although the cooperative solution is Pareto optimal, it is not an equilibrium with feedback control. The response strategy of a cooperative outcome is to increase investment with a lower threshold. It is then important to design a cooperative policy which retains the properties of equilibrium and generates outcomes that dominate the Nash feedback equilibrium.

The basic idea for constructing such an equilibrium is to design a new game with history dependent strategies and to construct a Nash equilibrium for this new game.

Now suppose firms can observe rival's actions with information time lag δ. The information structure with delayed action observation is then defined by function

$h^i(t) = \{x_t, n_t, u_{s-\delta}, s \in [0,t]\}, t \in [0,T]$, where $u_s = \{u_s^i, u_s^{-i}\}$, $u_s = u_0$, for $s \leq 0$. The information space for firm i, N_h^i, is induced by its information h^i.

We denote $\mathbb{P}_h(x_0, 0)$ as a 2-tuple differential game formulated by (1),(2) with the admissible control $u^i \in \mathbb{U}^i, i \in \{A,B\}$ and the information structure h. Denote the set of all feasible strategy profiles for $\mathbb{P}_h(x_0, 0)$ by S_h.

Firm i's expected payoff functional is defined as

$$V^i(t, h_t, \phi^i, \phi^{-i}) = E_t^Q \left[L^i(t, x_t, n_t, u^i, u^{-i}) | u^j(t) = \phi^j(t, h_t) \right], \phi \in S_h, j \in \{A,B\} \tag{5}$$

Specially, at initial time, the payoff functional might be conveniently denoted by $V^i(0, x_0, n_0, \phi^i, \phi^{-i})$.

Definition 4. A Nash equilibrium for the differential game $\mathbb{P}_h(x_0, n_0, 0)$ is a strategy profile $\phi^* = (\phi^{i*}, \phi^{-i*})$ such that for all $i \in \{A,B\}$ and all strategies $\phi^i \in S_h^{-i}(\phi^{-i*})$ it holds that $V^i(0, x_0, n_0, \phi^{i*}, \phi^{-i*}) \geq V^i(0, x_0, n_0, \phi^i, \phi^{-i*})$

We now introduce the trigger strategy $\psi \in S_h$ used to enforce a given target profile $\tilde{\phi} \in S_h$. At any time instant $s' \in [0, \infty)$, firm $i \in \{A,B\}$ can decide whether to cooperate and continue to play his target strategy $\tilde{\phi}^i$ or to defect by deviating from $\tilde{\phi}^i$.

We assume that if a firm defects from its target path at time s', its opponent will observe the deviation and start to punish it at time $s = s' + \delta$. [14] It is furthermore assumed that the punishment lasts forever. Under these assumptions a trigger strategy for firm i with target profile $\tilde{\phi} \in S_h$ can be defined as follows:

$$\psi^i(t, \cdot) = \begin{cases} \tilde{\phi}^i(t, \cdot) & \text{if no firm has defected before and at time } t - \delta \\ \varphi^i(t, \cdot) & \text{if a defection has occurred at or before time } t - \delta \end{cases} \tag{6}$$

[14] Defection of strategy may not lead to defection of action at some state region. As strategy is difficult to observe, we assume only action is observable.

where $\varphi = (\varphi^i, \varphi^{-i}) \in S_h$ is a strategy profile which we call the threats or the punishment strategies.

Denote the target path corresponding to the target profile $\tilde{\phi}$ by $\tilde{h}_t$. Now consider the decision problem of firm i at time t under the assumption that before time t no firm has defected. It can either continue to cooperate, in which case its discounted payoff over the remaining time horizon is $V^i(t, \tilde{h}_t, \tilde{\phi}^i, \tilde{\phi}^{-i})$, or it can defect at t. If we denote firm i's defection strategy by ϕ^i then we can write its discounted payoff over time interval $[t, \infty)$ in the case of defection as

$$\begin{aligned} & V^i_{DEF}(t, \tilde{h}_t, \phi^i, \tilde{\phi}^{-i}) \\ = \; & E_t^Q\Big[\int_t^{t+\delta} e^{-r(s-t)}(\hat{\zeta}^i_s \hat{\theta}^i X_s - \hat{u}^i_s I)ds + e^{-r\delta} V^i(t+\delta, \hat{h}_{t+\delta}, \varphi^i, \varphi^{-i})\Big]; \; i \in \{A, B\} \end{aligned} \tag{7}$$

where $\hat{\zeta}$, $\hat{\theta}$, $\hat{u}(\cdot)$ and $\hat{h}$ are the complete characteristic function, value parameter, control path, and information path, respectively, corresponding to the strategy profile $(\phi^i, \tilde{\phi}^{-i})$.

Proposition 1. *Let $\tilde{\phi}$ be a given target profile for the game $\mathbb{P}_h(x_0, 0)$ and let $\tilde{h}_t$ be the corresponding target path. The strategy profile $\psi = (\psi^A, \psi^B)$ defined in (6) constitutes a Nash equilibrium for the game $\mathbb{P}_h(x_0, 0)$ if and only if*

$$V^i_{DEF}(t, \tilde{h}_t, \phi^i, \tilde{\phi}^{-i}) \leq V^i(t, \tilde{h}_t, \tilde{\phi}^i, \tilde{\phi}^{-i}) \tag{8}$$

holds for all $i \in \{A, B\}$, all $t \in [0, \infty)$, and all feasible defection paths $\phi^i \in S^i_h(\tilde{\phi}^{-i})$.

There is only one condition to be satisfied by threats in order for a trigger strategy profile to constitute a Nash equilibrium: the threats must be effective as described by condition (8). However, in many situations the most effective threats may not be credible. A necessary condition for threats to be credible is that they constitute a subgame perfect Nash equilibrium. [15]

[15] A feedback Nash equilibrium φ for game $\mathbb{P}_\eta(x_0, n_0, 0)$ is subgame perfect if, for each $(x, n_t, t) \in \mathbb{X} \times \{0, 1, ...N\}^2 \times [0, \infty)$, the subgame $\mathbb{P}_\eta(x_t, n_t, t)$ admits a feedback Nash equilibrium ψ such that $\psi(y, n_s, s) =$

Let us assume that φ is a feedback perfect Nash equilibrium of the game $\mathbb{P}_\eta(x_0, n_0, 0)$. The expected payoff in the case of defection is then

$$\begin{aligned}
& V^i_{DEF}(t, \tilde{h}_t, \phi^i, \tilde{\phi}^{-i}) \\
= \;& E^Q_t[\int_t^{t+\delta} e^{-r(s-t)}(\hat{\zeta}^i_s \hat{\theta}^i X_s - \hat{u}^i_s I)ds + e^{-r\delta} V^i(t+\delta, \hat{h}_{t+\delta}, \varphi^i, \varphi^{-i})] \\
= \;& V^i(t, \tilde{h}_t, \phi^i, \tilde{\phi}^{-i}) + E^Q_t\{e^{-r\delta}[V^i(t+\delta, \hat{h}_{t+\delta}, \varphi^i, \varphi^{-i}) - V^i(t+\delta, \hat{h}_{t+\delta}, \phi^i, \tilde{\phi}^{-i})]\}
\end{aligned}$$

for all $i \in \{A, B\}$.

Proposition 2. *Let $\tilde{\phi}$ be a given target profile for the game $\mathbb{P}_h(x_0, 0)$ and let $\varphi \in S_\eta$ be a feedback perfect Nash equilibrium. Denote the corresponding target path to be $\tilde{h}_t$ and let $\hat{h}_t$ be as defined before. The strategy profile $\psi = (\psi^A, \psi^B)$ defined in (6) constitutes a Nash equilibrium for the game $\mathbb{P}_h(x_0, 0)$ if and only if*

$$\begin{aligned}
& V^i(t, \tilde{h}_t, \phi^i, \tilde{\phi}^{-i}) - V^i(t, \tilde{h}_t, \tilde{\phi}^i, \tilde{\phi}^{-i}) \le \\
& E^Q_t\{e^{-r\delta}[V^i(t+\delta, \hat{h}_{t+\delta}, \phi^i, \tilde{\phi}^{-i}) - V^i(t+\delta, \hat{h}_{t+\delta}, \varphi^i, \varphi^{-i})]\}
\end{aligned} \tag{9}$$

holds for all $i \in \{A, B\}$, all $t \in [0, \infty)$, and all feasible defection paths $\phi^i \in S^i_h(\tilde{\phi}^{-i})$.

3. Valuation

This section presents firms' value of noncooperative strategy and cooperative strategy by solving the coupled HJB equations.

$\varphi(y, n_s, s)$ holds for $(y, n_s, s) \in \mathbb{X} \times \{0, 1, ...N\}^2 \times [0, \infty)$. A feedback Nash equilibrium which is subgame perfect is also called a feedback perfect Nash equilibrium

3.1. Noncooperative Equilibrium

Proposition 3. *For a 2-tuple nonzero-sum stochastic differential game of prescribed fixed duration* $[0,T]$*, described by (1), (2), the admissible control* $u^i \in \mathbb{U}^i \subset \mathbb{U}, i \in \{A,B\}$ *and the information structure* η*, 2-tuple of feedback strategies* $\{\phi^{i*} \in S^i_\eta; i \in \{A,B\}\}$ *provides a Nash equilibrium solution if there exists suitably smooth functions* $J^i : [0,T] \times N^i_\eta \to \mathbb{R}, i \in \{A,B\}$*, satisfying the Hamiltonian-Jacobian-Bellman equation:*

$$\mathbb{D}J^i(t,x,n^i_t,n^{-i}_t) + \zeta^i_t \theta^i X_t \tag{10}$$

$$+ \sup_{u^i_t \in \mathbb{U}^i} \{u^i_t[\pi^i(n^i_t)(J^i(t,x,n^i_t+1,n^{-i}_t) - J^i(t,x,n^i_t,n^{-i}_t)) - I]\} \tag{11}$$

$$+ \phi^{-i*}\pi^{-i}(n^{-i}_t)[J^i(t,x,n^i_t,n^{-i}_t+1) - J^i(t,x,n^i_t,n^{-i}_t)] = 0 \tag{12}$$

$$J^i(T,x,n^i_T,n^{-i}_T) = F^i(x,n^i_T,n^{-i}_T) \tag{13}$$

where

$$\mathbb{D}J^i(t,x,n^i_t,n^{-i}_t) = \frac{1}{2}\sigma^2 x^2 J^i_{xx} + \mu x J^i_x + J^i_t - rJ^i \tag{14}$$

where the subscript J refers to the partial derivative.

Lemma 1. *Suppose* $\frac{\partial}{\partial x}(J^i(t,x,n^i_t+1,n^{-i}_t) - J^i(t,x,n^i_t,n^{-i}_t)) \geq 0, i \in \{A,B\}$*. Then firms have threshold strategy, i.e.,* $u^i = \phi^i(t,x_t,n_t) = 1_{x_t \geq x^{i*}(t,n_t)}$*,* [16] *with a threshold* $x^{i*}(t,n_t)$

When the feedback strategy is threshold strategy, $\phi^i(t,x_t,n_t) = 1_{x_t \geq x^i(t,n_t)}$, for threshold $x^i(t,n_t)$. It would be convenient to denote $V^i(t,x_t,n_t,x^i,x^{-i}) = V^i(t,x_t,n_t,\phi^i,\phi^{-i})$ for threshold pair x^i,x^{-i}. In addition, we may denote $V^i(t,x_t,n_t) = V^i(t,x_t,n_t,x^i,x^{-i})$ for simplicity.

[16] 1_{value} is a characteristic function. It equals 1 if value is true or 0 if value is false.

We start by assuming that one firm (the R&D Leader) invests not later than its rival (the R&D Follower), i.e., $x^{L*} \leq x^{F*}$, where x^{L*} and x^{F*} are Leader and Follower's thresholds. The R&D Leader or R&D Follower is not necessarily the market leader or market follower.

Theorem 1. *Suppose $n_t^i < N, i \in \{L, F\}$. For a 2-tuple nonzero-sum stochastic differential game of duration $[0, \infty)$, as described by (1), and (2), and under feedback information pattern, i.e., $u^i(t) = \phi^i(t, x_t, n_t)$. Let $V^{iss}(t, x_t, n_t), V^{ics}(t, x_t, n_t), V^{icc}(t, x_t, n_t)$ be functionals solved by HJB equations (10) with $(u^L, u^F) = (0,0), (u^L, u^F) = (1,0), (u^L, u^F) = (1,1)$, respectively, for $i \in \{L, F\}$. Suppose $\frac{\partial}{\partial x}(J^i(t, x, n_t^i + 1, n_t^{-i}) - J^i(t, x, n_t^i, n_t^{-i})) \geq 0, i \in \{L, F\}$. Then a 2-tuple of feedback strategies $\{\phi^{i*} \in S_\eta^i; i \in \{L, F\}\}$ provides a Nash equilibrium solution such that $\phi^{i*}(t, x_t, n_t) = 1_{x_t \geq x^{i*}(t, n_t)}$.*

The follower's value is

$$V^F(t,x,n_t) \quad = \quad \begin{cases} V^{Fss}(t,x,n_t) & x < x^{L*}(t,n_t).\ u^L = u^F = 0 \\ V^{Fcs}(t,x,n_t) & x^{L*}(t,n_t) \leq x < x^{F*}(t,n_t).\ u^L = 1, u^F = 0 \\ V^{Fcc}(t,x,n_t) & x^{F*}(t,n_t) \leq x.\ u^L = u^F = 1 \end{cases} \tag{15}$$

The leader's value is

$$V^L(t,x,n_t) \quad = \quad \begin{cases} V^{Lss}(t,x,n_t) & x < x^{L*}(t,n_t).\ u^L = u^F = 0 \\ V^{Lcs}(t,x,n_t) & x^{L*}(t,n_t) \leq x < x^{F*}(t,n_t).\ u^L = 1, u^F = 0 \\ V^{Lcc}(t,x,n_t) & x^{F*}(t,n_t) \leq x.\ u^L = u^F = 1 \end{cases} \tag{16}$$

with

$$V^L(t,0,n_t) = V^F(t,0,n_t) = 0 \tag{17}$$

$$\lim_{x \to \infty} V^L(t,x,n_t) \propto x \tag{18}$$

$$\lim_{x \to \infty} V^F(t,x,n_t) \propto x \tag{19}$$

when $x = x^{L*}(t, n^t)$,

$$V^{Lss}(t,x,n_t) = V^{Lcs}(t,x,n_t) \tag{20}$$
$$\frac{d}{dx}V^{Lss}(t,x,n_t) = \frac{d}{dx}V^{Lcs}(t,x,n_t) \tag{21}$$
$$V^{Fss}(t,x,n_t) = V^{Fcs}(t,x,n_t) \tag{22}$$
$$\frac{d}{dx}V^{Fss}(t,x,n_t) = \frac{d}{dx}V^{Fcs}(t,x,n_t) \tag{23}$$
$$\pi^L(V^L(t,x,n^L+1,n^F) - V^L(t,x,n^L,n^F)) - I = 0 \tag{24}$$

when $x = x^{F*}(t, n^t)$,

$$V^{Lcs}(t,x,n_t) = V^{Lcc}(t,x,n_t) \tag{25}$$
$$\frac{d}{dx}V^{Lcs}(t,x,n_t) = \frac{d}{dx}V^{Lcc}(t,x,n_t) \tag{26}$$
$$V^{Fcs}(t,x,n_t) = V^{Fcc}(t,x,n_t) \tag{27}$$
$$\frac{d}{dx}V^{Fcs}(t,x,n_t) = \frac{d}{dx}V^{Fcc}(t,x,n_t) \tag{28}$$
$$\pi^F(V^F(t,x,n^F+1,n^L) - V^F(t,x,n^F,n^L)) - I = 0 \tag{29}$$

The R&D Leader and R&D Follower's values follow from HJB equations (10) via some notation changes from A and B to L and F. Equation (17) to (19) are standard boundary conditions. The value matching conditions, (20,22,25,27) smooth pasting conditions (21,23,26,28) and transitional boundary conditions (24), (29) are sufficient to solve for the parameters. For a heuristic argument of the value matching conditions, and smooth pasting conditions see Dixit (1993, Section 3.8); a rigorous proof is in Karatzas and Shreve (1991, Theorem 4.4.9). The transitional boundary conditions follows from HJB equations (10).

Corollary 1. *Suppose* $n_t^i < N, i \in \{L, F\}$. *For a 2-tuple nonzero-sum stochastic differential game of duration* $[0, \infty)$*, as described by (1), and (2), and under feedback information pattern, i.e.,* $u^i(t) = \phi^i(t, x_t, n_t)$. *Let* $V^{iss}(t,x_t,n_t), V^{ics}(t,x_t,n_t), V^{icc}(t,x_t,n_t)$ *be functionals solved by*

HJB equations (10) with $(u^L,u^F)=(0,0),(u^L,u^F)=(1,0),(u^L,u^F)=(1,1)$*, respectively, for* $i\in\{L,F\}$*. Suppose* $\frac{\partial}{\partial x}(J^i(t,x,n_t^i+1,n_t^{-i})-J^i(t,x,n_t^i,n_t^{-i}))\geq 0, i\in\{L,F\}$*.*

(i)(Response) For some firm $i\in\{L,F\}$*, suppose the other firm's threshold strategy is given as* $\phi^{-i}(t,x_t,n_t)=1_{x_t\geq x^{-i}(t,n_t)}$*, then firm i's response strategy* $\phi^i(t,x_t,n_t)=1_{x_t\geq x^i(t,n_t)}$*, where* $x^i=R(x^{-i})=\arg\max_{x^i}V^i(t,x_t,n_t,x^i,x^{-i})$*. Leader and Follower's value functionals are solved from equation (15) to (29) by ignoring corresponding transitional Boundary Condition (24) or (29).*

(ii) For all $i\in\{L,F\}$*, suppose firm i' threshold strategy is given as* $\phi^i(t,x_t,n_t)=1_{x_t\geq x^i(t,n_t)}$*, then Leader and Follower's value functionals are solved from equation (15) to (29) by ignoring transitional Boundary Conditions (24) and (29).*

3.2. Cooperative Optimum

Now consider the case in which the two firms make their investment strategies cooperatively.

Proposition 4. *For a 2-tuple nonzero-sum stochastic differential game of prescribed fixed duration* $[0,T]$*, described by (1), (2), the admissible control* $u^i\in\mathbb{U}^i, i\in\{A,B\}$ *and the information structure* η*, a 2-tuple of feedback strategies* $\{\phi^{Ci*}\in S_\eta^i; i\in\{A,B\}\}$ *provides a cooperative equilibrium solution if there exists suitably smooth function functions* $J^C:[0,T]\times N_\eta\to\mathbb{R}$*, satisfying the Hamiltonian-Jacobian-Bellman equation:*

$$\begin{aligned}&\mathbb{D}J^C(t,x,n_t)+(\zeta_t^A\theta^A+\zeta_t^B\theta^B)X_t\\&+\sup_{u_t=(u_t^i,u_t^{-i})}\{u_t^i[\pi^i(n_t^i)(J^C(t,x,n_t^i+1,n_t^{-i})-J^C(t,x,n_t^i,n_t^{-i}))-I]\\&+u_t^{-i}[\pi^{-i}(n_t^{-i})(J^C(t,x,n_t^i,n_t^{-i}+1)-J^C(t,x,n_t^i,n_t^{-i}))-I]\}=0\end{aligned}\tag{30}$$

$$J^C(T,x,n_T^i,n_T^{-i}) = F^C(x,n_T^i,n_T^{-i}) \tag{31}$$

where

$$\mathbb{D}J^C(t,x,n_t) = \frac{1}{2}\sigma^2 x^2 J_{xx}^C + \mu x J_x^C + J_t^C - rJ^C \tag{32}$$

where the subscript parameter of J refers to the partial derivative.

Denote the cooperative Leader and Follower's thresholds as x^{CL*} and x^{CF*}. Then $x^{CL*} \leq x^{CF*}$.

Theorem 2. *For a 2-tuple nonzero-sum stochastic differential game of duration* $[0,\infty)$*, as described by (1), and (2), and under feedback information pattern, i.e.,* $u^i(t) = \phi^{Ci}(t,x_t,n_t^i,n_t^{-i})$*, for* $i \in \{L,F\}$*, let* $V^{Css}(t,x_t,n_t), V^{Ccs}(t,x_t,n_t), V^{Ccc}(t,x_t,n_t)$ *be functionals solved by HJB equations (30) with* $(u^L,u^F) = (0,0), (u^L,u^F) = (1,0), (u^L,u^F) = (1,1)$*, respectively. Suppose* $\frac{\partial}{\partial x}(J^C(t,x,n_t^i+1,n_t^{-i}) - J^C(t,x,n_t^i,n_t^{-i})) \geq 0, i \in \{L,F\}$*, Then a 2-tuple of feedback strategies* $\{\phi^{Ci*} \in S_\eta^i\}, i \in \{L,F\}$ *provides a cooperative equilibrium solution such that* $\phi^{Ci*}(t,x_t,n_t) = 1_{x_t \geq x^{Ci*}(t,n_t)}$.

The combined value to two cooperative firms is described by

$$V^C(t,x,n_t) = \begin{cases} V^{Css}(t,x,n_t) & x < x^{CL*}.\ u^L = u^F = 0 \\ V^{Ccs}(t,x,n_t) & x^{CL*} \leq x < x^{CF*}.\ u^L = 1, u^F = 0 \\ V^{Ccc}(t,x,n_t) & x^{CF*} \leq x.\ u^L = u^F = 1 \end{cases} \tag{33}$$

with

$$V^C(t,0,n_t) = 0 \tag{34}$$

$$\lim_{x \to \infty} V^C(t,x,n_t) \propto x \tag{35}$$

when $x = x^{CL*}$,

$$V^{Css}(x,0,0) = V^{Ccs}(x,0,0) \tag{36}$$

$$\frac{d}{dx}V^{Css}(x,0,0) = \frac{d}{dx}V^{Ccs}(x,0,0) \tag{37}$$

$$\pi^L(t,n_t)(V^C(t,x,n_t^L+1,n_t^F) - V^C(t,x,n_t^L,n_t^F)) - I = 0 \tag{38}$$

when $x = x^{CF*}$,

$$V^{Css}(x,0,0) = V^{Ccs}(x,0,0) \tag{39}$$

$$\frac{d}{dx}V^{Css}(x,0,0) = \frac{d}{dx}V^{Ccs}(x,0,0) \tag{40}$$

$$\pi^F(t,n_t)(V^C(t,x,n^L,n_t^F+1) - V^C(t,x,n_t^L,n_t^F)) - I = 0 \tag{41}$$

4. Solution

We now consider a one stage R&D process with infinite horizon and winner-take-all market environment, that is, $N = 1$, $T = \infty$, and $\theta_f = 0$. Without loss of generality, let $\theta_p = 1$.

The stationary strategy profile solution with feedback information patten is defined by $\phi^i(t,x,n_t) = 1_{x_t \geq x^i(n_t)}, i \in \{A,B\}$, where x^i refers to the corresponding investment threshold. Then it will be convenient to denote the payoff functional of firm i defined in (2) by $L^i(x_t,n_t,x^i,x^{-i})$, and then firm i's expected payoff functional can be written as $V^i(x_t,n_t,x^i,x^{-i}) = V^i(x_t,n_t,\phi^i,\phi^{-i})$.

At equilibrium point (x^{i*},x^{-i*}), the equilibrium value functional might be denoted as $V^i(x_t,n_t) = V^i(x_t,n_t,x^{i*},x^{-i*})$

4.1. Noncooperative Equilibrium

Proposition 5. *Suppose $T = \infty, N = 1, \theta_p = 1, \theta_f = 0$. Then for $i \in \{L, F\}$,*

$$V^i(x, n^i = 1, n^{-i} = 0) = \frac{x}{r - \mu} \tag{42}$$

$$V^i(x, n^i = 0, n^{-i} = 1) = 0 \tag{43}$$

At stage $n = (0,0)$, the R&D Follower's value is

$$V^F(x,0,0) = \begin{cases} a_{1F} x^{-\gamma_{1,r}} & x < x^{L*}.\ u^L = u^F = 0 \\ b_{1F} x^{-\gamma_{1,r+\pi^L}} + b_{2F} x^{-\gamma_{2,r+\pi^L}} & x^{L*} \le x < x^{F*}.\ u^L = 1, u^F = 0 \\ c_{2F} x^{-\gamma_{2,r+\pi^L+\pi^F}} + c_{3F} x + c_{4F} & x^{F*} \le x.\ u^L = u^F = 1 \end{cases} \tag{44}$$

The R&D Leader's value is

$$V^L(x,0,0) = \begin{cases} a_{1L} x^{-\gamma_{1,r}} & x < x^{L*}.\ u^L = u^F = 0 \\ b_{1L} x^{-\gamma_{1,r+\pi^L}} + b_{2L} x^{-\gamma_{2,r+\pi^L}} & \\ +b_{3L} x + b_{4L} & x^{L*} \le x < x^{F*}.\ u^L = 1, u^F = 0 \\ c_{2L} x^{-\gamma_{2,r+\pi^L+\pi^F}} + c_{3L} x + c_{4L} & x^{F*} \le x.\ u^L = u^F = 1 \end{cases} \tag{45}$$

where $\gamma_{1,y}$, $\gamma_{2,y}$ solve $\frac{1}{2}\sigma^2(-\gamma)(-\gamma-1) + \mu(-\gamma) - y = 0$, assume $y > 0$.

$$\begin{aligned} \gamma_{1,y} &= \frac{m - \sqrt{m^2 + 2y\sigma^2}}{\sigma^2} < 0, \\ \gamma_{2,y} &= \frac{m + \sqrt{m^2 + 2y\sigma^2}}{\sigma^2} > 0, \\ m &= \mu - \sigma^2/2 \end{aligned} \tag{46}$$

This solution allows us to derive analytical characterization of the value and optimal strategy. Here we can show closed form solutions for some special cases. In the general case the complexity of this formulation does not allow closed form solution. But it can be easily solved by numerical methods.

The monopoly case In the monopoly case, $x^B = \infty$. Then the threshold for the firm A becomes

$$x^A = \frac{I(r-\mu)}{\pi^A}\frac{(r+\pi^A-\mu)}{r+\pi^A}\frac{(\gamma_{2,r+\pi^A}-\gamma_{1,r})(r+\pi^A)-\pi^A\gamma_{2,r+\pi^A}}{(\gamma_{2,r+\pi^A}-\gamma_{1,r})(r+\pi^A-\mu)-(1+\pi^A)\gamma_{2,r+\pi^A}} \tag{47}$$

The R&D Follower case In the follower case, $x^B = 0$. [17] Then the threshold for firm A as the follower becomes

$$x^A = \frac{I(r-\mu)}{\pi^A}\frac{(r+\pi^A+\pi^B-\mu)}{r+\pi^A+\pi^B}\frac{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r+\pi^A})(r+\pi^A+\pi^B)-\pi^A\gamma_{2,r+\pi^A+\pi^B}}{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r+\pi^A})(r+\pi^A+\pi^B-\mu)-\pi^A(1+\gamma_{2,r+\pi^A+\pi^B})} \tag{48}$$

The symmetric case In the symmetric case, $\pi^A = \pi^B$, then $x^A = x^B$. Then the threshold for both firm A and firm B becomes

$$x^{N*} = \frac{I(r-\mu)}{\pi^A}\frac{(r+\pi^A+\pi^B-\mu)}{r+\pi^A+\pi^B}\frac{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r})(r+\pi^A+\pi^B)-\pi^A\gamma_{2,r+\pi^A+\pi^B}}{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r})(r+\pi^A+\pi^B-\mu)-\pi^A(1+\gamma_{2,r+\pi^A+\pi^B})} \tag{49}$$

4.2. Cooperative Optimum

Now consider the case in which the two firms decide on their investment strategies cooperatively.

[17] The follower case means the situation that the other firm will always invest.

Proposition 6. *Suppose $T=\infty, N=1, \theta_p=1, \theta_f=0$. Then $V^C(x,1,0)=V^C(x,0,1)=\frac{x}{r-\mu}$*

$$V^C(x,0,0) \quad = \quad \begin{cases} a_1 x^{-\gamma_{1,r}} & x < x^{CL*} \\ b_1 x^{-\gamma_{1,r+\pi^L}} + b_2 x^{-\gamma_{2,r+\pi^L}} + b_3 x + b_4 & x^{CL*} \leq x < x^{CF*} \\ c_2 x^{-\gamma_{2,r+\pi^L+\pi^F}} + c_3 x + c_4 & x^{CF*} \leq x \end{cases} \tag{50}$$

where

$$b_3 = \frac{\pi^L}{(r+\pi^L-\mu)(r-\mu)} \tag{51}$$

$$b_4 = -\frac{I}{r+\pi^L} \tag{52}$$

$$c_3 = \frac{\pi^L+\pi^F}{(r+\pi^L+\pi^F-\mu)(r-\mu)} \tag{53}$$

$$c_4 = -\frac{2I}{r+\pi^L+\pi^F} \tag{54}$$

The values of a_1, b_1 and b_2 are obtained by solving nonlinear equation (50) by optimization methods.

The symmetric case In the symmetric case, $\pi^A=\pi^B$, then $x^{CL*}=x^{CF*}$. Then the threshold for both firm A and firm B becomes

$$x^{C*} = \frac{I(r-\mu)}{\pi^A} \frac{(r+\pi^A+\pi^B-\mu)}{r+\pi^A+\pi^B} \frac{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r})(r+\pi^A+\pi^B)-2\pi^A\gamma_{2,r+\pi^A+\pi^B}}{(\gamma_{2,r+\pi^A+\pi^B}-\gamma_{1,r})(r+\pi^A+\pi^B-\mu)-2\pi^A(1+\gamma_{2,r+\pi^A+\pi^B})} \tag{55}$$

4.3. Analysis of noncooperative and cooperative solutions

We now analyze the noncooperative solution and the cooperative solution for two symmetric firms. [18]

Consider x^A as a response function of x^B, as $x^A(x^B)$, it is observed from (47), (48), (49) and (55) that $x^A(0) < x^{N*} < x^A(\infty) < x^{C*}$. Under uncertain future cash flow market condition, the investment thresholds of the two firms are positively correlated. [19]

Response Map Denote the response map $R : \mathbb{X} \to \mathbb{X}$ for firm i, i.e., $x^i(x^{-i}) = R(x^{-i}) = \arg\max_{x^i} V^i(t, x_t, n_t, x^i, x^{-i})$. The value of $R(x^{-i}), i \in \{A, B\}$ follows from Corollary 1 by replacing the transfer boundary condition (29) or (24) with the given threshold x^{-i} when i refers to Leader or Follower.

Figure 2 demonstrates the positive correlations between the two firms' thresholds. It is noted that the response function $R(x)$ is an increasing monotone function. $R(0) < R(x) < R(\infty), 0 < x < \infty$.

Notice that as showed in figure 2, cooperative solution $\{x^{C*}, x^{C*}\}$ is not a noncooperative equilibrium since $R(x^{C*}) < x^{C*}$.

Figure 3 illustrates the relations between the firms's value and their threshold under the response strategy. The response strategy threshold pair is $(x^A, x^B(x^A))$, where $x^B(x^A) = R(x^A))$. The value of a firm, V, is represented by the value of a_1 (refer to equation (44) or (45)) as a function of threshold x^A. The solid-dotted line refers to the value of firm A and the dash-dotted line refers to firm B. Firm A's optimal investment threshold for response strategy is x^{NA*}, then firm B's optimal investment threshold $x^{NB*} = x^B(x^{NA*})$.

[18] The asymmetric cooperative case has the difficulty in agreeing asymmetric investment rules and the need for side-payments implicit in cooperative outcome, so it is not easy to be considered for tacit collusion.

[19] Under certain future cash flow market condition, the investment thresholds of the two firms are independent.

Figure 3 also illustrates the relations between the symmetric firms's value and their threshold under the cooperative strategy. The cooperative threshold pair is (x^{CA}, x^{CB}). Note $V^{CA} = V^{CB}$, $x^{CB} = x^{CA}$ in the symmetric case. The value of a firm, V, is represented by the value of a_1 (refer to equation (50)) as a function of threshold x^{CA}. The solid line refers to the value of V^{CA} as a function of the cooperative threshold x^{CA}. The optimal investment cooperative threshold is $x^{CA*} = x^{CB*} = x^{C*}$. It is observed that $a_{1i}(x^{CA*}, x^{CB*}) > a_{1i}(x^{NA*}, x^{NB*})$, which implies the cooperative solution is more efficient than the noncooperative solution. It is also observed that $x^{C*} > x^{N*}$, which implies that the Pareto optimal solution leads to strategic delay.

The cooperative solution, however, is not an equilibrium with feedback strategy. Consider the response strategy of one firm, say firm B. $x^B(x^{CA*}) < x^{CB*}$. It is observed from Figure 3 that $a_{1B}(x^{CA*}, x^B(x^{CA*})) > a_{1B}(x^{CA*}, x^{CB*})$, $a_{1A}(x^{CA*}, x^B(x^{CA*})) < a_{1A}(x^{CA*}, x^{CB*})$. The implication is that firm B receives higher benefits through deviation, while firm A gets less benefits by firm B's deviation. It is also observed that $a_{1A}(x^{NA*}, x^{NB*}) > a_{1A}(x^{CA*}, x^B(x^{CA*}))$, which indicates that firm A has incentive to choose the feedback noncooperative solution facing the deviation of firm B. These observations suggest that both firms have incentive to deviate from the cooperative solution. They also indicate that the feedback noncooperative solution provides a credible threat.

These findings are also demonstrated in Figure 4, which illustrates the value of symmetric firms A and B with various investment strategies. When one and only one firm has deviated, the solid-pentagram line refers to the value of the firm that deviates, $V^{Di}(x; x^{D*}, x^{C*})$, and the solid-plus line refers to the value of the other firm which has not deviated $V^{-Di}(x; x^{D*}, x^{C*})$, where $x^{D*} = R(x^{C*})$. The solid-star line refers to the value of the firm pursuing a cooperative strategy $V^{Ci}(x; x^{C*}, x^{C*})$. The solid-dotted line refers to the value of the firm pursuing a non-cooperative equilibrium strategy $V^i(x; x^{N*}, x^{N*})$. It is observed that $V^{Di}(x; x^{D*}, x^{C*}) > V^{Ci}(x; x^{C*}, x^{C*}) > V^i(x; x^{N*}, x^{N*}) > V^{-Di}(x; x^{D*}, x^{C*})$. These observations imply that the cooperative optimum is more efficient than the noncooperative equilibrium, the cooperative optimum is not an equilibrium, and the threat to punish from being deviated to the noncooperative equilibrium is credible.

The interesting thing to note is that the Pareto optimal cooperative solution (x^{CA*}, x^{CB*}) is not a noncooperative equilibrium. We will show that a collusion equilibria with a trigger strategy generates payoffs which dominate those obtained via the classical noncooperative equilibrium.

4.4. Trigger Strategy

The trigger strategy is described as followed. Symmetric firms choose coooperative threshold x^{C*}, which is from Theorem 2, at the start of first period and continue pursuing it so long as no firm has ever deviated. If any firm has ever deviated, say firm i has deviated to a response $x^{D*} = R(x^{C*})$, then after time lag δ, both firms choose non-cooperative threshold x^{N*}, given by Theorem 1. Thus the threshold pairs for the target strategy $(\tilde{\phi}^i, \tilde{\phi}^{-i})$, deviation strategy $(\phi^i, \tilde{\phi}^{-i})$, and punitive strategy $(\varphi^i, \varphi^{-i})$ are (x^{C*}, x^{C*}), (x^{D*}, x^{C*}) and (x^{N*}, x^{N*}), respectively.

There are three phases when two firms exercise the trigger strategies, coordination phase, deviation phase and retaliation phase. In the coordination phase, both firms choose the Pareto optimal threshold x^{C*}, which exceeds the noncooperative feedback threshold x^{N*} of the punitive phase. These findings have implications for the understanding and assessment of empirical investment behavior.

For example, investment occurs late due to the strategic behavior of the firms who delay their investment in the fear of setting an R&D war to the deviation phase and then the retaliation phase. Hence, delay is due to strategic interactions between firms, not just the option effect of uncertainty. Investment is also more delayed than that when a single firm has the opportunity to invest. Such strategic delay is similar to that described by Weeds (2002) but arising for different reasons.

Similarly, firms exercise investment according to threshold strategies in both the coordination phase and the retaliation phase. It implies that investment will occur after a period of stagnation when market conditions rise and disappear after a period of investment activity when market conditions fall. In addition, firms may deviate to the deviation phase and result in the retaliation phase when the tacit collusion is not sustainable. This suggests that an R&D war may lead to a resumption of a discontinued R&D program, with a sudden burst of competitive activity, even when the market conditions are declining or unchanged, a phenomenon that contrasts strongly with the usual presumption that investment starts when market conditions increase.

5. Likelihood of Tacit Collusion on R&D

In this section, we analyze the characteristics that can affect the sustainability of collusion with a structural quantitative approach. The goal is to analyze the role of technical uncertainty and market uncertainty on the likelihood of tacit collusion. There are some basic structure variables, such as the number of competitors, barriers for entry and market transparency. In particular, we determine the impact of the probability of successful innovation, market growth drift and market volatility on the degree of market transparency that is necessary to sustain the collusion.

5.1. Measures of the degree of market transparency

The information time lag δ represents the degree of market transparency. We define two thresholds δ^{L*} and δ^{H*} to analyze the sustainability of collusion.

To check whether the trigger strategy constitutes a Nash equilibrium or the threat is effective for a given information lag δ, we can go one step further from proposition 2 with conditions $T=\infty, N=1, \theta_p=1, \theta_f=0$. Inequality equation (9) can then be written as

$$\begin{aligned} V^i(x_t,0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_t,0,0,\tilde{\phi}^i,\tilde{\phi}^{-i}) \leq \\ E_t^Q\{e^{-r\delta}1_{n_{t+\delta}=(0,0)}[V^i(x_{t+\delta},0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_{t+\delta},0,0,\varphi^i,\varphi^{-i})]\} \end{aligned} \tag{56}$$

Lemma 2.

$$1 \geq \Pr(n_{t+\delta}=(0,0)|n_t=(0,0),\phi^i,\tilde{\phi}^{-i}) \geq e^{-(\pi^A+\pi^B)\delta} \tag{57}$$

Theorem 3. *Suppose $T=\infty, N=1, \theta_p=1, \theta_f=0$. The strategy profile $\psi=(\psi^A,\psi^B)$ defined in (6) constitutes a Nash equilibrium for the game $\mathbb{P}_h(x_0,0)$ if*

$$\begin{aligned} V^i(x_t,0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_t,0,0,\tilde{\phi}^i,\tilde{\phi}^{-i}) \leq \\ e^{-(r+\pi^A+\pi^B)\delta}E_t^Q[V^i(x_{t+\delta},0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_{t+\delta},0,0,\varphi^i,\varphi^{-i})] \end{aligned} \tag{58}$$

holds for all $i \in \{A,B\}$, all $t \in [0,\infty)$, and all feasible defection paths $\phi^i \in S_h^i(\tilde{\phi}^{-i})$.

The strategy profile $\psi=(\psi^A,\psi^B)$ defined in (6) fails to constitute a Nash equilibrium for the game $\mathbb{P}_h(x_0,0)$ if there exits a feasible defection path $\phi^i \in S_h^i(\tilde{\phi}^{-i})$ for some $t \in [0,\infty)$, $i \in \{A,B\}$ such that

$$\begin{aligned} V^i(x_t,0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_t,0,0,\tilde{\phi}^i,\tilde{\phi}^{-i}) > \\ e^{-r\delta}E_t^Q[V^i(x_{t+\delta},0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_{t+\delta},0,0,\varphi^i,\varphi^{-i})] \end{aligned} \tag{59}$$

holds.

Denote

$$\begin{aligned} f1(x_t,\delta) = {} & V^i(x_t,0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_t,0,0,\tilde{\phi}^i,\tilde{\phi}^{-i}) \\ & - e^{-(r+\pi^A+\pi^B)\delta}E_t^Q[V^i(x_{t+\delta},0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_{t+\delta},0,0,\varphi^i,\varphi^{-i})] \end{aligned} \tag{60}$$

$$
\begin{aligned}
f2(x_t,\delta) = V^i(x_t,0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_t,0,0,\tilde{\phi}^i,\tilde{\phi}^{-i}) \\
- e^{-r\delta}E_t^Q[V^i(x_{t+\delta},0,0,\phi^i,\tilde{\phi}^{-i}) - V^i(x_{t+\delta},0,0,\varphi^i,\varphi^{-i})]
\end{aligned} \tag{61}
$$

$$
\delta^L(x) = \min_{\delta\geq 0}\{f1(x,\delta)\geq 0\} \tag{62}
$$

$$
\delta^{L*} = \min_x\{\delta^L(x)\} \tag{63}
$$

$$
\delta^H(x) = \max_{\delta}\{f2(x,\delta)\leq 0\} \tag{64}
$$

$$
\delta^{H*} = \min_x\{\delta^H(x)\} \tag{65}
$$

Corollary 2. *Suppose $T=\infty, N=1, \theta_p=1, \theta_f=0$. Let the threshold pairs for the target strategy $(\tilde{\phi}^i,\tilde{\phi}^{-i})$, deviation strategy $(\phi^i,\tilde{\phi}^{-i})$, and punitive strategy (φ^i,φ^{-i}) are (x^{Ci*},x^{-Ci*}), (x^{Di*},x^{-Ci*}) and (x^{i*},x^{-i*}), respectively. Suppose that $\frac{\partial fk(x,\delta)}{\partial\delta}>0$, for $k=\{1,2\}$. Then the corresponding strategy profile $\psi=(\psi^A,\psi^B)$ defined in (6) constitutes a Nash equilibrium for the game $\mathbb{P}_h(x_0,0)$ if $\delta\leq\delta^{L*}$. The corresponding strategy profile ψ fails to constitute a Nash equilibrium for the game $\mathbb{P}_h(x_0,0)$ if $\delta>\delta^{H*}$.*

Calculation of deviation condition In order to apply the inequality equation (58) and (59), it will be convenient to write $V^i(x_{t+\delta},0,0,\phi^i,\phi^{-i}) = V^i(x_{t+\delta},x^L,x^F)$, $\forall\phi\in S_h$, $\phi(.)=1_{x\geq x^i}$, with $i\in\{L,F\}$ refers to Leader or Follower, respectively, and x^L,x^F refers to the two thresholds for the Leader and Follower, respectively.

Then

$$
V^i(x,x^L,x^F) = \begin{cases}
a_{1i}x^{-\gamma_{1,r}} & x<x^L.\ u^L=u^F=0 \\
b_{1i}x^{-\gamma_{1,r+\pi^L}} + b_{2i}x^{-\gamma_{2,r+\pi^L}} + b_{3i}x + b_{4i} & x^L\leq x<x^F.\ u^L=1,u^F=0 \\
c_{2i}x^{-\gamma_{2,r+\pi^L+\pi^F}} + c_{3i}x + c_{4i} & x^F\leq x.\ u^L=u^F=1
\end{cases}
$$

Notice that

$$
\begin{aligned}
E_t^Q[e^{-r\delta}a_{1i}x_{t+\delta}^{-\gamma_{1,r}}1_{x_{t+\delta}<x^L}] &= E_t^Q[e^{-r\delta}a_{1i}e^{-\gamma_{1,r}y_{t+\delta}}1_{y_{t+\delta}<y^L}], \text{ Let } y=\ln x, y^L=\ln x^L \\
&= a_{1i}G(y^L;\delta,-\gamma_{1,r},1)
\end{aligned}
$$

where

$$
G(y;\delta,d,k)=E^Q[exp(-\int_0^\delta r_s ds)e^{d\cdot y_\delta}1_{k\cdot y_\delta\le y}] \tag{66}
$$

Then

$$
\begin{aligned}
E_t^Q[e^{-r\delta}V^i(x_{t+\delta},x^L,x^F)] = a_{1i}&G(\ln x^L;\delta,-\gamma_{1,r},1) \\
&+b_{1i}[G(\ln x^F;\delta,-\gamma_{1,r+\pi},1)-G(\ln x^L;\delta,-\gamma_{1,r+\pi^L},1)] \\
&+b_{2i}[G(\ln x^F;\delta,-\gamma_{2,r+\pi},1)-G(\ln x^F;\delta,-\gamma_{2,r+\pi^L},1)] \\
&+b_{3i}[G(\ln x^F;\delta,1,1)-G(\ln x^L;\delta,1,1)] \\
&+b_{4i}[G(\ln x^F;\delta,0,1)-G(\ln x^L;\delta,0,1)] \\
&+c_{2i}G(-\ln x^F;\delta,-\gamma_{2,r+\pi^L+\pi^F},1)+c_{3i}G(-\ln x^F;\delta,1,-1) \\
&+c_{4i}G(-\ln x^F;\delta,0,-1)
\end{aligned} \tag{67}
$$

So, if we can compute the function G, we can tell from proposition 3 whether a firm deviates or not with information lag δ.

When $r_t\equiv r$, $\mu_t\equiv\mu$, $\sigma_t\equiv\sigma$,

$$
G(y;\delta,d,k) = \begin{cases} e^{dy_0+(-r+(\mu-\frac{1}{2}\sigma^2)d+\frac{1}{2}d^2\sigma^2)\delta}\Phi(\frac{\frac{y}{k}-y_0-(\mu-\frac{1}{2}\sigma^2)\delta}{\sigma\sqrt{\delta}}-\sigma d\sqrt{\delta}) & k>0 \\ e^{dy_0+(-r+(\mu-\frac{1}{2}\sigma^2)d+\frac{1}{2}d^2\sigma^2)\delta}\Phi(-\frac{\frac{y}{k}-y_0-(\mu-\frac{1}{2}\sigma^2)\delta}{\sigma\sqrt{\delta}}+\sigma d\sqrt{\delta}) & k<0 \end{cases} \tag{68}
$$

where $\Phi()$ is the cumulative normal distribution.

Figure 5 illustrates the impacts of factors like information delay δ and x on the decision to deviate or not. The deviation decision only matters for $x \in [x^{D*}, x^{C*}]$. Deviations won't happen when $\delta \le \delta^L(x)$, while deviation will happen when $\delta > \delta^H(x)$, where $\delta^L(x)$ and $\delta^H(x)$ are defined in (62) and (64). It is also observed that given δ, the deviation is more likely to happen when x is near the middle of x^{D*} and x^{C*}, and less likely to happen when x is near x^{D*} or x^{C*}.

Figure 5 also demonstrates δ^{L*} and δ^{H*}, defined in (63) and (65), respectively. We can tell whether there is a tacit collusion equilibrium following corollary 2.

Sufficiently Long Information Lag δ [20] The cooperative solution that follows from Theorem 2, $\{x^{CA*}, x^{CB*}\}$ is not a noncooperative equilibrium for the game $P_h(x_0, 0)$.

Sufficiently Short Information Lag δ The cooperative solution following from Theorem 2, $\{x^{CA*}, x^{CB*}\}$, under trigger strategy, is a Nash equilibrium for the game $P_h(x_0, 0)$ for symmetric two firms.

In summary, these findings suggest that information or observation of rivals actions provide a scheme to induce tacit collusion.

5.2. Comparative Statics

We now examine the effects of changes in underlying parameters on the information time lag thresholds. In particular, we consider three potential influences on δ^{L*} and δ^{H*}: the success rate (π), the drift of market growth (μ), and the volatility of market (σ).

Consider the effect of the success rate on time lag thresholds. Figure 6 demonstrates that the effect of increasing likelihood of innovation success is a decrease in both information time

[20] For example, in the extreme case $\delta = \infty$, the two firms are required to commit to a threshold at the start of the game or they can't observe their rival's actions.

lag thresholds. The intuition is that if the probability of successful innovation is substantial, the firms then anticipate that innovation is coming soon and thus put less emphasis on the future retaliation and are more tempted to cheat on collusion. Therefore, the more likely innovation is, the more difficult it is to sustain collusion.

Now consider the effects of changes in the drift of market growth on the propensity of tacit collusion. Figure 7 illustrate that the effect of increasing absolute value of drift of market is a decrease in the two information time lag thresholds. It is also showed that a positive drift leads to lower information time lag thresholds than a negative drift does, given the same abstract drift value. A possible explanation to understand this is that the deviation only happens when x is in the deviation region, i.e., $x \in [x^{D*}, x^{C*}]$ and the retaliation only matters when x is in the retaliation region, i.e., $x \in [x^{N*}, x^{C*}]$. When $|\mu| > 0$, firms, anticipating that $x_{t+\delta}$ may be out of the retaliation region, put less emphasis on future retaliation and thus are more tempted to defect. In addition, $x_{t+\delta}$ is more likely out of the retaliation region when $\mu > 0$ than it is when $\mu < 0$, given the same absolute value. Therefore, the intuition is that collusion is easier to sustain in stagnating market, less likely to sustain in declining market and most difficult to sustain in growing market.

Finally consider the effects of changes in the volatility of market on the propensity of tacit collusion. Figure 8 illustrate that the effect of increasing volatility of market is first an increase and then a decrease in the two information time lag thresholds.

There are debates on the effects of demand volatility on likelihood of tacit collusion. [21] The implication from this analysis is that a simple “check list” factors method is not enough when the relation is not a simple one. This paper provides a structural quantitative approach to evaluate the likelihood of tacit collusion.

[21] For example, The European Court of First Instance (CFI) has recently overturned the decision by the European Commission to prohibit the merger between UK tour operators Airtours plc and First Choice plc. As regards market conditions, the Commission had argued that the volatility of demand which characterized the market was conducive to collusive behavior. However, the CFI noted that economic theory suggests the opposite (i.e. volatility of demand renders the creation of a collective dominant position more difficult) and that the Commission had failed to establish that economic theory did not apply or that volatility in demand was conducive to the creation of collective dominance. (Court of First Instance 6 June 2002)

6. Conclusion and Future Work

Motivated by *R&D* investments currently taking place in industry, we develop a stochastic differential game approach to analyze R&*D* investment under technical and market uncertainty with strategic interactions and ongoing costs. The model demonstrates that investment options might be exercised based on a trigger strategy and lead to either a tacit collusion equilibrium or a preemption equilibrium, depending on the length of the associated information time lag. In such a collusion equilibrium, a cooperative solution is played as long as the deviation never happens. If a deviation occurs, a punishment strategy derived from the preemption equilibrium will follow that forever. When the information lag is long, a preemptive equilibrium emerges in which the option values of delay are reduced by competition. When the information lag is sufficiently short, a collusion equilibrium emerges in which investment is delayed more than the single-firm counterpart and delayed less than that from a one-shot investment cost formulation like that in Weeds (2002).

The solution for this option exercise game provide an underlying theory from which one may begin to understand and assess empirical investment behavior. For example, some strategic R&D investments have been prone to be more delayed than the single firm counterpart. The model is able to isolate the conditions that make such phenomenons more or less likely. In addition, some R&D markets have been prone to overinvestment, where an R&D war may lead to a resumption of a previously discontinued R&D program even when market conditions might still be worse than when they were discontinued. While this is often regarded as irrational , the model provides a rational foundation for such exercise patterns.

We also provide implications of tacit collusion in R&D for antitrust and merger control policy. Tacit collusion has been dealt with under the notion of coordinated effects in a number of court decisions and corresponds to the "collective dominance" studied in Europe. Rather than a pure "check-list" of relevant factors, this paper provides an attempt of a structural quantitative analysis for a clear understanding of why each dimension is relevant, as well as how it affects collusion and it is affected by a merger. This helps to facilitate an overall assessment when several factors have a role and push in different directions.

The model of strategic R&D investments can be applied to a variety of settings. More recently, innovation market allegations have become common place, particularly in actions involving the pharmaceutical industry, where drugs are introduced only after years of laboratory and clinical testing and good information is available at least to the government about drugs in advanced stages of R&D. In other industries, practical difficulties often arise in applying the theory, given the secrecy of R&D. Consider also alternative technology hybrid electric cars in automotive industry. For example, GM stopped its hybrid project in 1998 and resumed it recently after observing Toyota's success in its hybrid car project of Prius.

Motivated by the sequential radical technology innovations like hybrid car and fuel cells vehicles currently being pursued in the automobile industry, Yao (2004) applies this stochastic differential game approach to product development and to project selection problem. That paper considers not only technical and market uncertainty as well as strategic interactions among firms but also interactions among multiple products. A more general analysis form of market share outcomes replaces the strong winner-take-all assumption employed here. That approach can be used to explain the phenomenons of multiple technology R&D races in the automobile and information technology industries.

A. Appendix

One player problem The one player objective functional, similar as two player's version (2), is denoted as:

$$L(t,x_t,n_t,u_t) = \int_t^T e^{-r(s-t)}(\zeta_s \theta X_s - u_s I)ds + e^{-r(T-t)}F(X_T,n_T). \tag{69}$$

The expected utility functional is defined as

$$V(t, x_t, n_t, u_t) = E_t^Q [L(t, x_t, n_t, u_t)] \tag{70}$$

From Ito's formular with jumps,

$$dV_t = \mu_V(t)dt + \sigma_V(t)dB_t^Q + \beta_S(t)dZ(t) \tag{71}$$

$$dZ(t) = \kappa(Z(t-)u(t, Xt)dN_t. \tag{72}$$

where

$$\kappa(0) = 1; \kappa(1) = 0$$

N_t is a Poisson process,

$$dN_t = \begin{cases} 1 & \text{with probability } \pi dt \\ 0 & \text{with probability } 1 - \pi dt \end{cases}$$

The process $Z(t) = n_t$ has two possible stages, say 0 and 1. When in stage 0, given the investment decision $u(t, X_t) = 1$, the process Z moves to stage 1 after a time whose probability distribution is exponential with parameter π. Stage 1 is an absorbing stage where $Z(t)$ will stay there forever.

Let $V_t = f(Z_t, X_t, t)$, then

$$dV_t = \mathbb{D} f dt + f_S \sigma_S(t) dB_t^Q + f(Z_t, X_t, t) - f(Z_{t-}, X_{t-}, t), \tag{73}$$

where $\mathbb{D} f = \frac{1}{2}\sigma_S^2 f_{SS} + \mu_S f_S + f_t$.

Moreover, $dV_t = \mu_V(t)dt + dY_t$, for Y a local martingale and

$$\mu_V(t) = \mathbb{D} f + \pi(t)G(t),$$

where

$$G(t) = f(Z_{t-} + \kappa(Z(t-)u(t,X_t),X_t,t) - f(Z_{t-},X_t,t)$$

Proposition 7. *For a one player stochastic differential equation of prescribed fixed duration $[0,T]$, described by (1), and the objective functional (69), the admissible control $u \in \mathbb{U}$ and the information structure η, a feedback strategy $\{\phi^* \in S_\eta\}$ provides a optimal control if there exists suitably smooth function functions $J : [0,T] \times N_\eta \to \mathbb{R}$, satisfying the Hamiltonian-Jacobian-Bellman equation:*

$$\mathbb{D}J(t,x,n_t) + \zeta_t \theta X_t + \sup_{u_t \in \mathbb{U}} \{u_t[\pi(n_t)(J(t,x,n_t+1) - J(t,x,n_t)) - I]\} = 0 \tag{74}$$

$$J(T,x,n_T) = F(x,n_T) \tag{75}$$

where

$$\mathbb{D}J(t,x,n_t) = \frac{1}{2}\sigma^2 x^2 J_{xx} + \mu x J_x + J_t - rJ \tag{76}$$

where the subscript parameter of J refers to the partial derivative.

Proof. This result follows from Fleming (1969) or Fleming and Rishel (1975) with application of Ito's formula with jumps. For a treatment of jumps see Duffie (2001, Appendix F). □

Proof of Proposition 1 Obvious.

Proof of Proposition 2 Obvious.

Proof of Proposition 3 This result follows from the definition of Nash equilibrium and from proposition 7, since by fixing all players' strategies, except the ith one's at their equilibrium choices, which are known to be feedback by hypothesis, we get a stochastic optimal control problem covered by proposition 7 and whose optimal solution is a feedback strategy.

Proof of Lemma 1 Notice $x^{i*}(t,n_t) = \arg\min_x\{J^i(t,x,n_t^i+1,n_t^{-i}) - J^i(t,x,n_t^i,n_t^{-i}) \geq 0\}$.

Proof of Theorem 1 The R&D Leader and R&D Follower's values follow from HJB equations (10) via some notation changes from A and B to L and F. Equation (17) to (19) are standard boundary conditions. The value matching conditions, (20)(22)(25)(27) smooth pasting conditions (21)(23)(26)(28) and transitional boundary conditions (24), (29) are sufficient to solve for the parameters. The value matching conditions, and smooth pasting conditions follow from Karatzas and Shreve (1991, Theorem 4.4.9). The transitional boundary conditions follows from HJB equations (10).

Proof of Proposition 4 This result follows from the definition of Nash equilibrium and from proposition 7, with a two elements array replacing the one dimension variable such as defining $u = (u^{CL}, u^{CF})$.

Proof of Theorem 2 The values follow from HJB equations (30) via some notation changes from A and B to CL and CF. The value matching conditions, smooth pasting conditions and transitional boundary condition are sufficient to solve for the parameters. The value matching conditions, and smooth pasting conditions follow from Karatzas and Shreve (1991, Theorem 4.4.9). The transitional boundary conditions follows from HJB equations (30).

Proof of Lemma 2 $1 \geq \Pr(n_{t+\delta} = (0,0)|n_t = (0,0), \phi^i, \tilde{\phi}^{-i})$ is obvious. To see the second inequality, notice $e^{-(\pi^A+\pi^B)\delta} = \Pr(n_{t+\delta} = (0,0)|n_t = (0,0), u^i \equiv 1)$.

Proof of Theorem 3 Plug (57) in (56).

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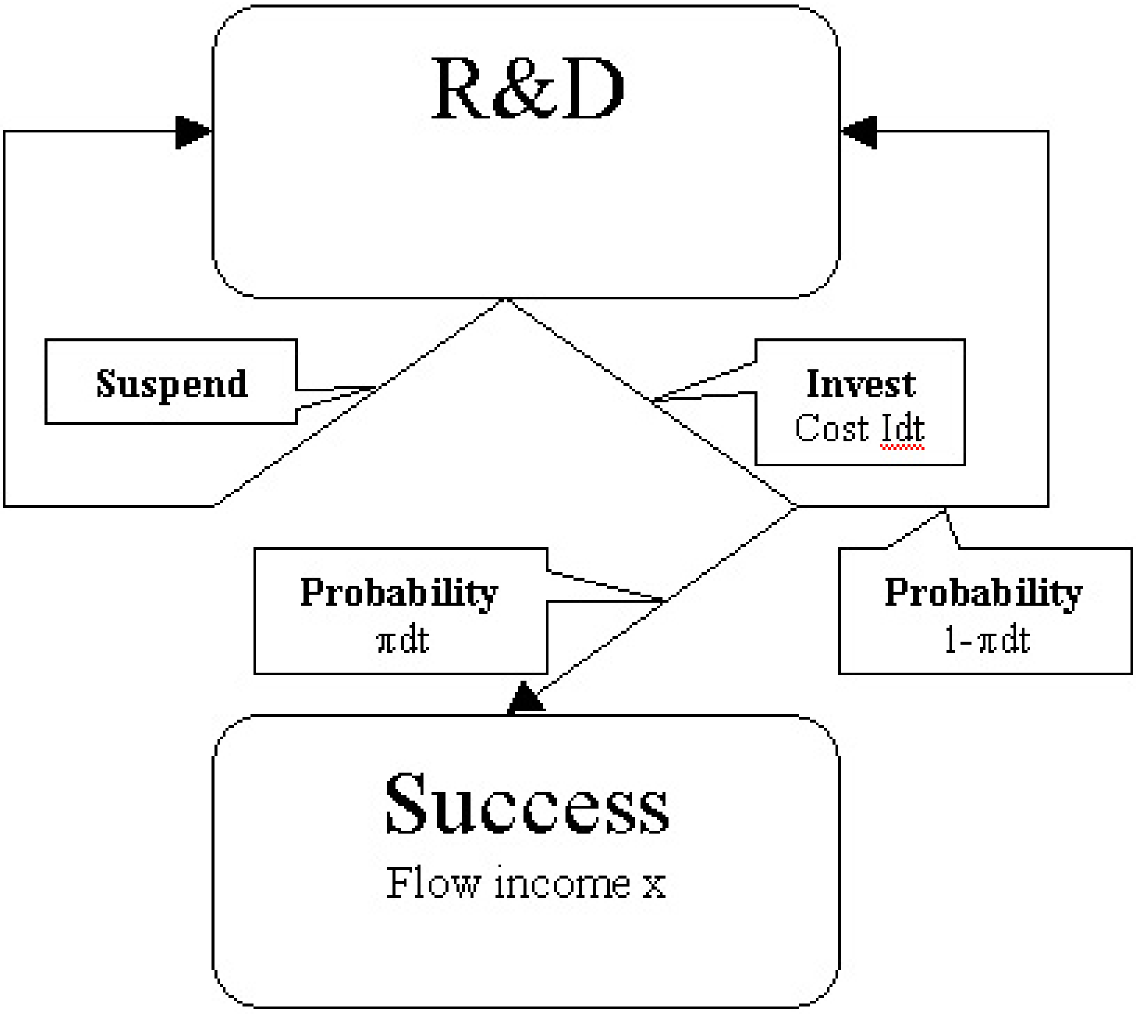

Figure 1. The flow chart of the model

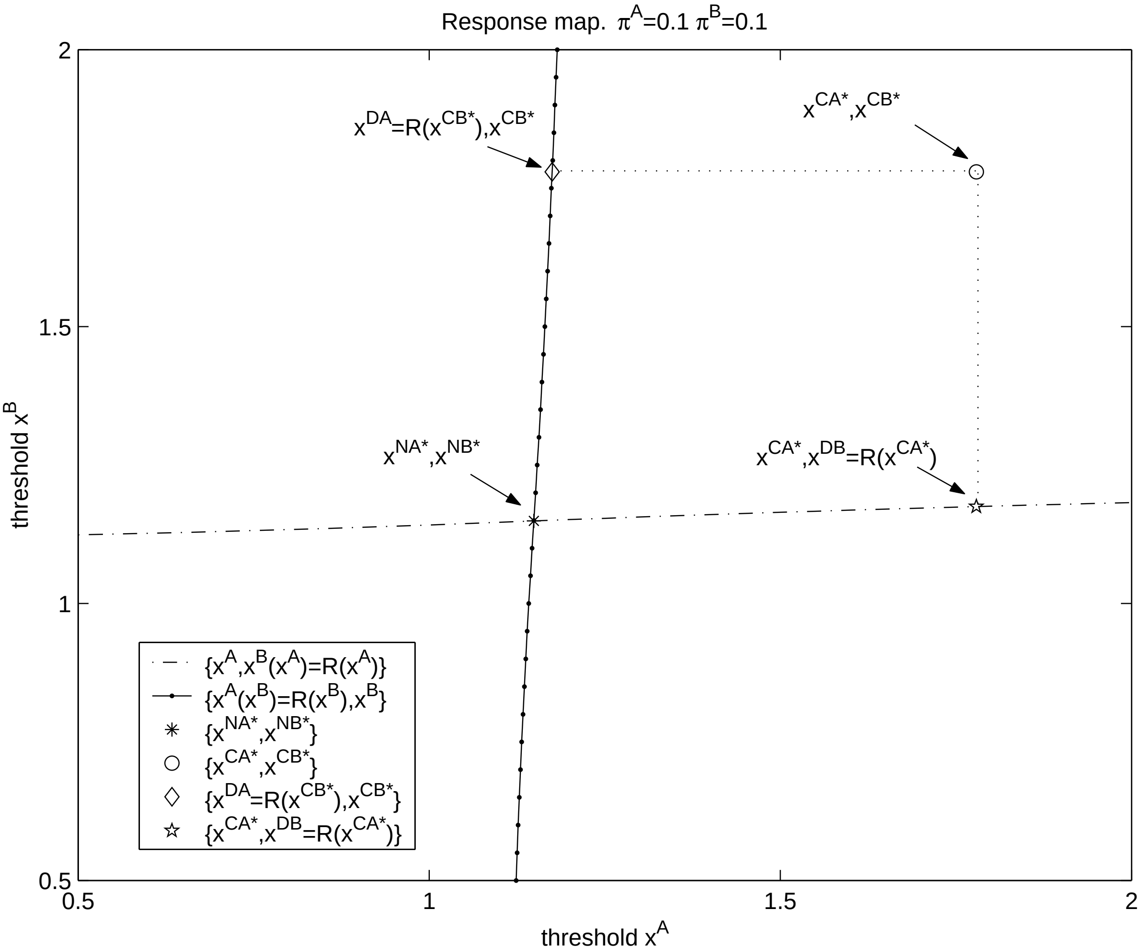

Figure 2. Response Maps (Reaction Curves). This figure illustrates the relations of the strategy thresholds x^A and x^B. The dash-dotted line refers to firm B's threshold as a response function of firm A 's threshold $x^B(x^A)$. The solid-dotted line refers to firm A 's threshold as a response function of firm B 's threshold $x^A(x^B)$. The non-cooperative equilibrium threshold pair is (x^{A*}, x^{B*}). The cooperative optimum threshold pair is (x^{CA*}, x^{CB*}). The pairs of one firm's cooperative threshold and the other firm's responsive threshold are $\{x^{DA} = R(x^{CB*}), x^{CB*}\}$ and $\{x^{CA*}, x^{DB} = R(x^{CA*})\}$, where $R(.)$ is a responsive function, which imply that the cooperative optimum is not an equilibrium as both firms have incentives to deviate.

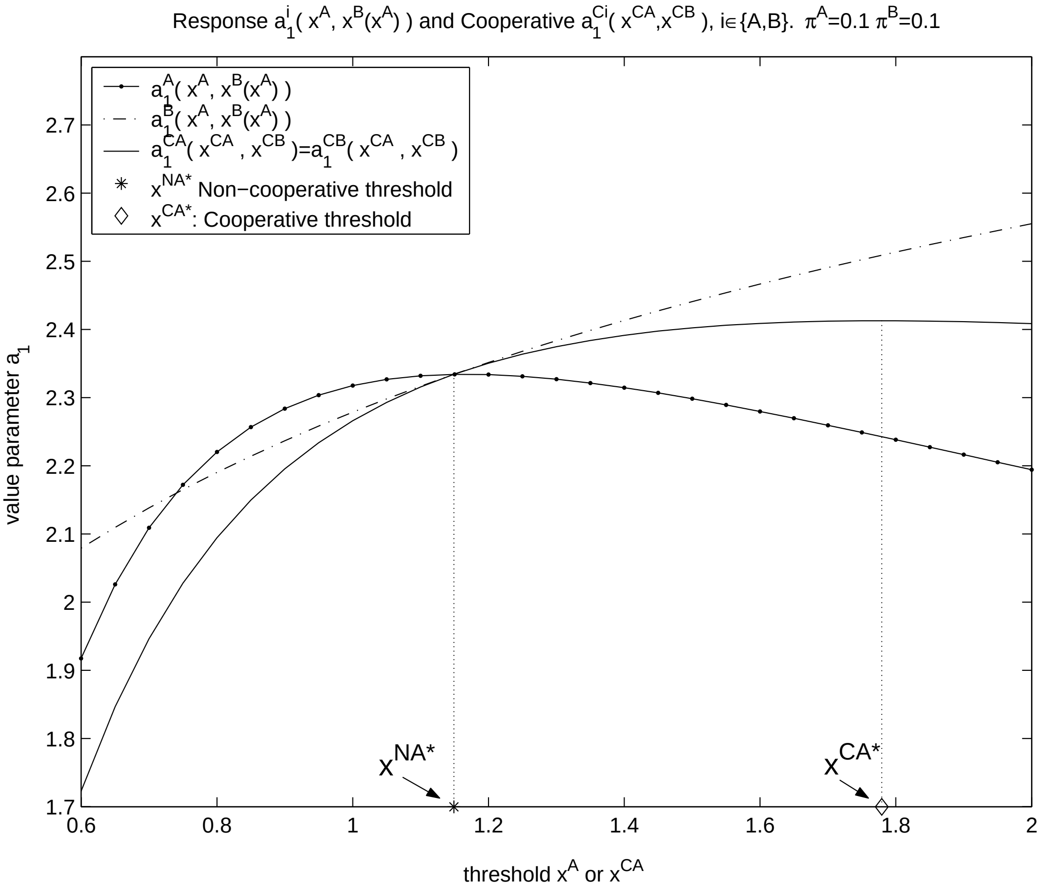

Figure 3. The effect of threshold on values with response and cooperative strategies. This figure illustrates the relations between the symmetric firms's value and their threshold, under the response strategy or cooperative strategy, respectively. In the response strategy, firm A choose its threshold first, firm B then select a threshold as a response of firm A's. The response strategy threshold pair is $(x^A, x^B(x^A) = R(x^A))$, where $R(.)$ is a response function. In the cooperative strategy, symmetric firms A and B choose the same threshold. The cooperative threshold pair is (x^{CA}, x^{CB}), where $x^{CB} = x^{CA}$. The value of a firm, V, is represented by the value of a_1 (refer to equation (44), (44) or (50)) as a function of threshold x^A or x^{CA}. The solid-dotted line refers to the value of firm A and the dash-dotted line refers to firm B, both as functions of threshold x^A. The response optimum for symmetric firms is exactly the noncooperative equilibrium. Thus firm A's optimal investment threshold for the response strategy is x^{NA*}, firm B's optimal response threshold $x^{NB*} = x^B(x^{NA*})$. The solid line refers to the value of V^{CA} as a function of the cooperative threshold x^{CA}. Note $V^{CA} = V^{CB}$, in the symmetric case. The optimal investment cooperative threshold is $x^{CA*} = x^{CB*}$. It is observed that $a_1^{Ci}(x^{CA*}, x^{CB*}) > a_1^i(x^{NA*}, x^{NB*})$, $i \in \{A, B\}$, which implies that the cooperative optimum is more efficient than the noncooperative equilibrium. It is also observed that $a_1^B(x^{CA*}, x^B(x^{CA*})) > a_1^{Ci}(x^{CA*}, x^{CB*}) > a_1^A(x^{CA*}, x^B(x^{CA*}))$, which indicates that the cooperative optimum is not an equilibrium.

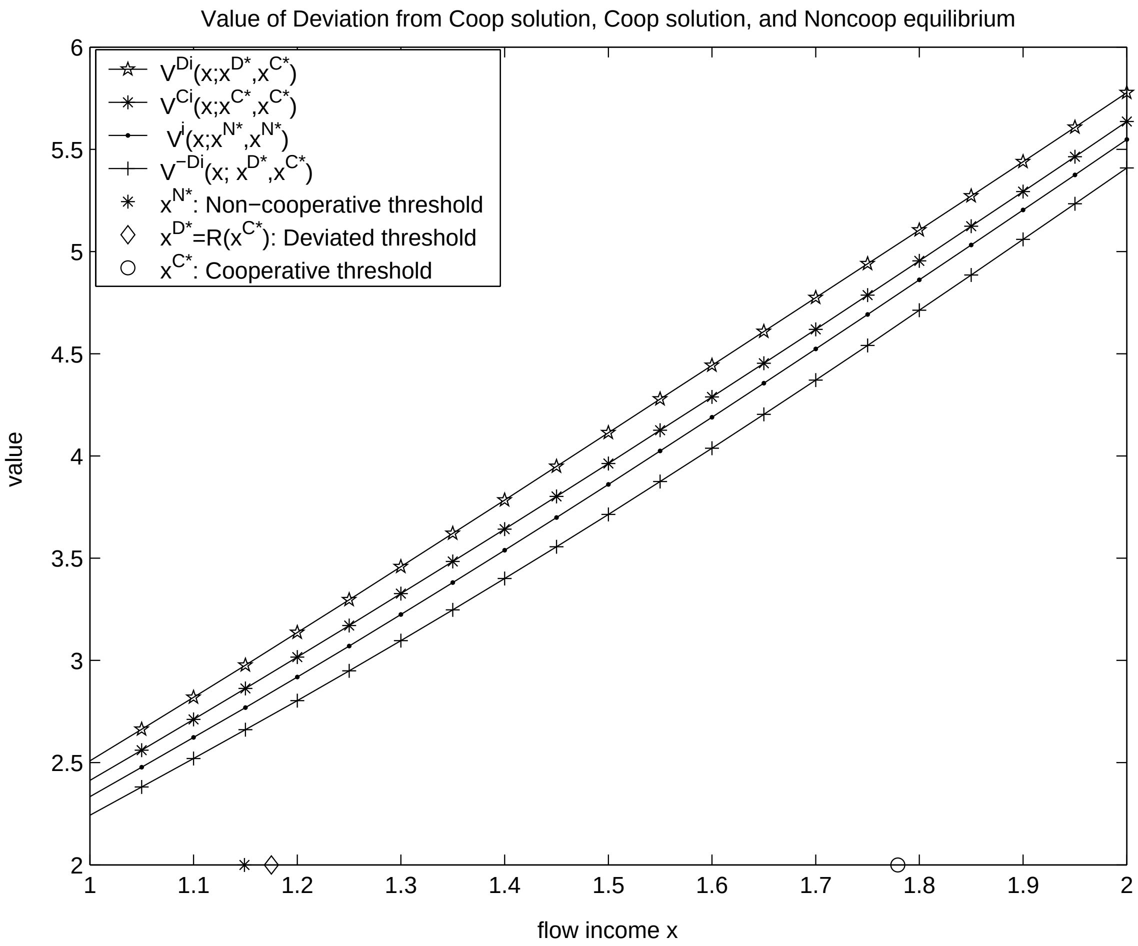

Figure 4. Noncooperative equilibrium, cooperative optimum, and deviation from cooperative optimum. This figure illustrates the value of symmetric firms A and B with various investment strategy. When one and only one firm has deviated, the solid-pentagram line refers to the value of the deviated firm $V^{Di}(x;x^{D*},x^{C*})$, and the solid-plus line refers to the value of the other firm which has not deviated $V^{-Di}(x;x^{D*},x^{C*})$. The solid-star line refers to the value of the firm with cooperative strategy $V^{Ci}(x;x^{C*},x^{C*})$. The solid-dotted line refers to the value of the firm with non-cooperative equilibrium strategy $V^{i}(x;x^{N*},x^{N*})$. It is observed that $V^{Di}(x;x^{D*},x^{C*}) > V^{Ci}(x;x^{C*},x^{C*}) > V^{i}(x;x^{N*},x^{N*}) > V^{-Di}(x;x^{D*},x^{C*})$. These observations imply that the cooperative optimum is more efficient than the noncooperative equilibrium, the cooperative optimum is not an equilibrium, and the threat to punish from being deviated to the noncooperative equilibrium is credible.

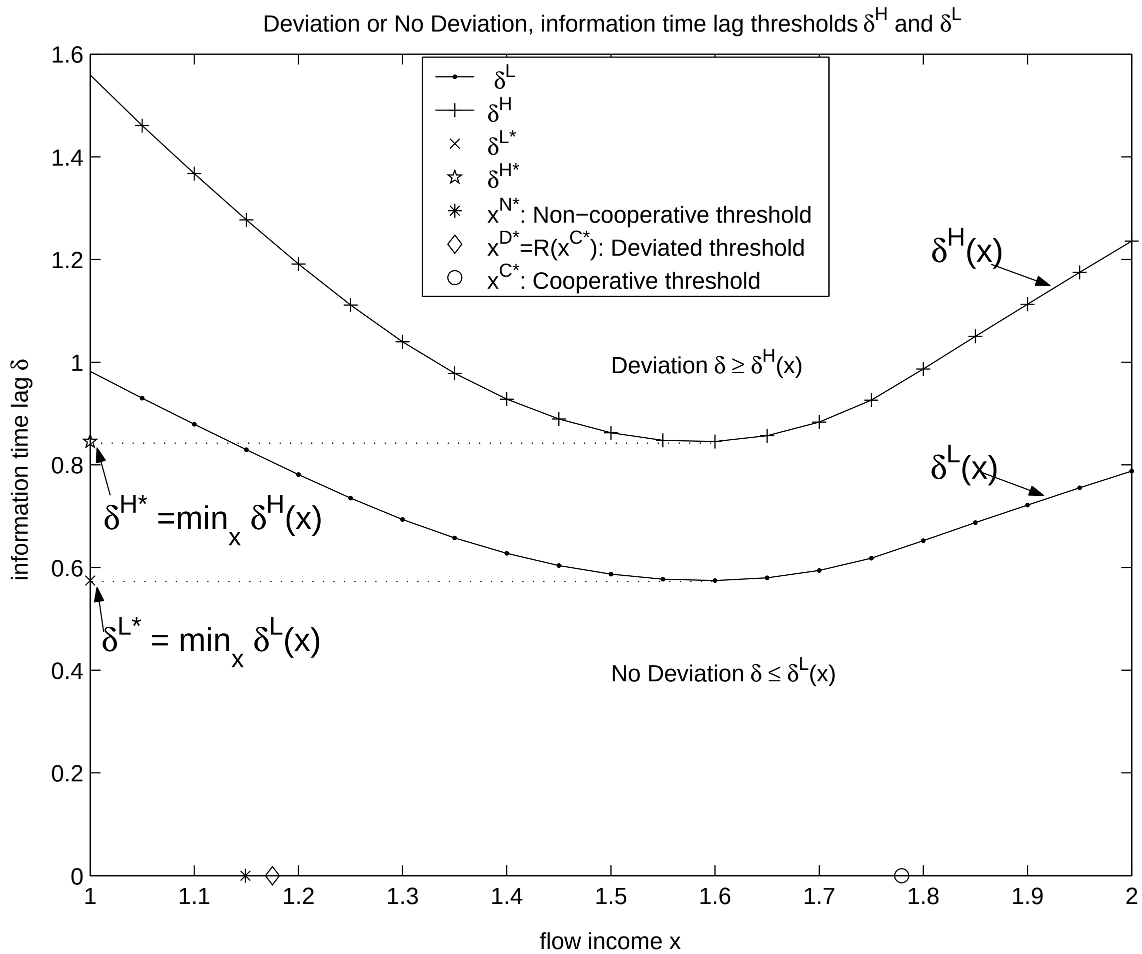

Figure 5. The effect of information time lag and market flow income on the deviation decisions. This figure illustrates the impacts of factors like information delay δ and flow income x on the decisions to deviate or not. The deviation decision only matters for x in deviation region, i.e., $[x^{D*}, x^{C*}]$. Deviation won't happen when $\delta \leq \delta^L(x)$, while deviation will happen when $\delta > \delta^H(x)$, where $\delta^L(x)$ and $\delta^L(x)$ are defined in (62) and (64). It is also observed that given δ, the deviation is more likely to happen when x is near the middle of x^{D*} and x^{C*}, and less likely to happen when x is near x^{D*} or x^{C*}.

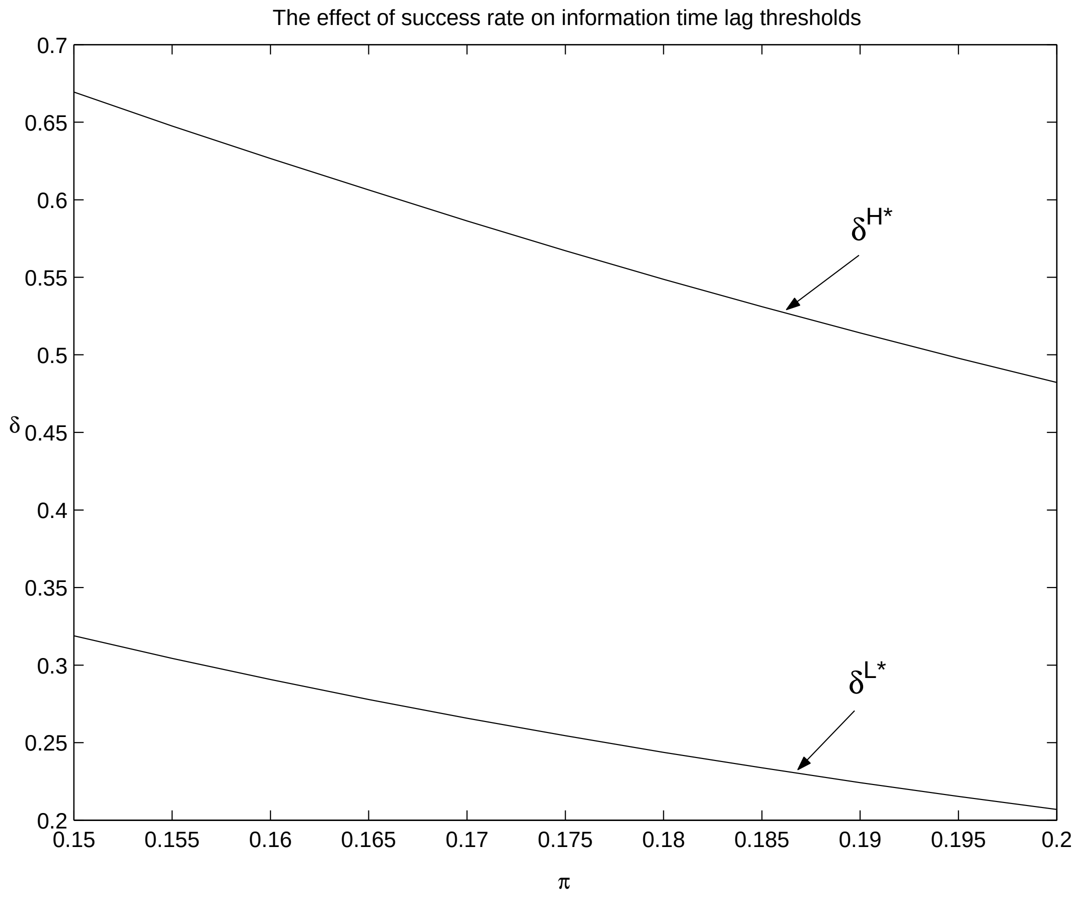

Figure 6. The effect of success rate on information time lag thresholds. This figure demonstrates that the effect of increasing likelihood of innovation success is a decrease in both information time lag thresholds.

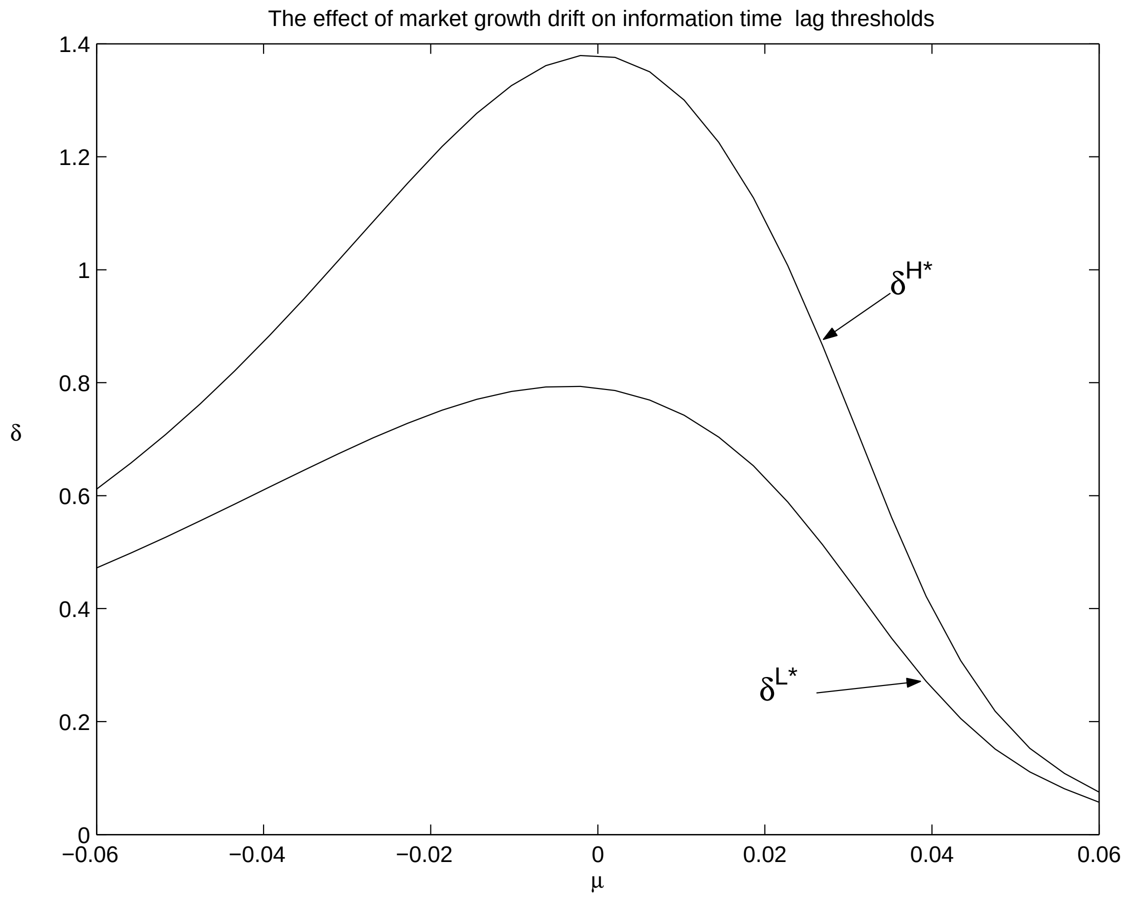

Figure 7. The effect of market growth drift on information time lag thresholds. This figure illustrate that the effect of increasing absolute value of drift of market is a decrease in the two information time lag thresholds.

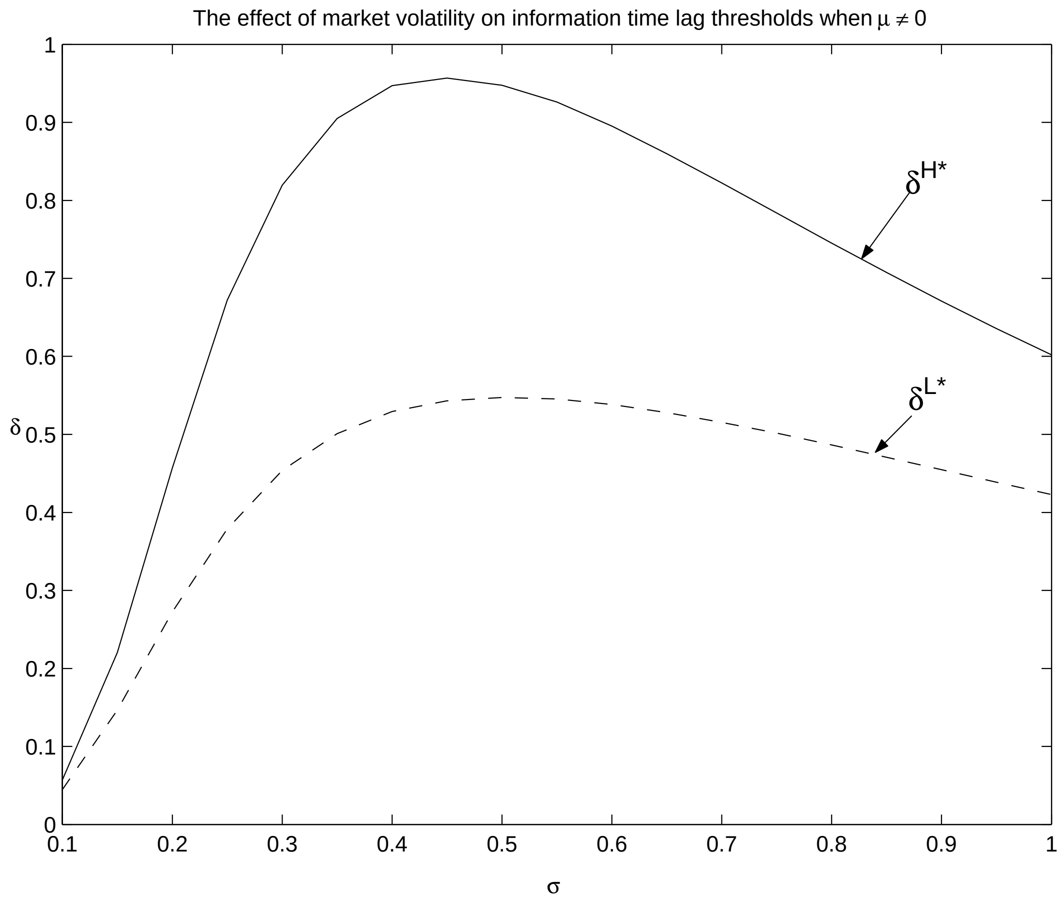

Figure 8. The effect of market volatility on information time lag thresholds. This figure illustrate that the effect of increasing volatility of market is first an increase and then a decrease in the two information time lag thresholds.