The Option to Sell a Real Asset, and The "Grace Rate" Rule

Abstract:

This paper explores an option to sell a real asset (a put option on a real asset), under general assumptions accepted in the literature about investment under uncertainty. First, general differential equation for the put option value is derived, and appropriate boundary conditions are spelled out. Next, a behavior of this real option value is examined, while looking at this value as a function of asset's parameters, such as volatility of cash flows, and a discount rate. Then, as an analog for the hurdle rate rule in case of investing in the real asset, the "grace rate" rule is introduced as a proxy for the optimal selling (scrapping) decision. The finding is that, similarly to the hurdle rate rule, the "grace rate" rule is robust enough, and may lead to a near optimal decision for a wide range of the project's parameters.

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Introduction

In literature that deals with investment under uncertainty problem, we find many papers with comprehensive analysis of the option to wait with investment. Usual scenario considers a firm which faces a decision to invest in a risky project with future stochastic cash flows. One of the most important conclusions about this situation is that the "zero NPV" rule should not be used. Instead, the firm should wait until the value of the project under consideration will rise up to a value called the "investment trigger", and then to invest. The issue here is that the firm has an option to wait with investment. This option is a real option, because it is associated with a real asset. The option to wait has a value, which can be estimated, and it causes the "zero NPV" rule not to work. This situation has been examined in details in Dixit and Pindyk (1994), and McDonald (1986), (1999).

This paper is based on the work of Dixit and Pindyk (1994), McDonald (1999), and Jagannathan and Meier (2002). All these sources contribute to understanding the optimal capital budgeting policy by exploring primarily the option *to wait with investment*. I will try to look at the option *to wait with selling* an existing project. Other names for this option are "option to scrap", or just a "put option".

Let us consider the following example. There is an old, but profitable plant, located in the downtown area, where prices on land are high. The piece of land occupied by the plant is very attractive, and has a great potential. In this situation, the decision to scrap this business is just a matter of time. The scrapping value would be the price of the land, less, maybe, demolishing costs. A classic financial approach says that the business should be abandoned whenever the present value of future cash flows equals the scrapping price. Real options method, on the other hand, considers the option to sell the real asset for a fixed price, and rules that the value which will trigger a scrapping decision should be adjusted by the value of this put option. This means, the optimal scrapping policy would be to sell the business when its value falls below the scrapping price, and to utilize the option value as much as possible.

Valuation of a real option may not be a simple task, and managers, responsible for scrapping decision may be looking for some approximation, which will lead to an optimal outcome. This paper introduces one of such possible "shortcuts" for an optimal scrapping decision. I would call it a "grace rate" rule, as an analog to the hurdle rate rule in case of investment decision.

The subsequent chapters are arranged as following. In the beginning, a general differential equation for the put option is derived and solved, by using approach from Dixit and Pindyk (1994). Then, this put option function is analyzed, in the same way the option to wait with investment is examined in McDonald (1999). Finally, the grace rate rule is presented, and tested for robustness by varying parameters such as volatility, discount rate and risk-free rate.

Assumptions

The set of assumptions is the same as in Dixit and Pindyk (1994):

- The value of a project, *V*, is spanned by existing assets in the economy. That means, that one can construct a dynamic portfolio of assets, that would be perfectly correlated with *V*.
- The value of a project, *V,* is a function of more basic asset *P*. This basic asset *P* may be interpreted as an instantaneous cash flow from the project.

  In McDonald (1999), it is referred by letter *C*.
- The price of basic asset follows the geometric Brownian motion process:

  $dP = \alpha P dt + \sigma P dz$

  The value of the project *V(P)*, in turn, follows

  $$dV = \alpha V dt + \sigma V dz \qquad (1)$$
- Dixit and Pindyk (1994) derive the value of the option to invest as a function of *P*, while McDonald (1999), referring to former, re-writes it as a function of *V*, making a statement that the model may be expressed either in terms of *P* (*C* in McDonald's text) or *V*.

Dixit and Pindyk (1994) on page 182 note that the equations that link the option to invest *F*, and *V* may be quite complicated, but we will follow McDonald (1999), and will write the option as a function of *V*, rather than of *P*.

Derivation of the equation for put option value

Dixit and Pindyk (1994) derive the equation for the option to wait with investment (call option) by constructing *a risk-free portfolio* that contains an option to invest, and a short position in project output. Similarly, we will construct a portfolio of the option to sell, and a long position in project's output. This portfolio also will be risk free, upon determining the number of units of output to purchase. To define it, let us write the expression for the value of such portfolio.

The value of portfolio may be written as

$F + nV,$

where *F* is an option to sell, and *V* is the value of the project.

Following Dixit and Pindyk (1994), page 150, let $n = -F^{'}(V)$. (In their text, $n = F^{'}(V)$, since that portfolio has a short position in *V*). Interestingly, the explanation of why this should hold is given only on page 180. It is required to get rid of the *dz* component (Wiener process), when the value of portfolio is expressed using Ito's lemma. Thus, the value of portfolio is

$F - F^{'}(V)V$

While holding the portfolio for the short period of time *dt*, the long position will pay the dividend of $-(\delta V * F^{'}(V))$, where δ is a dividend rate. Note that the dividend rate is defined as

$\delta = \rho - \alpha$,

where ρ is a discount rate on the project, and α is a drift coefficient from (1). The value of α defines an expected rate of change in project's cash flows.

The portfolio return over the short time dt is a sum of capital gain and dividend payment:

$$\{dF - F'(V)dV\} - \delta V * F'(V)dt \qquad (*)$$

Using Ito's lemma,

$$dF = F'(V)dV + \frac{1}{2}F''(V)(dV)^2$$

Also, from (1)

$$(dV)^2 = \sigma^2 V^2 dt$$

Statement (*) for the payoff becomes

$$\frac{1}{2}\sigma^2 V^2 F''(V)dt - \delta V * F'(V)dt$$

This payoff should be risk free, and be equal to

$$r(F(V) - F'(V)V)dt$$

where r is a risk-free rate of return.

Thus,

$$\frac{1}{2}\sigma^2 V^2 F''(V)dt - \delta V * F'(V)dt = r(F(V) - F'(V)V)dt$$
$$\frac{1}{2}\sigma^2 V^2 F''(V)dt + (r-\delta)VF'(V)dt - rF(V)dt = 0$$

Dividing by dt gives the equation

$$\frac{1}{2}\sigma^2 V^2 F''(V) + (r-\delta)VF'(V) - rF(V) = 0 \qquad (2)$$

This is exactly the equation derived by Dixit and Pindyk. It appears many times in their book, for example, as equation (23) on page 151 and (6) page 183.

As one can see, the same equation defines two different options: option to buy (invest), and option to sell (disinvest, scrap). Two different risk-free portfolios were constructed, and the resulting statement was the same in both cases. However, it is worth to point the difference in the process of getting the equation (2) for two cases. The number of units of output, which makes portfolio risk free, is different in cases of call and put options.

In case of short position in portfolio (case of call option, or option to invest), this number is equal to $F'(V)$, which implies the positive sign of $F'(V)$. This means that *F(V)* is an increasing function of *V*. Since *F* in this case is the option to invest (call option), this is very reasonable: the price of call option increases when the price of underlying asset goes up.

In turn, when another portfolio was constructed to derive the value of put option, the number of units of output in long position was set to $-F'(V)$. Since the number of units in long position cannot be negative, $F'(V)$ itself must be negative. This implies that $F(V)$ decreases as *V* grows, which perfectly complies with the *F (V)* being the option to scrap (put option): as the price of underlying asset goes up, the price of put option goes to zero.

Put option value function, and boundary conditions

The solution for (2) is given in the book of Dixit and Pindyk. It has a form of

$$F(V) = A_1 V^{b_1} + A_2 V^{b_2}, \tag{2a}$$

where $b_1 > 1$ and $b_2 < 0$.

The expression for b_2, that will be used later, is

$$b_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[\frac{r-\delta}{\sigma^2} - \frac{1}{2}\right] + \frac{2r}{\sigma^2}} \tag{2b}$$

Boundary conditions for the case of the option to sell will be:

1. When $V \to \infty$, $F(V) \to 0$. To assure this condition, we will set A_1, the coefficient for the positive degree of *V*, to zero. So, $A_1 = 0$.

2. The value-matching condition at the trigger value of $V = V^* \equiv V_L$. This is the value of the project when the option to sell is exercised, and the project is sold. This condition requires

$F(V_L) = I - V_L$,

where *I* is a selling price (scrap value).

3. The smooth-pasting condition requires

$F^{'}(V_L) = (I - V)^{'}|_{V=V_L} = -1$

Using condition 2, the value of A_2 is obtained as

$$A_2 = \frac{I - V_L}{V_L^{b_2}}$$

Substituting this statement into (2) gives the following expression for the put option:

$$F(V) = (I - V_L)\left(\frac{V}{V_L}\right)^{b_2} \tag{3}$$

The trigger value V_L may be derived from conditions 2 and 3 as following:

Condition 2: $A_2 V^{b_2} = I - V_L$ (4)

Condition 3: $b_2 A_2 V^{b_2 - 1} = -1$ (5)

Dividing (4) by (5),

$$\frac{V_L}{b_2} = V_L - I$$

$$V_L\left(1 - \frac{1}{b_2}\right) = I$$

$$V_L = \frac{b_2}{b_2 - 1} I \tag{6}$$

Note that since $b_2 < 0$, $\frac{b_2}{b_2 - 1} < 1$, so V_L is always less than *I*.

Put option characteristics

Now, having equations for the value of the option to sell (3), the trigger value (6), and for the value of b_2 (2a) which is used in (3), we can have a closer look to the characteristics of this put option. We will follow McDonald (1999) in methodology, having in mind that the option under consideration is not a call, but put option.

Figure 1 shows the value of the put option, as a function of the discount rate used for the project valuation. Describing the similar chart for the call option (investment timing option), McDonald (1999) notes: " *Each point on the graph should be thought as a separate project with different discount rate, each of which currently has a zero NPV, i.e. V = I = 1.*
A firm which invests immediately at zero NPV would therefore lose the full value of the investment option depicted in the figure."

Here the picture is symmetric: each point on Figure 1 graphs is a project with value of 1, which may be sold for the price of 1. If the firm will sell it now, it will lose the value of the put (sell) option, shown on the axis Y. This value was computed using equation (3), when $V = I = 1$. Risk free rate, as in McDonald (1999), was assumed to be 8%.

The Figure 1 also shows dependency of the option value on project cash flow growth rate, α, and volatility of cash flow, σ. Quite expectable, the option value is a decreasing function of α, and an increasing function of σ.

The critical trigger value V_L, as a function of project discount rate, for different values of α and σ, is shown on the Figure 2. The greater is the volatility of a project, the lower should fall its value before the project will be sold. Higher dividend rate, $\rho - \alpha$, will result in the same effect.

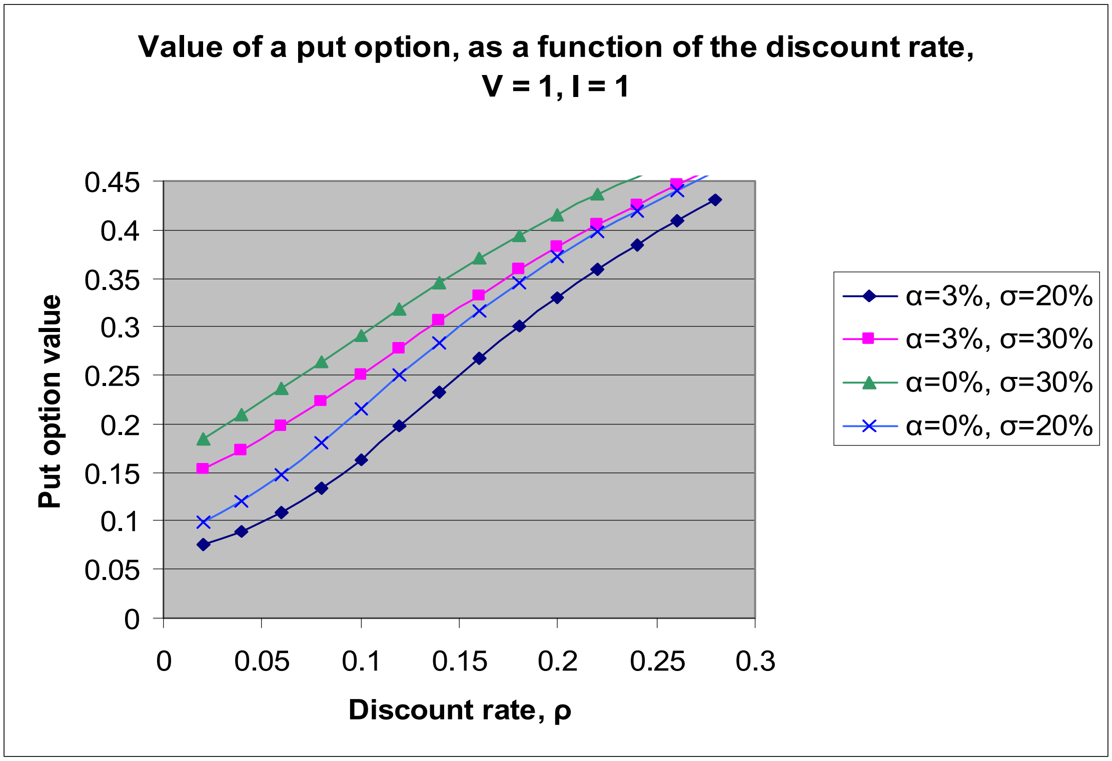

Figure 1

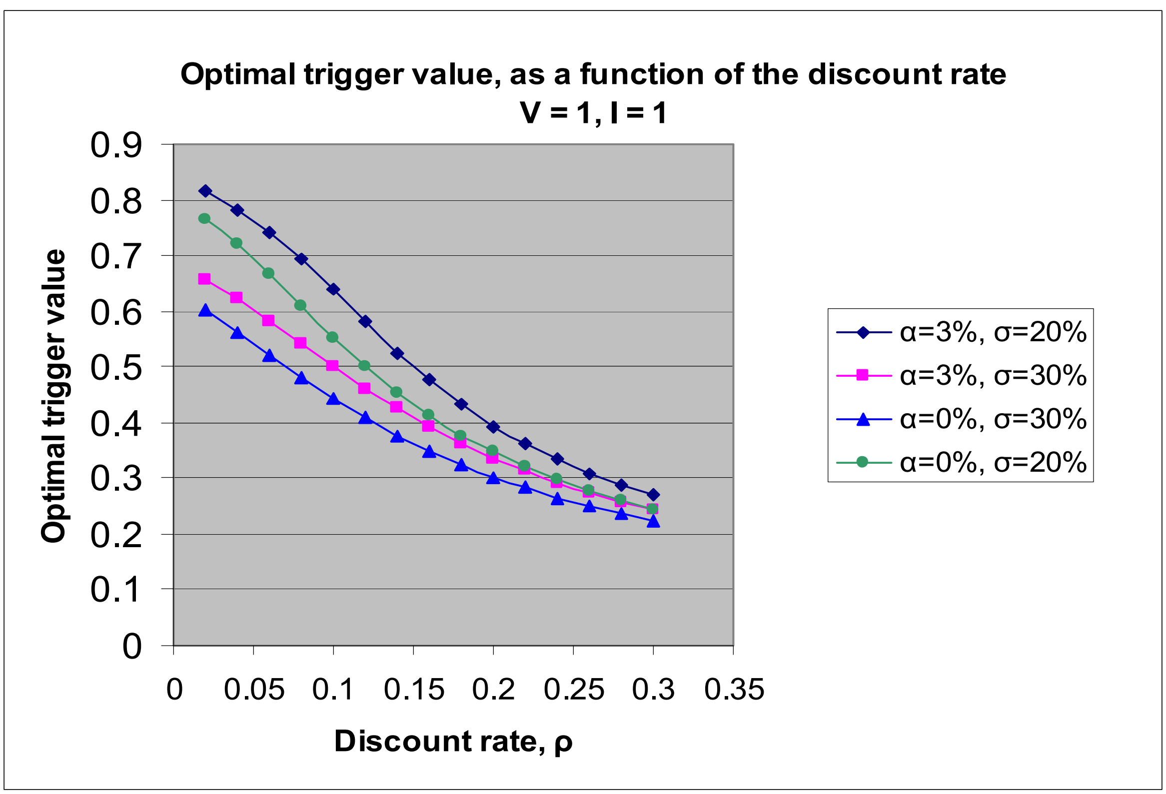

Figure 2

Next two figures demonstrate the cost of non-optimal scrapping policy. As R. McDonald asks in his paper, "*If a manager does not explicitly calculate the equation* <for the optimal trigger value>, *is it possible that a seemingly arbitrary investment rule comes close, in the sense that the value lost from the approximation is small?*" The next two charts show the answer – for the selling, not investment option. In both cases, growth rate, α, was assumed to be zero.

Figure 3 shows cases for fixed discount rate, and different volatilities, and Figure 4 depicts the same for constant volatility with various discount rates. Two pictures look very similar, and show same tendencies. Both, discount rate and volatility, when increased, cause the trigger value to decline, and the option value – to boost. Almost linear for the values just below selling price ($I = 1$), graphs reach the maximum at the optimal point of V_L, which may be calculated using formula (5), and then fall to zero as the trigger value goes to zero. The latter phenomenon may be explained by saying that a put option, once it has reached its maximum value, should be exercised. Further waiting will only cause the option value to decline. If an arbitrary trigger value is less than the optimal trigger value, the project value may fall to it only *after* passing the optimal point. (This assumes that the project value is a continuous function of time, and is differentiable at least once with respect of time). The fact that at some point of time the project is valued below the optimal scrapping trigger value means that the firm "waited too long" to sell it. As a result, a part of the option value has disappeared.

From the figures it may look like lower values of volatility and discount rate create better conditions for making a decision that will be closer to optimal, because the graphs for smaller numbers of σ and ρ are more "flat" in the vicinity of optimal trigger value, V_L. As a result, the range of arbitrary trigger values, which will capture same percentage of option value (say, 90%), will be wider. However, closer examination shows that this range stays more or less the same, due to fact that the maximal option value increases, as the graphs become more and more concave.

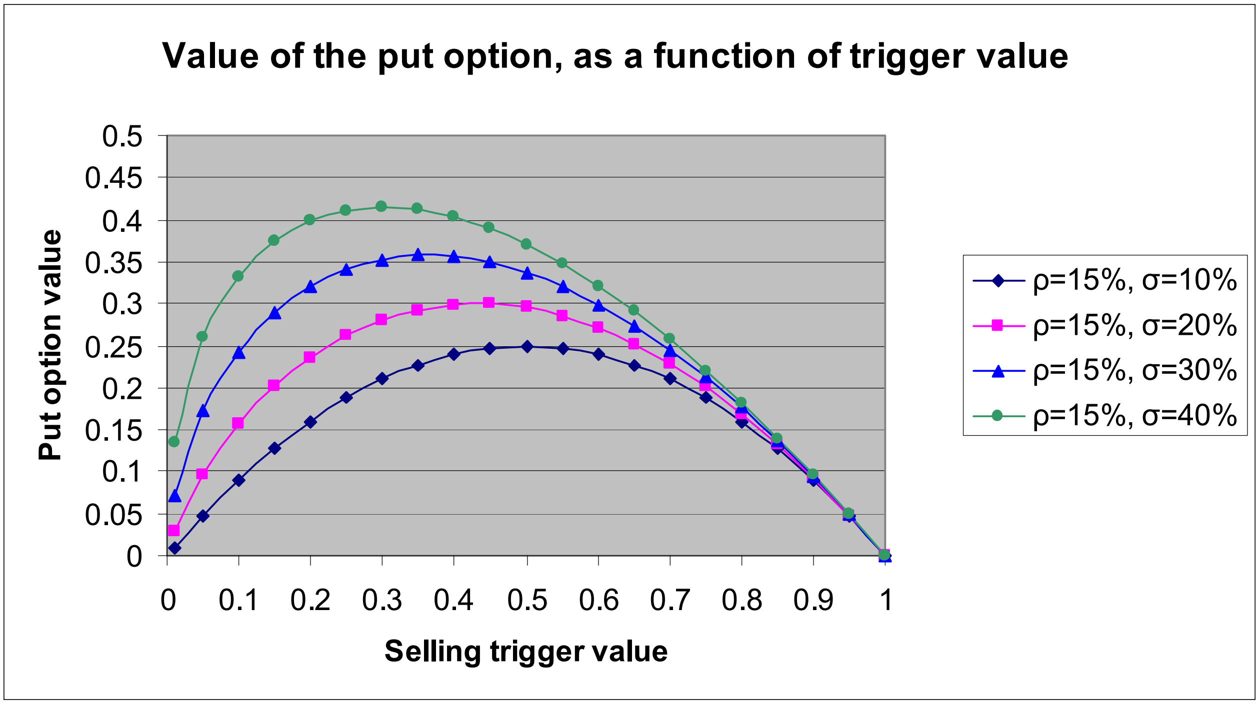

Figure 3

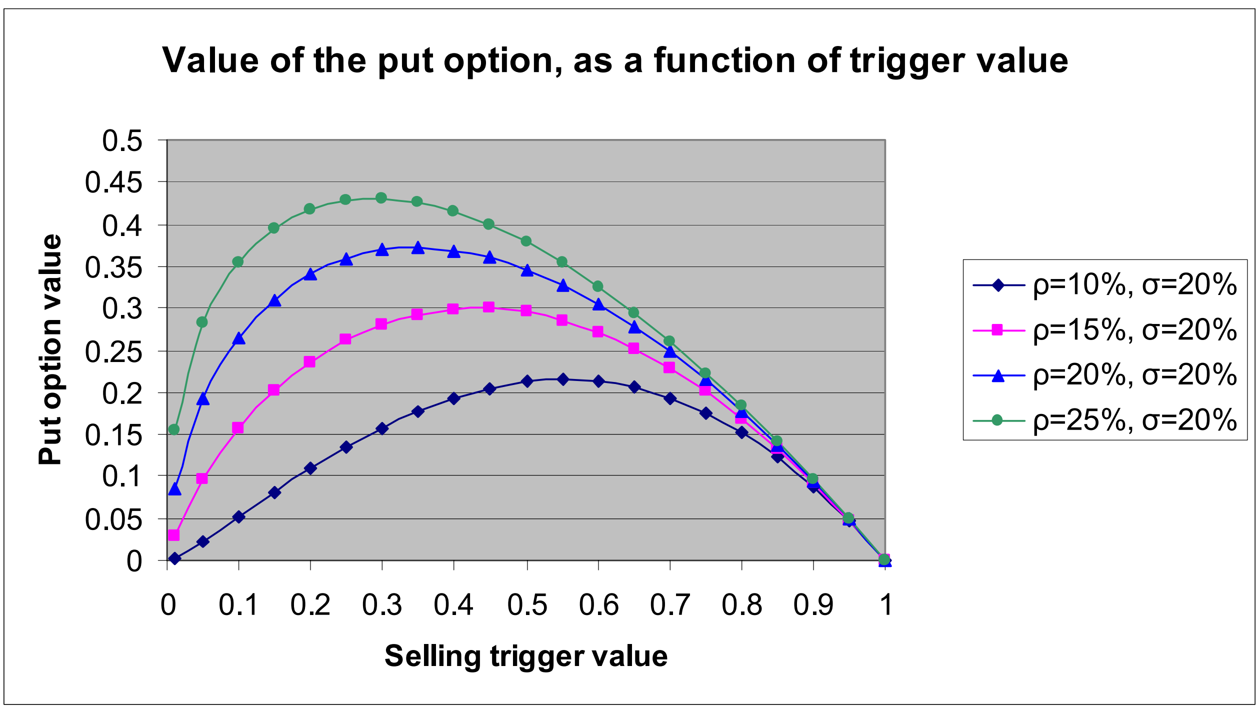

Figure 4

"Grace" rate for selling (scrapping) decision

R. McDonald (1999), repeating Dixit (1992), makes a note about the usage of hurdle rates for making an optimal investment decision.

In case of investment decision, the value of hurdle rate will be higher than the discount rate, and the investment rule says to invest whenever the IRR of the project exceeds the value of hurdle rate.

The equation for hurdle rate, γ, is given by equation (9) in McDonald (1999):

$$\gamma_H = \alpha + (\rho - \alpha)\frac{b_1}{b_1 - 1} ,$$

where b_1 is a positive degree value from the equation (2a).

The fact that γ is greater than ρ, may be verified by recalling that $b_1 > 1$, hence $\frac{b_1}{b_1 - 1} > 1$.

For the case of scrapping decision, the equation for γ will be the same, as above, but instead of b_1 we will use b_2 :

$$\gamma_L = \alpha + (\rho - \alpha)\frac{b_2}{b_2 - 1} \tag{7}$$

Let us call this rate the "grace" rate, rather than the hurdle rate, since it is less than ρ ($b_2 < 0$, thus $\frac{b_2}{b_2 - 1} < 1$). In a way similar to the hurdle rule for investment, the grace rate rule will dictate to scrap the project whenever IRR falls below γ_L . For an arbitrary chosen γ_L , the firm will sell the project when its value will fall to

$$V_\gamma = \frac{\gamma - \alpha}{\rho - \alpha} I \tag{8}$$

The equation (8) is a replica of formula (8) from McDonald (1999). Here I is the selling price, while the "original" equation has as I the required investment.

Table 1 demonstrates how the optimal grace rate behaves depending on three project parameters: volatility, growth rate and discount rate. Risk-free rate was assumed 8%.

As shown in the table, higher volatility implies lower numbers for grace rates, which, in turn, will cause lower values of V_L, according to (8). Dependency of V_L on σ was shown on Figure 2. Increase in a growth rate as well as in a discount rate causes the optimal grace rate to go up. The reason for this: equation (7) assumes an arbitrary value of trigger value – the optimal trigger, and it should stay constant. Interestingly, most values for grace rate are less than the risk-free rate. Those that are greater than the risk-free rate (> 8%) are shown in red.

Table 1 : Optimal "grace" rate values for a range of project parameters

		Discount Rate			
Cash Flow volatility, σ	Cash Flow growth rate, α	8%	12%	16%	20%
20%	-3%	2.78%	3.48%	3.87%	4.11%
20%	0%	4.88%	6.00%	6.60%	6.94%
20%	3%	6.47%	8.23%	9.18%	9.70%
30%	-3%	1.67%	2.44%	2.95%	3.30%
30%	0%	3.84%	4.90%	5.59%	6.05%
30%	3%	5.70%	7.15%	8.10%	8.72%
40%	-3%	0.79%	1.53%	2.08%	2.49%
40%	0%	3.06%	4.00%	4.69%	5.19%
40%	3%	5.12%	6.32%	7.19%	7.83%

As emphasized in McDonald (1999), the main question while using a hurdle (grace) rates is the question of optimal policy. How close to the optimal will be the investment (scrapping) decision? To decide about the "quality" of the decision, one can have a look at the fraction of the option that will be captured, when a decision is made by using a grace rate rule. Figure 5 shows this fraction for different projects. This figure is a "twin" of Figure 4 from the paper of McDonald.

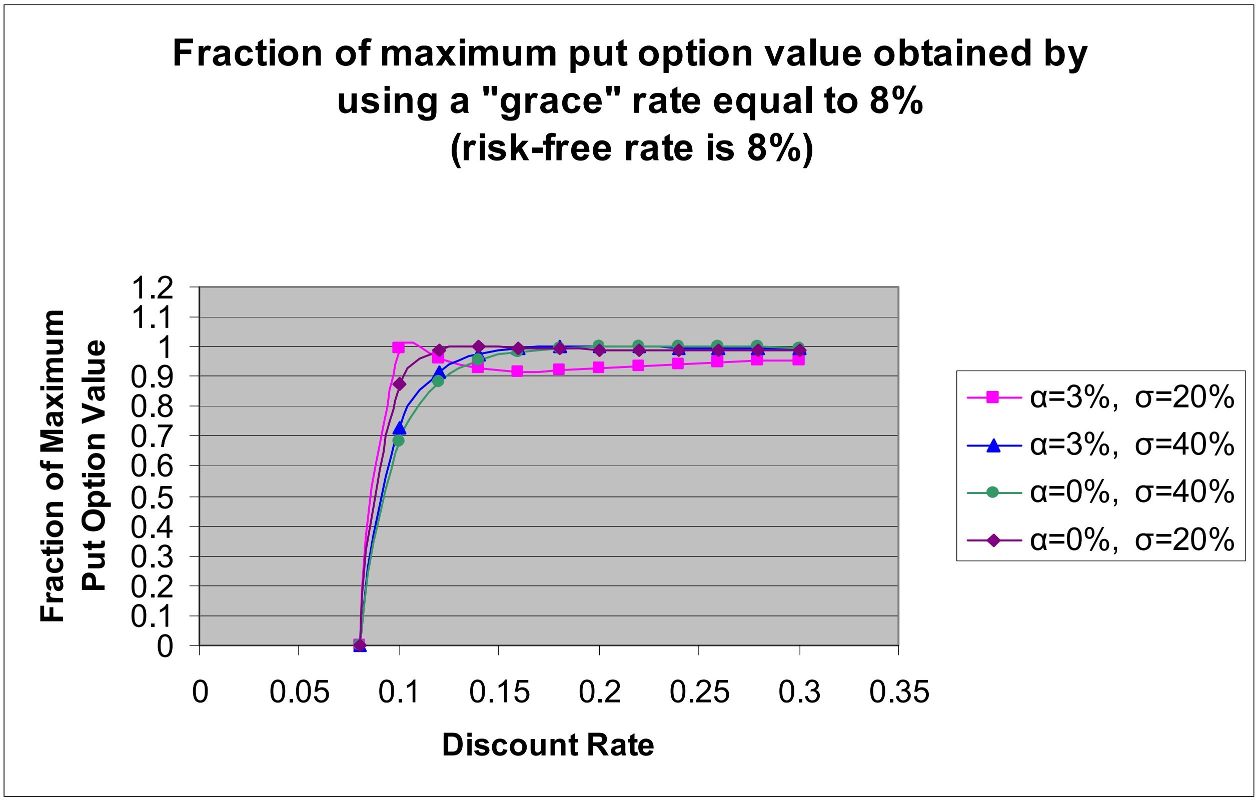

Figure 5

Figure 5 was constructed by calculating a ratio of

$$\frac{F(V_\gamma)}{F(V_L)}.$$

Here, numerator is the value of option to sell a project calculated using a trigger value from (8), while grace rate was assumed 8%. Denominator is the value of option to sell a project calculated using the *optimal* trigger value, from (6). Project value was assumed to be 1, and risk free rate was 8%. The selling price taken was 0.8, but any another value will give the same picture, because the above ratio does not depend on selling price.

Having risk free rate of 8%, the grace rate of 8% is nearly optimal for any discount rate greater than 12%: captured fraction of a put option will be 90% or greater. This means, with the chosen setup, usage of 8% grace rate is a good proxy for an optimal decision regarding selling the project.

However, if we want to decide about a "rule of thumb", using a terminology of McDonald (1999), for selling decision, it is not obvious at all, how different values of grace rate will behave, and how robust the "grace rate rule" will be with respect to project parameters.

How optimal is the grace rate rule?

To address these concerns, we conducted calculations and built graphs similar to those from Jagannathan and Meier (2002). In this paper, which is largely based on the paper of McDonald (1999), the authors calculated the "bandwidth" of hurdle rates, and volatility values, which will guarantee near optimal investment decision. Investment decision is 'near optimal' if at least 90% of the option value is captured when a firm invests into a project. Like in McDonald (1999), the option examined in Jagannathan and Meier (2002) is the option to wait.

Next three figures show a fraction of a put option, which is captured while using a grace rate rule, as it was done for Figure 5. Each figure has different project parameters as axes X and Y, while the fraction value is shown on axis Z, which cannot be viewed on XY plane. To depict values on axis Z, different colors are used.

Figure 6 shows the fraction of the optimal (maximal) put option value, which is captured by using a grace rate, as a function of discount rate and grace rate, having volatility and risk free rate fixed. The dark-red area corresponds to at least 90% fraction of the put option value. Grace rate rule looks here very robust: for example, the grace rate of 8% will "work" for any project with a discount rate greater than 11%. At the same time, a wide range of grace rates may be applied for a single project, resulting in near optimal decision. For example, if the discount rate is 20%, any rate between 6% and 12% may be used a grace rate. From Table 1, the optimal grace rate for this project is 8.72%.

Next, let us have a look at the same value – fraction of a put option, but in different axes: discount rate vs. volatility. Figure 7 shows this case. It looks even more promising: most of the area shows that the fraction captured will be at least 90% of the maximal put option value.

For discount rates greater than 21%, a grace rate of 8% will guarantee a near optimal decision, regardless of project volatility.

Finally, Figure 8 shows how the relation between risk free rate and grace rate influences the portion a put option captured when a selling decision is based on the grace rate rule. For

a project with a discount rate 20% and volatility 30%, a risk free rate of 8% "allows" the grace rate to be between 7.5% and 14%. Any grace rate inside this range will lead to a capture of at least 90% of the option value, when the project is sold. Another conclusion may be made from Figure 8: knowing the risk free rate, we can construct a simple proxy for a grace rate, without carrying out a calculation of formula (7). One possible example is shown: the straight line on Figure 8 is a graph of $\gamma = r + 1\%$. For any risk free rate above 4%, such approximation while choosing a grace rate will lead to a near optimal decision.

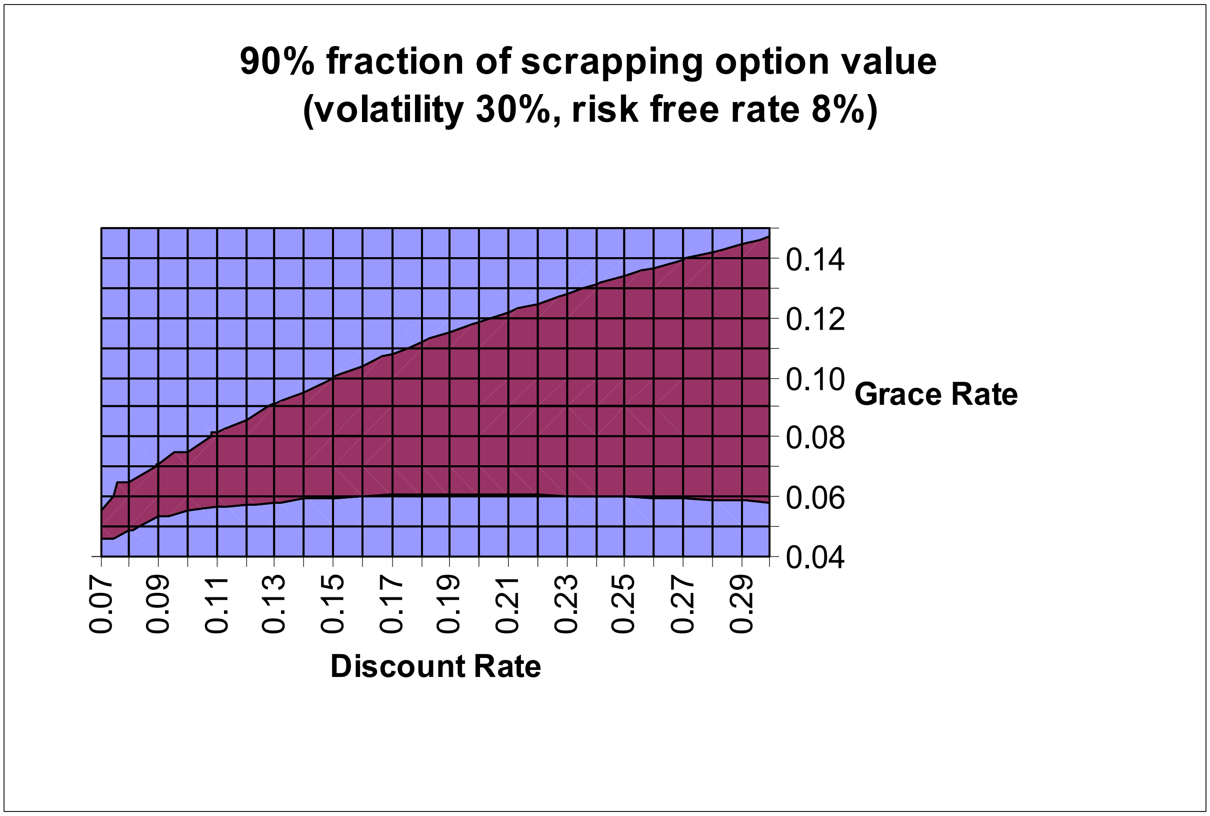

Figure 6

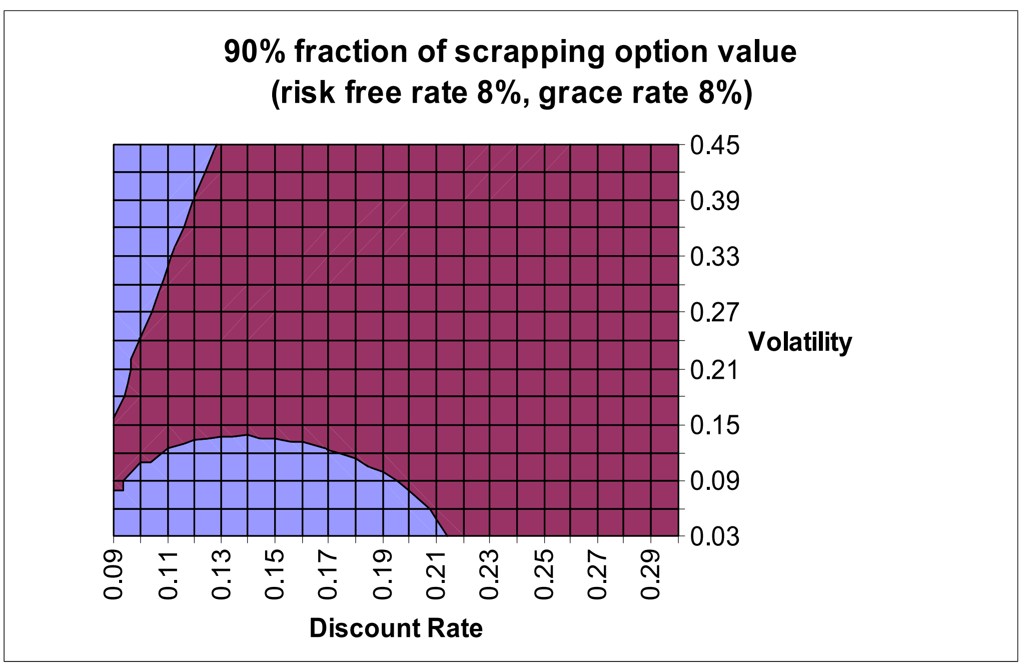

Figure 7

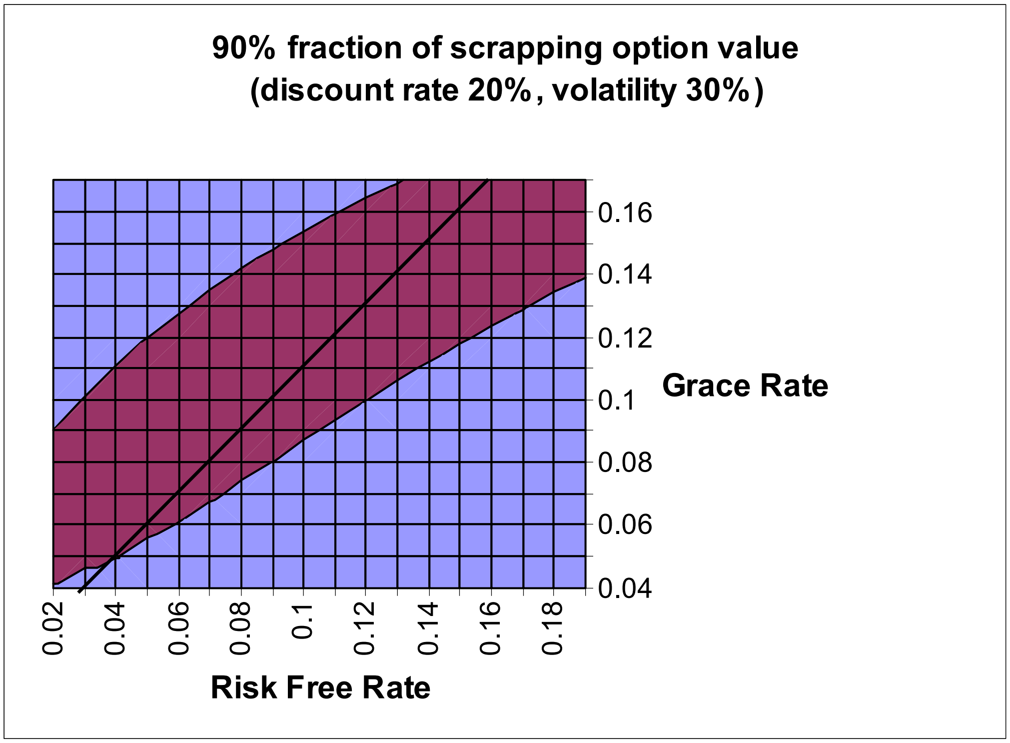

Figure 8

Conclusion

In this paper, I looked at the option to sell a real asset. At first, the equation for the option to sell value was derived. The resulting equation is the same as for the option to wait. The difference comes from different boundary conditions that were worked out as well. Following McDonald (1999), and using same approach, I looked at the characteristics of the option to sell, which is, basically, a put option on the real asset.

McDonald (1999), and Jagannathan and Meier (2002) analyze the hurdle rate rule as an alternative to carrying out mathematical calculations in order to get to an optimal investment decision. In this paper, I introduce the "grace rate rule" – the rule, analogous to a hurdle rate rule, which may be utilized in the case of selling a real asset. Grace rate is the rate that may be used by a firm in purpose to determine the project value, while deciding about the selling of the project.

The finding is that the grace rate rule may be very robust for a wide range of project parameters, and it will lead to a near optimal decision. This outcome is similar to the results in Jagannathan and Meier (2002) in regards to the hurdle rate rule. It also shown in this paper, that a risk free rate may be easily used for constructing a simple proxy for the optimal grace rate. The decision based on this approximate value will be near optimal for a wide range of risk free rate values.

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