The Optimal Decision to Invest in a Duopoly Market for (Two) Positioned Companies when there are Hidden Competitors

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Abstract

The aim of this paper is to study the option to invest in a duopoly market, allowing for more competitors to enter the market.

In fact, we relax the common assumption which states that (only) *two firms* compete for *the two places* in the market. In the existing models, the problem consists of, basically, defining which one will be the leader, which will be the follower, and when. We can say that, in these settings, the investment opportunities are *semi-proprietary*, since the follower's position is, at least, guaranteed for both firms.

As we said, our approach relaxes this assumption, allowing for more than two competitors for the positions on the duopoly. This additional competition has, as we will see, a major impact on the decision to invest.

We also allow for both *ex-post* symmetry and *ex-post* asymmetry, and for asymmetrical investment costs for the leader and for the follower.

Keywords: Real Options, Duopoly, Hidden Competition.

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1 Introduction

A major difference between a financial and a real option, is that the later is, most of the times, shared with other competitors. If a real option is held in isolation by a single firm, no competitive interactions should be considered and so, the decision to invest is not influenced by any rival's actions (models to value this type of options can be found in Dixit and Pindyck [1]).

However, most of the real investment opportunities are shared with other firms. Then, when this is the case, the models should incorporate this competitive dimension, since it could have a dramatic impact on the decision to invest.

Generally, the analysis of the option to invest in a duopoly is carried out as in Smets [6], Grenadier [2], Weeds [8], Paxson and Pinto [4], and Tsekrekos [7].

In fact, the problem is, basically, the following: *two competing firms* have the option to invest and enter in a *duopoly market*; the first company to do so (the leader) may benefit, after investing, from a temporary or permanent competitive advantage over the other firm (the follower), by securing, for example, a higher market share (Paxson and Pinto [4], and Tsekrekos [7]). Then the main question becomes: when should they invest? While incorporating competition between firms, these models treat the investment opportunities as *semi-proprietary* real options, since the follower's position is, at least, guaranteed for both firms; in these models, after the leader's entrance, the other firm receives a proprietary (and perpetual) option to enter the market as a follower.

However, we think a more realistic model can be developed. In fact, instead of being (only) *two firms* competing for *two places* in the market, we allow for more competition, and so the problem no longer is to define the roles between two firms, but to define those roles in a context where they (both) can be taken by some others rivals. In our setting, these rivals are called "hidden competitors".

So, we have two types of competing firms: two companies, which explicitly compete for a place in the duopoly market (we call them "positioned companies"), and hidden competitors, not yet revealed, but with the capacity to enter the market. An hidden competitor might be a firm which can produce the same product, or some perfect substitute. In both cases, their actions have an impact (eventually major) on the option to invest of the positioned firms.

As we will see later, the positioned firms are assumed to be identical and well informed about each other; in other words, they have the same investment costs and the same expectations about the market, and they both know that. On the contrary, in our setting, the hidden competitors are not explicitly competing for a place in the market: they could be *completely hidden* (meaning that none of the positioned firms knows who they are), or, at least, the information about them is very scarce, particularly, their investment costs and their expectations about the market[1]. In both cases, a non-zero probability for their existence is assumed.

[1] This applies, also, to the hidden competitors among themselves.

The hidden competitors' actions are exogenous events (modeled as Poisson jumps). The market roles of the leader and the follower are obtained endogenously, but conditional on these exogenous events.

Firstly, we study the decision to invest with hidden competitors, assuming *ex-post* symmetry (meaning that the competitive advantage for the leader is temporary). After, we also relax this assumption allowing for *ex-post* asymmetry, i.e.: allowing for a leader's permanent competitive advantage. In this later section, we allow asymmetry as to the investment costs. Not for the same reasons as those presented by Pawlina and Kort [3], but for another important reason: since the leader will secure a permanent market share, it seams plausible (and more realistic) to assume that the leader will need more installed capacity than that needed by the follower, and so, the leader's positions will be more expensive than that of the follower's[2]. Note that the investment costs are different not because of some firm-specific reason, but because that for being the leader a firm must sink more money.

This paper is organized as follows: in the next section we derive the model for the decision to invest with hidden competition and *ex-post* symmetry; in section 3, the model is developed to allow permanent competitive advantage (*ex-post* asymmetry) and also asymmetric investment costs; in both of these sections a numerical example is presented; finally, in section 4, the conclusions are derived.

2 The Decision to Invest with Hidden Competitors and *ex-post* Symmetry

Consider two firms facing an opportunity to invest, sinking the investment cost K, in a duopoly market. These two firms are assumed to be risk-neutral, identical[3], and well informed about each-other. As we said, we call them "positioned companies", since they are in the front-line to enter the market.

Let x be the net cash-flow for the whole market, which evolves stochastically according to a geometric Brownian motion, as follows:

$$dx_t = \alpha x_t dt + \sigma x_t dZ \tag{1}$$

where $x_t > 0$, α and σ corresponds, respectively, to the drift parameter and to the instantaneous volatility, $\alpha \in [0, r)$ where r is the risk-free rate, and dZ is the increment of the Wiener process.

The *total net cash-flow* for an operating firm depends, not only on x, but also on the number of firms already in the market. Let $D(C)$ be a deterministic

[2] Paxson and Pinto [4], and Tsekrekos [7], assume the same investment costs for the leader and for the follower, even though a permanent competitive advantage is assumed.

[3] Meaning that both companies have the same expectations about the project cash-flows and investment costs.

parameter which multiplied by x give that total net cash-flow for the firm; here, $D(C)$ represents the market share[4].

In our duopoly market, $C \in \{1, 2\}$; $D(1)$ is the market share for the leader, if alone in the market, and $D(2)$ is the market share for both the leader and the follower, after the entrance of the later. If a company stands alone in the market, in a monopoly position, its total net cash-flow is $xD(1)$; after the entrance of the follower the installed firm must share the market with the new firm, so the total cash-flow decreases to $xD(2)$, which is equal for both firms[5]. In order to guarantee a first mover advantage we impose that $D(1) > D(2)$.

Let us now introduce the core aspect of our approach. Instead of being (only) two companies competing for two places, in a duopoly market, we assume that there is a non-zero probability for the existence of some hidden competitors[6], which may enter the market before the positioned firms, reducing or eliminating the (two) available places. If the entrance of any of the hidden firms occurs before of anyone of these two firms' entrance, then, after this event, they both compete for one more place in the market; but if the entrance of the hidden competitor happens after the entrance of the leader, then the second positioned firm loses the chance to invest.

So, in our approach, the follower doesn't have a proprietary right for being a follower, nor they both have (at least) the follower's position as guaranteed. Additionally, the entrance of an hidden competitor is assumed to be an exogenous event, corresponding to a Poisson jump with the intensity λ.

As with the related models, a leader's competitive advantage over the follower is also assumed, which means that, for some interval of the state variable, both companies will compete for this position, trying to preempt its rival. In this section the competitive advantage is temporary, meaning that after the follower's entrance the advantage disappears, and both companies become identical again (in terms of market share)[7].

This problem is solved backwards, starting with the follower. In our approach, we must distinguish between two situations, depending on the leader, since it can be either one of the two positioned firms or an hidden competitor.

2.1 The Value Function and the Trigger for the Follower

As we said, this problem is solved backwards, starting with the follower, and assuming that the leader is already in the market. But, on contrary to the other real duopoly models, in this one we have to distinguish between two situations, which leads to two different solutions: firstly, we analyze the situation of a leader

[4] Alternatively, $D(C)$ can capture, for example, the monopolistic price for the goods, during the period when the leader is alone in the market.

[5] Remember that we are assuming a temporary first-mover advantage. This will be relaxed later, in this paper.

[6] As we said, the two positioned companies do not know who those hidden competitors are, or, at least, the information about them is very scarce.

[7] As an extension, we will derive a model, also under this "hidden competition environment", but assuming *ex-port* asymmetry, i.e.: asymmetry between the leader and the follower, after the entrance of the later.

which is one of the two positioned companies, and next, the situation of a leader which is an hidden competitor.

2.1.1 Situation One: A Positioned Firm has Already Entered the Market as a Leader

Let both the diffusion of x and $D(C)$ be as previously defined. The follower's value function, $F(x)$, given the leader has already entered the market, must satisfy the following ODE, during the continuation period (when it is not yet optimal to invest):

$$\frac{1}{2}\sigma^2 x^2 F''(x) + \alpha x F'(x) - (r+\lambda)F(x) = 0 \tag{2}$$

subject to the boundary conditions:

$$F(0) = 0 \tag{3}$$

$$F(x_F) = \frac{x_F D(2)}{r-\alpha} - K \tag{4}$$

$$F'(x_F) = \frac{D(2)}{r-\alpha} \tag{5}$$

where $\lambda \in [0,1)$[8] is the *mean arrival rate* of an hidden competitor; during an infinitesimal period of time dt, the probability for the entrance is given by λdt; x_F is the trigger value for the follower.

The solution for the equation (2) takes the form:

$$F(x) = Ax^\beta \tag{6}$$

where $A = \frac{D(2)}{\beta x_F^{\beta-1}(r-\alpha)}$, and $\beta = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{(-\frac{1}{2} + \frac{\alpha}{\sigma^2})^2 + \frac{2(r+\lambda)}{\sigma^2}} > 1$.

Finding x_F, using the boundaries:

$$x_F = \frac{\beta}{\beta-1}\frac{(r-\alpha)}{D(2)}K \tag{7}$$

Note that the existence of a non-zero (and, of course, positive) λ increases the value of β. As λ increases, β also increases, and the factor $\frac{\beta}{\beta-1}$ decreases, decreasing the trigger value for the follower. So, we can say that the higher the probability of entrance of an hidden competitor, the lower the threshold for the follower, in other words, the lower its interest in waiting to invest. Also, the

[8] As we will demonstrate, this must be imposed in order to guarantee the uniqueness of the leader's trigger.

follower wants to preempt the hidden competitor (since the entrance of the later eliminates the option to invest), but this interest in preemption depends upon the probability of entrance of the hidden competitor.

As a consequence, the follower's optimal acting is to invest immediately when $x(t) \geq x_F$ (paying K, and receiving $\frac{xD(2)}{r-\alpha}$); until then, the firm has a (non-proprietary) American option to invest, which can suddenly disappear with an hidden competitor's entrance[9], whose value is $\frac{K}{\beta-1}\left(\frac{x}{x_F}\right)^{\beta}$. So, the closed-form solution for $F(x)$ is as follows:

$$F(x) = \begin{cases} \frac{K}{\beta-1}\left(\frac{x}{x_F}\right)^{\beta} \text{ for } x < x_F \\ \\ \frac{xD(2)}{r-\alpha} - K \text{ for } x \geq x_F \end{cases} \tag{8}$$

2.1.2 Situation Two: An Hidden Competitor has Already Entered the Market as a Leader

Let us now analyze the second possibility: the leader has entered the market, but it is not one of two positioned firms.

Since the first company to enter the market was none of the two positioned firms, these ones compete now for the last available place in the market: they both want to enter as the follower. Additionally, both positioned companies must consider the possible action of *another* hidden competitor.

If both positioned companies *only* consider this later possibility, they found optimal to invest when x hits x_F.

Recalling our assumption that these two firms are identical and well informed about each-other, they both have x_F as an optimal trigger for investing. However, precisely because of this, one firm will want to invest a little bit sooner than the optimal trigger, say $x_F - \varepsilon$, in order to preempt its rival; but anticipating this, the other firm will act even more sooner, say $x_F - 2\varepsilon$, to not be preempted.

This fear of preemption leads to the *full preemption*, which means that this game only stops when an additional ε turns the project worthless. So the *new* trigger is simply the value for x which implies a zero-NPV (the traditional Marshallian trigger):

$$x_F^M = \frac{(r-\alpha)}{D(2)}K < x_F \tag{9}$$

After observing the leader's position occupied by an hidden competitor, both positioned firms will decide to invest simultaneously for any $x(t) \geq x_F^M$, but only one of them will effectively enter the market (each firm has 1/2 probability to achieve this objective, and the same probability to lose the investment opportunity).

[9] This is an American option with *random* maturity (see, Pereira and Armada [5]).

Note that, in this case, competition completely erodes the value of the option to defer the project implementation.

2.2 The Value Function and the Trigger for the Leader

Let us now look to the leader's position. If the leader has already exercised the option to invest, entering the market, its value function, $L(x)$, must satisfy, prior to the entrance of the follower, the following nonhomogeneous ODE:

$$\frac{1}{2}\sigma^2 x^2 L''(x) + \alpha x L'(x) - rL(x) + xD(1) + \lambda x[D(2) - D(1)] = 0 \tag{10}$$

This equation is similar to those that appear in some related models, however it has an additional term, $\lambda x[D(2) - D(1)]$. This term captures the expected loss, of the leader's value function, due to the entrance of an hidden competitor in the market, as a follower, in a moment when x has not yet achieved the trigger x_F. If that happens, the leader is no more alone in the market, and its (temporary) monopoly advantage disappears sooner than expected.

We can rearrange equation (10) to appear as follows:

$$\frac{1}{2}\sigma^2 x^2 L''(x) + \alpha x L'(x) - rL(x) + (1-\lambda)xD(1) + \lambda x D(2) = 0 \tag{11}$$

Two boundaries must be placed[10]:

$$L(0) = 0 \tag{12}$$

$$L(x_F) = \frac{x_F D(2)}{r - \alpha} \tag{13}$$

The solution for this ODE is, after considering the first boundary (12), as follows:

$$Bx^\beta + \frac{(1-\lambda)xD(1) + \lambda x D(2)}{r - \alpha} \tag{14}$$

where β is as previously defined, and $B = \frac{(1-\lambda)x_F D(2) - (1-\lambda)x_F D(1)}{x_F^\beta (r-\alpha)}$.

Note that, since $x_F = \frac{\beta}{\beta-1}\frac{(r-\alpha)}{D(2)}K$, then $B = \frac{\beta}{\beta-1}(1-\lambda)\left(1 - \frac{D(1)}{D(2)}\right)K\frac{1}{x_F^\beta}$.

Accordingly, and considering that the leader pays K at the moment he invests, the solution for $L(x)$ can be expressed as follows:

[10] Differently as to follower's ODE, here we only need two boundaries, because we only have two unknowns, coming from the solution to the homogeneous part of this ODE. The typical third unknown, the trigger value for the leader, is obtained by indifference, rather than by optimization.

$$L(x) = \begin{cases} \frac{(1-\lambda)xD(1)+\lambda xD(2)}{r-\alpha} + (1-\lambda)\frac{\beta}{\beta-1}\left(1-\frac{D(1)}{D(2)}\right)K\left(\frac{x}{x_F}\right)^{\beta} - K \text{ for } x < x_F \\ \frac{xD(2)}{r-\alpha} - K \text{ for } x \geq x_F \end{cases} \tag{15}$$

The trigger for the leader, x_L ($< x_F$) must be such that, for that value of x, it will be indifferent for both positioned firms to be the leader or the follower. This happens when the value function of the leader mets the value function of the follower:

$$L(x_L) = F(x_L) \tag{16}$$

This value x_L exists and is unique for $\lambda \in [0, 1)$ (see Appendix A for the proof). Additional properties are:

$$L(x) < F(x), \text{ for } x < x_L \tag{17}$$

since for a x lower that x_L, both firms prefer to be the follower, and

$$L(x) \geq F(x), \text{ for } x > x_L \tag{18}$$

because the leader's value function is above the follower's until x hits x_F [$L(x) > F(x)$, for $x_L < x < x_F$], and then both functions permanently met[11] [$L(x) = F(x)$, for $x \geq x_F$][12].

2.3 The Equilibria

If no hidden competitor has already entered the market, two types of strategic equilibrium can be considered, depending upon the initial value for x, i.e.: x_0. If $x_0 \in [x_L, x_F)$ then both firms have the incentive to become the leader, so they invest sequentially, one preempting the other[13]. This conducts to a *preemption equilibrium.* Note that if $x_0 \in [0, x_L)$, none of the positioned firms enters the market, because, for that level of x, they (both) prefer to be the follower. This means that they will wait until x hits x_L, leading, also, to sequential entrances. For $x_0 \geq x_F$ both firms will be interested to invest immediately, leading to the so-called *simultaneous equilibrium.*

As we will, these two types of equilibria can be strongly influenced by the presence of hidden competition.

[11] Note that we are assuming that the firms are *ex-post* symmetric. This assumption will be relaxed in the next section.

[12] Another way of interpreting these properties is to look at the numerical example on section 2.4, in particular to Figure 4.

[13] As we will see, in this case both firms will act in order to become the leader, however only one will (randomly) achieve this objective.

2.3.1 The Preemption Equilibrium, Assuming no One in the Market to begin with

As we said, the *preemption equilibrium* occurs when the initial level of the state variable is lower than that of the follower's trigger, i.e.: when $x_0 < x_F$. If this is the case, the optimal action for both positioned firms is to preempt each other. Consequently, they both decide to invest immediately, if $x_0 \in [x_L, x_F)$, or as soon as x hits x_L, if $x_0 \in (0, x_L)$.

Like some related models, also it is assumed here that only one of the firms can win the leader's position. So, one positioned firm turns to be the leader, and the other has the chance to be the follower. We assume that this happens randomly, and they both have the same chances for that.

After losing the leader's position, the other positioned firm, in order to act optimally, waits until the state variable hits x_F. Remember that, on the contrary to the other models, in this one, the follower's position is not proprietary, so we assume that there is non-zero probability for the entrance of an hidden competitor, which is incorporated on x_F.

2.3.2 The Simultaneous Equilibrium, Assuming no One in the Market to begin with

If the initial level of the state variable is higher than, or equal to, the follower's trigger, i.e.: $x_0 \geq x_F$, then, the optimal strategic action for both positioned companies is to invest immediately. This is called the *simultaneous equilibrium*. Since we are in presence of *ex-post* symmetry, then, under this equilibrium, the roles for the two positioned firms are irrelevant, because they both have the same value[14].

2.3.3 The Impact of the Hidden Competitors on the Equilibria: An Integrated Analysis

As we will see, if an hidden competitor moves, that movement has a major impact on the previous presented equilibria.

Suppose that no one is in the market, and $x \in (0, x_L)$. The preemption equilibrium will no longer holds, if an hidden competitor enters the market. With this movement, the leader's position will be occupied, which implies that the two positioned firms compete now for the follower's position. As we said previously, this competition completely erodes the value of the option to defer, reducing the follower's trigger (x_F) to the level of the classic Marshallian trigger (x_F^M). Since both compete for the follower's position, only one will (randomly) achieve this objective. Both firms have 1/2 probability to enter the market as a follower, and the same probability to lose the chance to invest. Note that there is an additional possibility, a second hidden competitor can enter the market, occupying the last available place, before x hits x_F^M. This eliminates the option

[14] As we will see later, this will not be case under *ex-post* asymmetry and asymmetric investment costs.

to invest for both firms, and there is nothing they can do prevent this. None of of the positioned firms will be interested to invest for a level of x lower than x_F^M , since this is the value for x that leads to a zero-NPV.

If $x \in [x_L, x_F)$, and the leader is one of the two positioned firms, then the other positioned firm will optimally wait until x hits x_F. Meanwhile, this firm may face an undesired entrance of an hidden competitor, which has a catastrophic impact on its option to invest. As we said, the trigger x_F must incorporate the probability for that occurrence. Figure 1, below, shows the impact of the probability of entrance of an hidden competitor on the follower's trigger. As expected from equation (7), the grater the λ, the lower the trigger x_F. Is interesting to see that x_F rapidly decreases for lower levels of λ. This means that, even if the probability of an exogenous entrance is lower, the firm will want to invest much sooner, so this "risk", has a major impact on the option to wait.

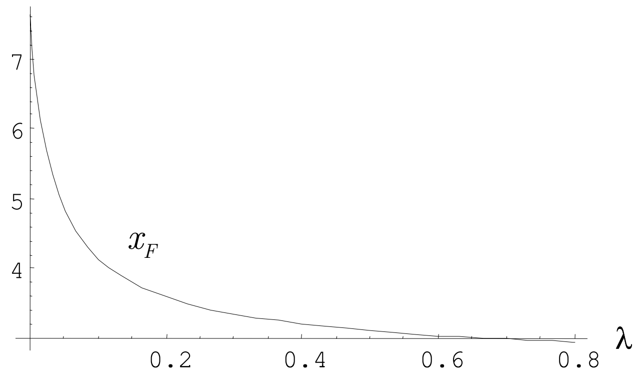

Figure 1: The impact of the probability of entrance of an hidden competitor on the followe's trigger. $x_F^M = 2.4$. The parameters are: $D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda$ from 0 to 0.8.

Another important finding of this paper is related to the impact of λ on the trigger of the leader. As opposed to its impact on that of the follower, which permanently decreases as λ increases, the leader's trigger (also) decreases, but only until a particular level of λ, and *beyond that level*[15], the trigger of begins to increase, converging to the follower's trigger (see Figures 2 and 3).

What can explain this? Assume, for a moment, that there is no hidden competition. The leader's disposition to invest earlier (*refusing* the benefits from deferring) is justified with the benefits of the temporary competitive advantage. So the leader will be interested to invest earlier, in order to receive monopolistic cash-flows for a period of time, until the entrance of the follower. This is reflected on the leader's value function. Let us look to the solution of $L(x)$ [equation (15)] for $x < x_F$, with $\lambda = 0$. The first part of the equation gives the present value

[15] Which depends on the parameters.

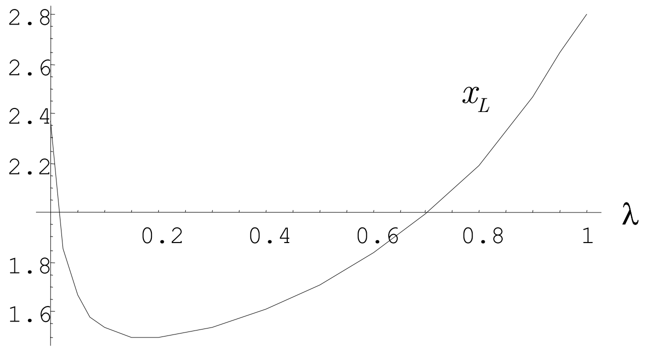

Figure 2: The impact of the probability for the entrance of an hidden competitor on the leader's trigger. Parameters are: $D(1) = 1; D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda$ from 0 to 0.999.

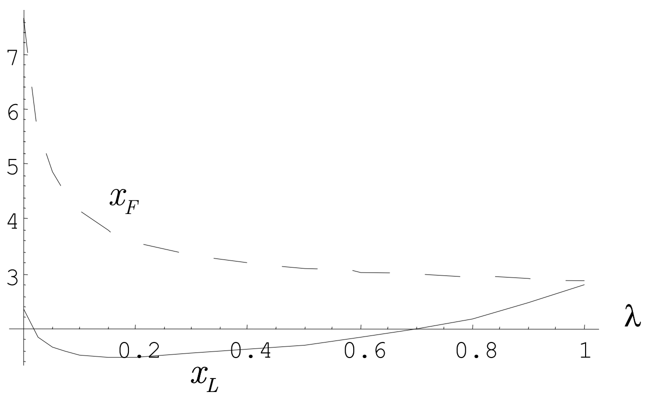

Figure 3: A simultaneous analysis of the impact of the probability of entrance of an hidden competitor on the triggers. λ from 0 to 0.999. The remaining parameters are: $D(1) = 1; D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25$.

of the monopolistic cash-flows, if they last forever; the second part is a negative increment (note that $\frac{D(1)}{D(2)} > 1$) which captures the loss that occurs with the entrance of the follower, which *only* happens when x hits x_F.

Let us return to our "hidden competition environment". The expected net cash-flow for the next period of time is $(1-\lambda)xD(1) + \lambda xD(2)$, and its present value is given by the first part of the solution of $L(x)$, for $x < x_F$ [again, see equation (15)]. Note that the higher the λ, the lower this present value will be, and so the lower the monopolistic benefits. The competitive advantage is expected to last (much) less than they would last, if no hidden competition is assumed. In the limit, the net cash-flow for the next period of time will be $xD(2)$, in a moment when x_F has not yet been reached.

In other words, the expected temporary competitive advantage for the leader will be lower as λ increases, and so less interested the leader will be to invest earlier. As $\lambda \rightarrow 1$[16], the leader and the follower will almost have the same trigger, meaning they will tend to invest almost at the same time. As we will see later, if the competitive advantage is permanent, the leader's trigger have a similar behavior as the one presented here, but it stays well bellow the follower's trigger.

2.4 Numerical Example

Let us present an hypothetical example, in order to implement the model. Let the inputs be: $D(1) = 1; D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

Assuming that no hidden competitor is in the market, the triggers for the leader and for the follower are:

$$\begin{aligned} x_L &= 1.494 \\ x_F &= 3.592 \end{aligned}$$

This means that, if the state variable x is below 1.494, neither of the firms invests because they both prefer to be a follower (note $F(V)$ dominates $L(x)$ for $x \in [0, x_L)$), and so they will wait until x hits 1.494. If x is higher than 1.494 and bellow 3.592, then a preemption equilibrium occurs: one of the positioned firm enters as a leader, and the other waits, investing only when x_F is achieved. If x is above 3.592 both firms invest simultaneously.

In Figure 4 we plot the leader's and the follower's value functions, where these points can be visually identified.

If the leader's position is occupied by an hidden competitor, then the two positioned companies compete for the last available place in the market. The fear of preemption leads, as we said previously, to the *full preemption.* In this case, the *new* trigger turns to be x_F^M, the value of x which gives a zero-NPV project (see Figure 5).

[16] Excluding 1 (see footnote 8, and Appendix A.1).

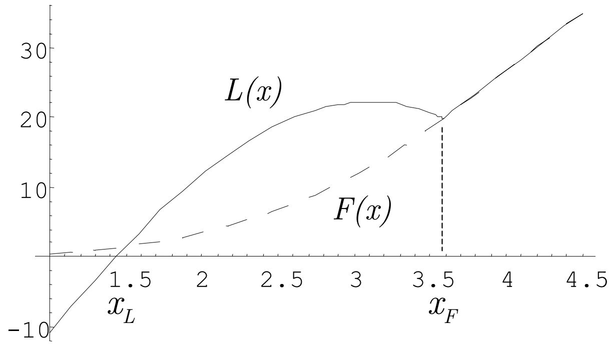

Figure 4: The value functions and the triggers for the leader and for the follower. The parameters are: $D(1) = 1; D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

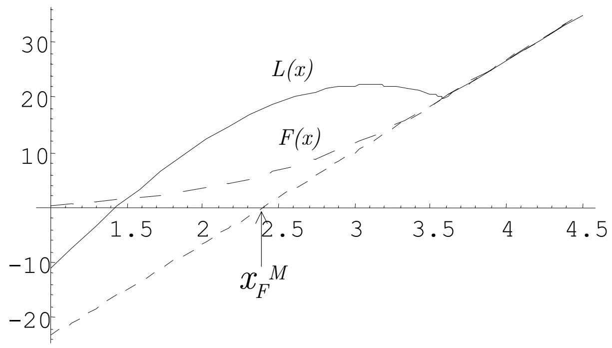

Figure 5: The trigger for both positioned companies, if an hidden competitor is the leader.

If no hidden competition is assumed (i.e.: if $\lambda = 0$) the triggers are much higher, mainly for the follower:

$$\begin{aligned} x_L &= 2.367 \\ x_F &= 7.644 \end{aligned}$$

In Figure 6 we plot $L(x)$ and $F(x)$ with both $\lambda = 0$ and $\lambda = 0.2$ (x_L and x_F when $\lambda = 0.2$; x'_L and x'_F when $\lambda = 0$). We can easily verify significant impact of the "hidden competition" on the triggers.

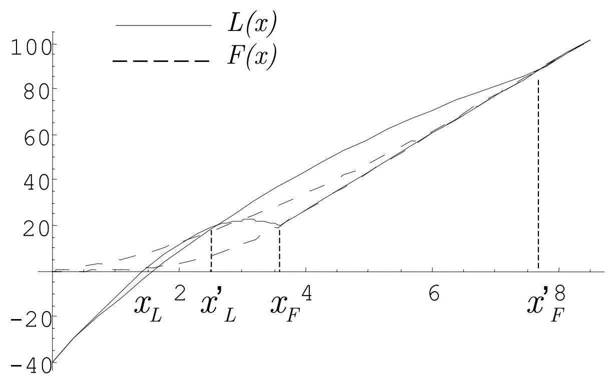

Figure 6: The value functions and the triggers for the leader and for the follower. $\lambda = 0$ and 0.2. Remaining parameters are: $D(1) = 1; D(2) = 0.5; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25$.

3 The Decision to Invest with Hidden Competitors and ex-post Asymmetry

The assumption that the first mover advantage is temporary, will be relaxed in this section, while maintaining the "hidden competition environment". We do so in order to incorporate the possibility for a permanent market advantage for the leader. This advantage may come from the relation between the leader and market, which may result in an higher market share for the leader, even after the entrance of the follower. This type of first mover advantage turns the companies asymmetric *ex-post*.

Let x, $D(C)$, and $xD(C)$ be as previously defined. In order to incorporate the *ex-post* asymmetry, let $xD(1)$ be the net cash-flow for the leader, if it is the only one in the market, $xD^L(2)$ the net cash-flow for the leader after the entrance of the follower, and $xD^F(2)$ the net cash-flow for the follower. The first mover advantage, within the context of this new framework, will always be permanent (guaranteed) because we will always have $D(1) > D^L(2) > D^F(2)$.

Additionally, in this context, we assume, as opposed to what is assumed in the literature so far[17], that the leader has higher investment costs than those of the follower. This is so, for two possible reasons: firstly, when the leader is the only one in the market it will have an installed capacity to respond to the total market demand, and secondly, even after the entrance of the follower, the leader will sell more than that of the follower and, consequently, an higher investment cost must be sunk from the leader. In other words, we assume that the leader invests k (%) more than the follower[18].

3.1 The Value Function and the Trigger for the Follower

Starting with the follower, we assume that the leader is already in the market. As in section 2.1, here we distinguish between two possibilities: firstly, the leader is one of the two positioned firms, and secondly, the leader's position has been occupied by an hidden competitor. Let us star by the former.

If the leader is one of the two positioned companies then, the other company, has the (non-proprietary) option to be the follower. The value function $F(x)$ must satisfy the ODE (2) subject to the same type of boundaries, but (4) and (5) should read now:

$$F(x_F) = \frac{x_F D^F(2)}{r-\alpha} - K \tag{19}$$

$$F'(x_F) = \frac{D^F(2)}{r-\alpha} \tag{20}$$

Using the same procedures, we find that the follower's trigger, and its value function are as follows:

$$x_F = \frac{\beta}{\beta-1}\frac{(r-\alpha)}{D^F(2)}K \tag{21}$$

$$F(x) = \begin{cases} \frac{K}{\beta-1}\left(\frac{x}{x_F}\right)^{\beta} \text{ for } x < x_F \\ \\ \frac{xD^F(2)}{r-\alpha} - K \text{ for } x \geq x_F \end{cases} \tag{22}$$

If the leader's position has been occupied by an hidden competitor, then, as in 2.1.2, the two positioned companies compete for the last available place. As we saw then, this competition, and the fear of preemption, completely erodes the option to defer the project implementation. This means that both firms will decide to invest immediately when $x(t) \geq x_F^M = \frac{(r-\alpha)}{D^F(2)}K$ (the so-called Marshallian trigger, which is the value for x that leads to a zero-NPV).

[17] See, as an example, the footnote 2.

[18] Our model is flexible enough in order to incorporate the possibility of $k(\%) = 0$, meaning that both positioned firms might have the same investment costs.

3.2 The Value Function and the Trigger for the Leader

The value function of the leader, $L(x)$, must satisfy the following ODE:

$$\frac{1}{2}\sigma^2 x^2 L''(x) + \alpha x L'(x) - rL(x) + (1-\lambda)xD(1) + \lambda x D^L(2) = 0 \tag{23}$$

which is similar to the equation (11), except for the last term of the left-hand side of the equation. Note that, with the entrance of the follower in the market, the leader's net cash-flow drops from $xD(1)$ to $xD^L(2)$[19].

The boundary (12) remains the same in this case, but the other boundary should be:

$$L(x_F) = \frac{x_F D^L(2)}{r-\alpha} \tag{24}$$

Solving this ODE, considering the boundaries and incorporating the investment costs, we arrive at the following solution:

$$L(x) = \begin{cases} \frac{(1-\lambda)xD(1)+\lambda x D^L(2)}{r-\alpha} + (1-\lambda)\frac{\beta}{\beta-1}\left(\frac{D^L(2)-D(1)}{D^F(2)}\right) K \left(\frac{x}{x_F}\right)^{\beta} - K_L & \text{for } x < x_F \\ \frac{xD^L(2)}{r-\alpha} - K_L & \text{for } x \geq x_F \end{cases} \tag{25}$$

where $K_L = (1+k)K$. The factor $(1+k)$ reflects the additional price for being the leader, instead of the follower.

As previously, the leader's trigger is obtained by indifference; the trigger exists and it is the unique value bellow x_F where the leader's and the follower's value function both met:

$$L(x_L) = F(x_L) \tag{26}$$

with the following properties:

$$L(x) < F(x), \text{ for } x < x_L \tag{27}$$

$$L(x) > F(x), \text{ for } x > x_L \tag{28}$$

Meaning that if the state variable is bellow x_L both firms prefer to be the follower (27), and that the leader's value function is above the follower's for all $x > x_L$, ensuring the permanent competitive advantage (ensuring the *ex-post* asymmetry) for the leader (28).

It can be easily proved that, in order not to violate the later property, a restriction must be imposed to kK (which is the additional amount of money that a firm must sink in order to enter the market as a leader). This restriction can be presented as follows:

[19] And not to $xD(2)$, as in equation (11).

$$kK < \frac{x_F[D^L(2) - D^F(2)]}{r - \alpha} \tag{29}$$

so we say that $k \in [0, k^*)$, where:

$$k^* = \frac{x_F[D^L(2) - D^F(2)]}{(r - \alpha)K} \tag{30}$$

Under these properties and restrictions, the x_L exists and is unique, as proved in Appendix A.2.

3.3 The Equilibria

Here the approach in order to derive the equilibria is similar to the one presented in the section 2. If no hidden competitor has entered as a leader, the equilibrium can be either a preemption equilibrium or a simultaneous one, depending upon the initial level of the state variable. However, both k and the permanent competitive advantage for the leader, will have an impact on its trigger, and so on the equilibrium.

As to the *preemption equilibrium*, if $x_0 < x_L$, none of the positioned firms will be interested in exercise immediately the option to invest, since they both prefer to be a follower. They will not act until x hits x_L. As we can see from Figures 7 and 8, x_L increases as k increases, and decreases as the permanent competitive advantage increases, *caeteris paribus*. This means that, the existence of some additional costs for being the leader, will induce the firms to invest later; and, on the contrary, the existence of a permanent competitive advantage for the leader, will induce the firms to invest sooner.

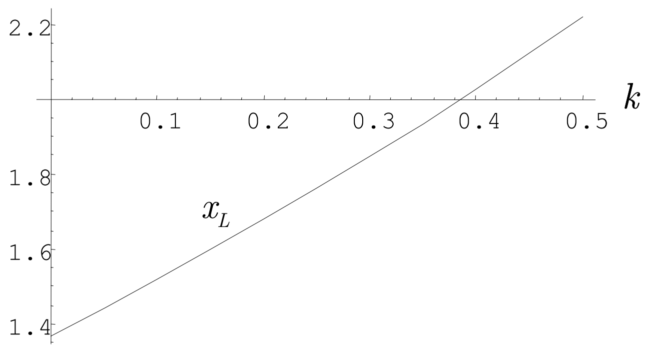

Figure 7: The impact of the parameter k on the leader's trigger. The values for k from 0 to 0.5. The remaining parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

If $x_0 \in [x_L, x_F)$ then, both positioned firms want to enter the market as the leader, but, as previously, only one of them will (randomly) achieve this

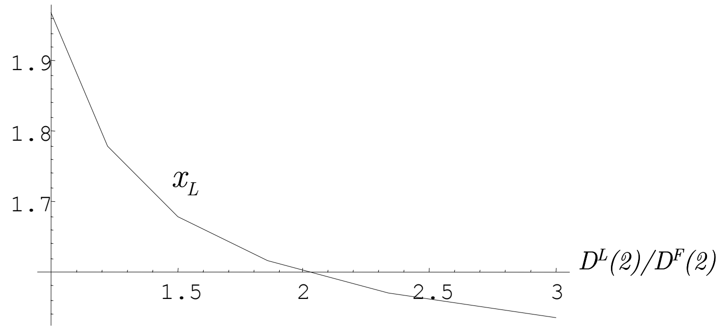

Figure 8: The impact of the competitive advantage on the leader's trigger. The advantage comes in the form of the ratio: $\frac{D^L(2)}{DF(2)}$; the higher this ratio, the higher the competitive advantage. $\frac{D^L(2)}{D^F(2)}$ from 1 to 3. The remaining parameters are: $K = 40; k = 0.2; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

objective, and both firms have the same probability for that. After the entrance of the leader, the other positioned firm receives a *non-proprietary* option to be the follower. The later will act optimally, waiting until x hits x_F. As in the previous section, the follower's trigger, x_F, incorporates the probability for the entrance of an hidden competitor, occupying the follower's position. The higher this probability the lower x_F will be.

For a $x_0 \geq x_F$, the *simultaneous equilibrium* occurs. In this case, both positioned firms have a strong preemptive incentive, since $L(x) > F(x)$, for $x > x_F > x_L$, so they will both invest immediately. Here, on the contrary to the *ex-post* symmetry situation, the roles of the positioned firms will be very important. So, randomly, one of them gets the *leader's* position (securing a higher market share, but paying a higher investment cost for that purpose), and the other becomes the *follower* (with a lower market share, and a lower investment cost).

Let us now analyze the impact of the hidden competition on the equilibria. To avoid repeating the same arguments, we say that the analyses presented in 2.3.3 is also valid here. One difference, being the impact of λ on the leader's trigger. As in 2.3.3, increments in λ reduces x_L until a particular level of λ, but after that level, x_L begins to increase as λ increases. However, because of the permanent competitive advantage, x_L never hits x_F, even for higher values of λ (see Figure 9).

The entrance of an hidden competitor, in a moment in time when x_F has not yet been reached, does not have (here) a so-deep impact on the leader's value function as in the case of a temporary advantage; and so, in order to guarantee this permanent competitive advantage, the leader will still be interested to invest earlier, even for a higher value of λ.

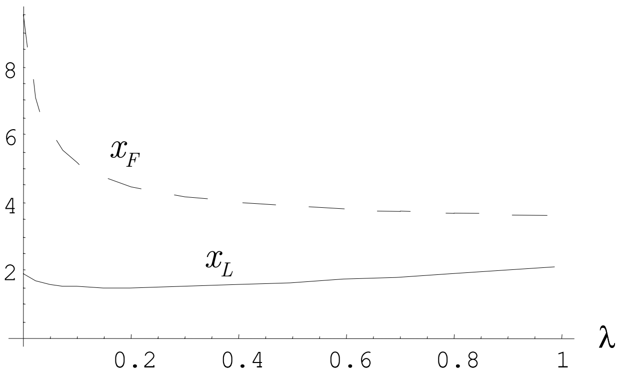

Figure 9: A simultaneous analysis of the impact of the probability for the entrance of an hidden competitor on the triggers. $\lambda = 0$ to 1. The remaining parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 40; k = 0.2; r = 0.05; \alpha = 0.02; \sigma = 0.25$.

3.4 Numerical Example

Let us implement the model using a numerical example. Let the inputs be: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 40; k = 0.2; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

For this parameters, the leader's and follower's value functions, and their triggers, are presented in Figure 10.

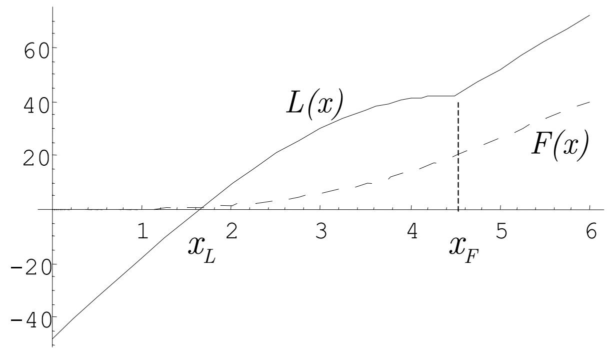

Figure 10: The value functions and the triggers for the leader and for the follower. The parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 40; k = 0.2; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

As opposed to the situation where the competitive advantage for the leader is temporary (only during the *monopolistic period*), here the advantage is permanent, which can be verified looking to the leader's value function. In fact, $L(x)$ is always above $F(x)$ for all $x \geq x_L$. The triggers for the leader and for the follower are, respectively, 1.679 and 4.489. Note that, with no hidden competition, those triggers would be 2.195 and 9.555, respectively.

The impact of k on the leader's value function and on its trigger is reported on Figure 11.

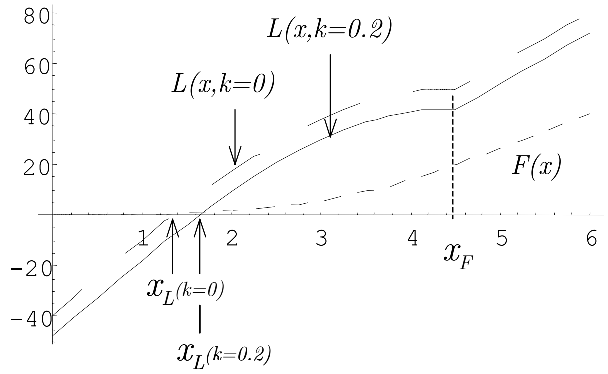

Figure 11: The impact of k on the value function and on the trigger of the leader. $k = 0$ and 0.2. Remaining parameters are: $D(1) = 1; D^L(2) = 0.6; D^F(2) = 0.4; K = 40; r = 0.05; \alpha = 0.02; \sigma = 0.25; \lambda = 0.2$.

A higher cost for the leader, results on a lower value function, and so to a higher level for x for which is optimal to enter the market.

4 Conclusions

The aim of this paper was to develop an approach to value real options in a duopoly setting, introducing more competition in the process.

In fact, the common assumption, which states that (only) *two firms* compete for *the two positions* in the market, was relaxed. We called this common view as a *semi-proprietary* option, since the follower's position is, at least, guaranteed for both firms.

Our approach relaxes this assumption, allowing for more than two competitors for the positions of the duopoly market. The rivals were divided in two categories, the positioned firms, and the hidden competitors.

The additional competition has, as we saw, a major impact on the decision to invest (both on the firms' value functions and on their triggers), and, in some cases, completely erodes the value of the option to defer.

The developed approach allows for both *ex-post* symmetry and *ex-post* asymmetry, and also for asymmetrical investment costs for the leader and for the follower.

A Appendix

A.1 The Proof of the Uniqueness of x_L Under *ex-post* Symmetry

In this Appendix, we will prove the uniqueness of $x_L \in (0, x_F)$, assuming *ex-post* symmetry.

Let $H(x) = L(x) - F(x)$:

$$\begin{aligned} H(x) &= \frac{(1-\lambda)xD(1) + \lambda xD(2)}{r-\alpha} + \\ &+ (1-\lambda)\frac{\beta}{\beta-1}\left(1 - \frac{D(1)}{D(2)}\right)K\left(\frac{x}{x_F}\right)^{\beta} - K - \frac{K}{\beta-1}\left(\frac{x}{x_F}\right)^{\beta}. \end{aligned}$$

Calculating $H(x)$ at $x = 0$, and at $x = x_F$ we obtain:

$$H(0) = -K$$

$$H(x_F) = 0$$

Calculating now the derivative of $H(x)$ at x_F:

$$\frac{\partial H(x)}{\partial x}\Big|_{x=x_F} = \frac{(\lambda-1)(\beta-1)[D(1)-D(2)]}{r-\alpha} < 0, \text{ for } \lambda < 1$$

which means that $H(x)$ must have at least one root in the interval $(0, x_F)$, for $\lambda < 1$.

To prove uniqueness, we only need to demonstrate strict concavity of $H(x)$ over the interval $(0, x_F)$. Calculating the second derivative of $H(x)$:

$$\frac{\beta K}{x^2}\left[-\left(\frac{x}{x_F}\right)^{\beta} + \frac{(\lambda-1)\left(\frac{x}{x_F}\right)^{\beta}\beta[D(1)-D(2)]}{D(2)}\right] < 0, \text{ for } \lambda < 1$$

Thus, noting that λ is nonnegative, x_L is unique over the interval $(0, x_F)$ for $\lambda \in [0, 1)$.

A.2 The Proof of the Uniqueness of x_L Under *ex-post* Asymmetry

In this Appendix, we will prove the uniqueness of $x_L \in (0, x_F)$, assuming *ex-post* asymmetry.

Let $H(x) = L(x) - F(x)$:

$$H(x) = \frac{(1-\lambda)xD(1)+\lambda xD(2)}{r-\alpha} + +(1-\lambda)\frac{\beta}{\beta-1}\left(\frac{D^L(2)-D(1)}{D^F(2)}\right)K\left(\frac{x}{x_F}\right)^\beta - (1+k)K - \frac{K}{\beta-1}\left(\frac{x}{x_F}\right)^\beta .$$

Calculating $H(x)$ at $x=0$, and at $x=x_F$ we obtain:

$$H(0) = -(1+k)K$$

$$H(x_F) = K\left[\frac{\beta}{\beta-1}\left(\frac{D^L(2)}{D^F(2)}-1\right)-k\right] > 0 \text{ for } k \in [0,k^*)$$

where $k^* = \frac{x_F[D^L(2)-D^F(2)]}{(r-\alpha)K}$ (see equations (29) and (30)). So, $H(x)$ must have at least one root, over the interval $(0, x_L)$.

To prove uniqueness, we only need to demonstrate strict concavity of $H(x)$ over the interval $(0, x_F)$. Calculating the second derivative of $H(x)$:

$$\frac{\beta K}{x^2}\left[-\left(\frac{x}{x_F}\right)^\beta + \frac{(\lambda-1)\left(\frac{x}{x_F}\right)^\beta \beta\left[D(1)-D^L(2)\right]}{D^F(2)}\right] < 0, \text{ for } \lambda < \lambda^*$$

where $\lambda^* = 1 + \frac{D^F(2)}{D(1)-D^L(2)} > 1$.

Thus, noting that λ is nonnegative, x_L is unique over the interval $(0, x_F)$ for $\lambda \in [0, \lambda^*)$ and $k \in [0, k^*)$.

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