The Value and Timing of *Shared* Real Options with Random Maturity

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Abstract

In this paper we present an original model, with the purpose of determining the value and timing of an investment opportunity (IO) that is *shared* by several companies, in a competitive environment. We assume that the market can accommodate a finite number of firms (N), which are assumed to enter the market stochastically. Accordingly, the IO matures with the last "admitted" company's entrance and, after that moment, the option to invest disappears. Since companies can invest anytime up to, but not including, the *random* maturity date, it's important to determine the optimal timing to invest, in this context. We provide an example that shows the value and the optimal timing for investing for several situations. As expected, the value of the IO, and its trigger value for investing, decreases as the available "places" in the market decreases, or (and) as the probability of a competitor entrance increases. The example also shows that the IO's value and the optimal timing tend to the value and to the optimal timing of a perpetual American option as N assumes higher values or (and) when the probability of a competitor entrance tends to zero. At the end of the paper we present several possible extensions to the basic model.

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1 Introduction

To the traditional question: *"Should the company invest in the project?"*, modern capital budgeting, influenced by the insights of the real options literature, says it must be complemented with another, extremely important, one: *"When should the company invest?"* In other words it is important to determine at what point in time is it optimal to pay the investment cost K in order to receive the gross value V. Instead of saying: "Invest if $V > K$" (NPV rule), the real options theory says: "Invest if (when) $V \geq V^*$, where V^* is some critical value for V.

A problem may arise because, commonly, and in addition of being deferrable, the investment decisions can be, at least in part, irreversible, since the investors can't, without penalty, step back and recover the position prior sinking the implementation cost[1]; however, the decision to defer the project implementation is reversible: the company can, anytime, decide to stop deferring, choosing to exercise the option to invest. This asymmetry has a major impact, both on the project's value and on the (optimal) timing to invest[2].

To act (invest) optimally, the company must trade-off the benefits from deferring (in order to obtain more information about the project, reducing the uncertainties) and the costs that the decision to postpone introduces (e.g.: the lost cash-flows).

However, in this context, the firm has to determine, or to predict (as in most real world situations), until when it will have the capacity to defer the project implementation. After that particular moment - that is called the maturity date - the option disappears, and the company loses the chance to invest.

Related to this, there are three possibilities: (*i*) there is no finite maturity for the option to invest, meaning the company has perpetual right to invest; (*ii*) the maturity is finite, but its date is known with full certainty; (*iii*) finally, and possibly the most common, the maturity date is finite but its occurrence is, for several reasons, *uncertain*.

There is some prior research that focus (theoretically or empirically) on the (optimal) timing to invest, for example: Titman (1985), McDonald and Siegel (1986), Paddock, Siegel and Smith (1988), Pindyck (1988), Bjerksund and Ekern (1990), Trigeorgis (1990a), Ingersoll and Ross (1992), Kemna (1993), Lee (1997), McDonald (2000) and Pereira, Armada and Kryzanowski (2002). But basically, in all these papers, investment opportunities are treated like European calls, American

[1] In fact, the decision to invest now kills the option to invest later in the same project, which means that an irreversible project that can be deferred competes with itself delayed in time. This introduces a mutually exclusive problem between the project now and itself later in the future.

[2] For details, see two excellent surveys: Dixit and Pindyck (1994), and Trigeorgis (1996b)

calls, perpetual (American) calls or American exchange options, so it is assumed that either the maturity does not exist, or, if it is finite, it is known with certainty (a deterministic maturity is assumed).

Excluding the perpetual situation, and on contrary to the financial options[3], the determination of maturity date plays an important role in the real option valuation, particularly if we think of it as a random variable.

Since the maturity is the moment after which the investment opportunity disappears, it's important to think about what contributes to the maturity's occurrence, and, also important, to its (possible) randomness. Starting with the non-stochastic situations, we can say that maturity can depend on some (pre-defined) contractual length of time to explore some natural resource - the concession period (e.g.: offshore oil properties, mining exploration, etc.), or to start producing some protected product (patents), where the protection ends in some (known) future date.

Additionally, there are some (maybe few) situations where the option to invest lives forever; a typical example is the option to construct (e.g.: a building) in a vacant land, since the owner of the land has the perpetual right to postpone the construction.

But, in the majority of the situations, the maturity date is neither contractual (and, as a consequence, certain) nor perpetual. In most real world investment opportunities, the maturity date depends on some exogenous (to the firm) factors that, after occurring, vanish the option to invest. That's the case, for example, when, in a monopoly, some firm enters earlier, destroying the chance for all the others[4].

In practice, possibly as a consequence of being strongly influenced by actions of others and by the competition among companies, *the maturity date tends to be random*.

Additionally, this competition aspect is frequently ignored, since most of the real options models assume that the firm has an exclusive right on the investment opportunities. However, the companies' real options are rarely proprietary; on contrary, investment opportunities are (most of the times) shared options.

As a consequence, and in addition to the traditional factors that influence the real options value, the firm must incorporate the competitors' actions, due to its impact on the option to invest. That's what some authors do when study the preemption possibility in a monopoly market[5], or study the leader/follower optimal action in duopoly market[6].

[3]The financial options' maturity is known in advance, as part of the traded contract.

[4]That's also the case for the *R&D* competition, since the first to do the discovering gets an exclusive right against its competitors (in this *R&D* context the maturity occurrence, corresponding to the discovery occurrence, is a "good" event for the company, but a "bad" one for its competitors).

[5]See Trigeorgis (1991b).

[6]See, for example, Smit and Ankum (1993), Trigeorgis (1996b), Paxson and Pinto

By introducing a new methodology[7] for valuing and determining the optimal timing of shared investment opportunities, our work makes a contribution, dealing with the problem in a more general way: we assume the market can accommodate N firms, and not only one or two[8].

As we will note, the "shared" characteristic of the real options, and its random maturity are, in our model, intimately related: if the market can accommodate N companies, then the maturity occurs exactly when the N^{th} competitor enters the market. At this (critical) random moment, the option to invest disappears, and the company loses the chance to undertake the project; in other words, at this moment, the investment opportunity turns worthless.

In the next section (section 2), the assumptions and the model are presented. Section 3 presents some numerical results for some hypothetical examples. Finally, in section 4 some conclusions are derived and some avenues for future research are presented.

2 The Model

2.1 Basic Characteristics and Assumptions

A firm is facing an opportunity to invest in a project. This investment opportunity is shared with some other competitors (note that a problem treated in this paper only exists if the number of possible entrants is higher than the "available places" in the market). In this model, we assume the market can "accommodate" N companies exploring the same "business". If N=1 the market is monopolistic, if N=2 the market if duopolistic, and if N assumes higher values that means more companies can be placed tending to a more competitive market. As we can easily see, a company having some shared investment opportunity can defer the decision to invest until the number of the companies already in place equals N; until then the company has the chance to enter, because the market stills having available "space". This means that, after N entrances (which will be modeled as Poisson jumps), the option to invest becomes worthless, because the market has no capacity to accommodate more firms. At that moment, the maturity occurs. Note that, since the entrance of other companies is stochastic (Poisson "events"), then the maturity date is, also, stochastic.

Accordingly, the company can invest anytime until all the "admitted" firms have already entered the market. If all the companies that the market can ac-

(2003).

[7]Some very basic ideas where taken from Carr (1998). However our specific objectives, assumptions and formulae are totally different.

[8]Monopolies and duopolies could be looked as particular cases of this model. Note however that, as we will see, we assume that the actions of the competotors are exogenous to the model.

commodate are already in, the option to invest turns worthless, since there is no possibility to invest afterwards. This type of option is similar to an American option with an important difference: at the maturity date, the option to invest does not have the traditional payoff *max[V-K, 0]*. At that moment the company cannot invest because the last admitted firm has already arrived, so, at the maturity date the value of the option to invest is zero. In other words, the option to invest is not entirely like an American option, because the company can invest anytime up to, but not including, the random maturity date.

As we said, the entrances of the competitors are modeled as Poisson jumps, and the maturity happens with the N^{th} jump. Additionally, the jumps are assumed to be independent from each other, independent from the projects' value, and constant in time.

There is, also, no time decay associated with the option to invest. In our model, the company gets closer to maturity, not as time passes, but as jumps occurs.

Finally, it is also assumed that the market is "homogeneous", meaning that all admitted (or installed) firms produce (or sell) the same type of goods (or services); and the firms are identical in terms of market dimension and profitability.

We can present several examples of businesses that can be modeled this way, but we just mention three: *(i)* franchise opportunities (the franchiser generally defines how many franchise units are allowed in a region; when all units are in place, no more will be allowed); *(ii)* construction industry (in the presence of some large vacant land, the construction companies have the shared option to buy a part of the land and construct a building there; but after completely urbanized there's no more chance for construction); *(iii)* malls and shopping centers (usually, local authorities define the allowed number of this large commercial spaces, when all are in place there's no chance to construct more).

2.2 Building-In the Model

A risk-neutral firm has the chance to invest in a project, sinking the investment cost K, whose gross value V is assumed to evolve according to the following geometric Brownian motion with drift:

$$dV = (r - \delta)V dt + \sigma V dZ \tag{1}$$

where r is the risk free rate, δ is the "dividend-yield" (corresponding to the opportunity cost of the decision to defer), σ is the volatility of V, and dZ is the increment of the Wiener process.

Additionally, as we said, the value of the investment opportunity is, also, function of N (the available "places" in the market[9]) that follows a Poisson process of

[9] At the beginning, when no company has entered the market, N represents the total number of firms that it can accommodate; after, N represents the available "places".

the form:

$$dN = -dq \tag{2}$$

where:

$$dq = \begin{cases} 1 & \text{with probability} \quad \lambda dt \\ 0 & \text{with probability} \quad 1-\lambda dt \end{cases} \tag{3}$$

Each jump corresponds to the entrance of a new company (the Poisson "event") reducing the available places in the market by 1, and this happens with the intensity λ, where λ represents the mean arrival rate of the event during the infinitesimal period of time dt.

Under risk-neutrality, the option's value, which is a function of V and N, $F(V,N) \equiv F_N(V)$, must satisfy the following equilibrium condition:

$$rF_N(V)dt = E\left[dF_N(V)\right] \tag{4}$$

Expanding $dF_N(V)$ using Ito's lemma, we get the differential equation (5) which must be satisfied by $F_N(V)$, during the "continuation" period (whenever $V< V_N^*$, where V_N^* is the option's trigger value[10]):

$$\frac{\sigma^2}{2}V^2\frac{\partial^2 F_N(V)}{\partial^2 V} + (r-\delta)V\frac{\partial F_N(V)}{\partial V} + \lambda\left[F_{N-1}(V) - F_N(V)\right] = rF_N(V) \tag{5}$$

It's important to note that $\lambda\left[F_{N-1}(V) - F_N(V)\right]$ captures the *real option's expected loss due to the entrance of a new company in the market*, during the infinitesimal length of time dt. If a company joins the market, the *shared* option, instead of maturing at the N^{th}, jumps will mature after N^{th}-1. As a consequence, both the real option's value and the trigger value will be lower after the entrance (because the *time*-to-maturity[11] has been, suddenly, reduced).

Rearranging (5) we get:

$$\frac{\sigma^2}{2}V^2\frac{\partial^2 F_N(V)}{\partial^2 V} + (r-\delta)V\frac{\partial F_N(V)}{\partial V} - (\lambda + r)F_N(V) = -\lambda F_{N-1}(V) \tag{6}$$

Equation (6) must be solved considering the following boundary conditions[12]:

$$\lim_{V \to 0} F_N(V) = 0 \tag{7}$$

[10]The company should only invest when (or if) V reaches V_n^*, n=1,...,N.

[11]It may be more adequate to use the expression *jumps*-to-maturity.

[12]The reason why we need three conditions is because we have three unknowns, two of them arising from the solution of the second-order differential equation, and the third one is the optimal value for acting.

$$\lim_{V \to V_N^*} F_N(V) = V_N^* - K \tag{8}$$

$$\lim_{V \to V_N^*} \frac{\partial F_N(V)}{\partial V} = 1 \tag{9}$$

The boundary (7) ensures that the option to invest is worthless when V=0; (8) is called the value-matching condition, and (9) is the smooth-pasting condition.

Fortunately, this type of problem can be solved analytically, although recursively, starting at n=1 [with the value of $F_1(V)$], and then passing on to n=2, 3,...,N [calculating the values for $F_2(V)$, $F_3(V)$,..., and finally $F_N(V)$].

When n=1, the option matures at the next jump, which can occur either if the market is monopolistic or if N-1 companies are already in, and just one more is admitted. The option value must then satisfy (6) whose version, when n=1, becomes:

$$\frac{\sigma^2}{2}V^2\frac{\partial^2 F_1(V)}{\partial^2 V} + (r-\delta)V\frac{\partial F_1(V)}{\partial V} - (\lambda + r)F_1(V) = -\lambda F_0(V) \quad for\ V < V_1^* \tag{10}$$

As we already said, when n=0 the option to invest is worthless, because the last admitted company has entered the market, meaning $F_0(V) = 0$[13]. So equation (10) becomes:

$$\frac{\sigma^2}{2}V^2\frac{\partial^2 F_1(V)}{\partial^2 V} + (r-\delta)V\frac{\partial F_1(V)}{\partial V} - (\lambda + r)F_1(V) = 0 \quad for\ V < V_1^* \tag{11}$$

which is a *Cauchy-Euler* homogeneous differential equation, bounded by the following conditions (where V_1^* is the trigger value for the option that matures at the next jump):

$$\lim_{V \to 0} F_1(V) = 0 \tag{12}$$

$$\lim_{V \to V_1^*} F_1(V) = V_1^* - K \tag{13}$$

$$\lim_{V \to V_1^*} \frac{\partial F_1(V)}{\partial V} = 1 \tag{14}$$

The solution for (11) is:

$$F_1(V) = A_{1,1}V^{\beta_1} + A_{1,2}V^{\beta_2} \tag{15}$$

where $A_{1,1}$ and $A_{1,2}$ are constants to be determined, and

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2\ (r+\lambda)}{\sigma^2}} > 1 \tag{16}$$

[13] After that last jump the company loses the chance to enter, because the option has disappeared.

$$\beta_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{r-\delta}{\sigma^2}\right)^2 + \frac{2\,(r+\lambda)}{\sigma^2}} < 0 \tag{17}$$

In order to respect the boundary condition (12), and noting that $A_{1,2}V^{\beta_2} \to +\infty$ as $V \to 0$, $A_{1,2}$ must be zero. The other two conditions are used to determine $A_{1,1}$ (or simply A_1 hereafter) and V_1^*.

Then, the value of the investment opportunity that matures just after the next company enters the market is given by the following expression:

$$F_1(V) = \begin{cases} A_1 V^{\beta_1} & for \quad V < V_1^* \\ V - K & for \quad V \geq V_1^* \end{cases} \tag{18}$$

where β_1 is the same as in (16), and

$$A_1 = \frac{1}{\beta_1 V_1^{*\beta_1 - 1}} \tag{19}$$

$$V_1^* = \frac{\beta_1}{\beta_1 - 1} K \tag{20}$$

As one can easily see from (18-20), the value of an option that matures at the next entrance, is similar to a perpetual American option with the underlying asset (for instance X) following a mixed Brownian motion/jump process of the form $dX = \alpha X dt + \sigma dZ - X dq$ [see McDonald and Siegel (1986)].

When n=2, the value of the option, that matures at the 2^{nd} jump, $F_2(V)$, must satisfy the following differential equation:

$$\frac{\sigma^2}{2} V^2 \frac{\partial^2 F_2(V)}{\partial^2 V} + (r-\delta) V \frac{\partial F_2(V)}{\partial V} - (\lambda + r) F_2(V) = -\lambda F_1(V) \quad for\ V < V_2^* \tag{21}$$

bounded by the following conditions:

$$\lim_{V \to 0} F_2(V) = 0 \tag{22}$$

$$\lim_{V \to V_2^*} F_2(V) = V_2^* - K \tag{23}$$

$$\lim_{V \to V_2^*} \frac{\partial F_2(V)}{\partial V} = 1 \tag{24}$$

Equation (21) is a non-homogeneous *Cauchy-Euler* differential equation. The general solution corresponds to the sum of the solution to the homogeneous part of the equation with its particular solution. So the value of $F_2(V)$ is as follows, taking into account (22):

$$F_2(V) = \begin{cases} A_2 V^{\beta_1} - [\int \frac{\lambda\omega A_1 V^{\beta_1}}{V(\beta_1-\beta_2)} dV] V^{\beta_1} + [\int \frac{\lambda\omega A_1 V^{\beta_1-\beta_2-1}}{\beta_1-\beta_2} dV] V^{\beta_2} & for\ V < V_1^* \\ \\ B_2 V^{\beta_1} - [\int_0^{V_1^*} \frac{\lambda\omega A_1 V^{\beta_1}}{V(\beta_1-\beta_2)} dV] V^{\beta_1} + [\int_0^{V_1^*} \frac{\lambda\omega A_1 V^{\beta_1-\beta_2-1}}{\beta_1-\beta_2} dV] V^{\beta_2} - \\ [\int_{V_1^*}^{V} \frac{\lambda\omega(V-K)V^{-\beta_1-1}}{\beta_1-\beta_2} dV] V^{\beta_1} + [\int_{V_1^*}^{V} \frac{\lambda\omega(K-V)V^{-\beta_2-1}}{-\beta_1+\beta_2} dV] V^{\beta_2} & for\ V \in [V_1^*, V_2^*[\\ \\ V - K & for\ V \geq V_2^* \end{cases} \tag{25}$$

where A_2 and B_2 are constants to be determined, and $\omega \equiv \frac{2}{\sigma^2}$.

Solving the integrals, then the solution takes the form:

$$F_2(V) = \begin{cases} A_2 V^{\beta_1} + a_1(V) V^{\beta_1} + a_2(V) V^{\beta_2} & for\ V < V_1^* \\ \\ B_2 V^{\beta_1} + a_1(V_1^*) V^{\beta_1} + a_2(V_1^*) V^{\beta_2} + \bar{a_1}(V) V^{\beta_1} + \\ \bar{a_2}(V) V^{\beta_2} - [\bar{a_1}(V_1^*) V^{\beta_1} + \bar{a_2}(V_1^*) V^{\beta_2}] & for\ V \in [V_1^*, V_2^*[\\ \\ V - K & for\ V \geq V_2^* \end{cases} \tag{26}$$

where

$$a_1(x) = -\frac{\lambda\omega A_1}{\beta_1 - \beta_2} log(x)$$

$$a_2(x) = \frac{\lambda\omega A_1}{(\beta_1 - \beta_2)^2} x^{\beta_1-\beta_2}$$

$$\bar{a_1}(x) = -\frac{\lambda\omega\left(\frac{x}{1-\beta_1} + \frac{K}{\beta_1}\right)}{\beta_1 - \beta_2} x^{-\beta_1}$$

$$\bar{a_2}(x) = \frac{\lambda\omega\left(\frac{x}{1-\beta_2} + \frac{K}{\beta_2}\right)}{\beta_1 - \beta_2} x^{-\beta_2}$$

and x is either V or V_1^*

In order to simplify (26), an taking into account that $\bar{a_1}(V)V^{\beta_1} + \bar{a_2}(V)V^{\beta_2} = \lambda\omega(-\frac{V}{(\beta_1-1)(\beta_2-1)} + \frac{K}{\beta_1\beta_2})$, then we have:

$$F_2(V) = \begin{cases} A_2 V^{\beta_1} + a_1(V) V^{\beta_1} + a_2(V) V^{\beta_2} & for\ V < V_1^* \\ \\ B_2 V^{\beta_1} + a_1(V_1^*) V^{\beta_1} + a_2(V_1^*) V^{\beta_2} + \\ \lambda\omega(-\frac{V}{(\beta_1-1)(\beta_2-1)} + \frac{K}{\beta_1\beta_2}) - \bar{a_1}(V_1^*) V^{\beta_1} - \bar{a_2}(V_1^*) V^{\beta_2} & for\ V \in [V_1^*, V_2^*[\\ \\ V - K & for\ V \geq V_2^* \end{cases} \tag{27}$$

where

$$A_2 = \frac{1-\frac{\lambda\omega A_1}{\beta_2-\beta_1}V_2^{*\beta_1-1}-\frac{\beta_1\lambda\omega A_1}{(\beta_1-\beta_2)^2}V_2^{*\beta_1-1}-\frac{\beta_1\lambda\omega A_1}{\beta_2-\beta_1}log(V_2^*)V_2^{*\beta_1-1}}{\beta_1 V_2^{*\beta_1-1}}$$

$$B_2 = \frac{1-\beta_1[a_1(V_1^*)-\bar{a_1}(V_1^*)]V_2^{*\beta_1-1}-\beta_2[a_2(V_1^*)-\bar{a_2}(V_1^*)]V_2^{*\beta_2-1}+\frac{\lambda\omega}{(\beta_1-1)(\beta_2-1)}}{\beta_1 V_2^{*\beta_1-1}}$$

and V_2^* is the solution to the equation:

$$V_2^* + \lambda\omega\left(-\frac{V_2^*}{\beta_2-1}+\frac{K}{\beta_2}\right)+\frac{\lambda\omega A_1}{\beta_1-\beta_2}V_1^{*\beta_1-\beta_2}V_2^{*\beta_2}+$$
$$-\lambda\omega\left(\frac{V_1^*}{1-\beta_2}+\frac{K}{\beta_2}\right)V_1^{*-\beta_2}V_2^{*\beta_2}-\beta_1(V_2^*-K)=0 \tag{28}$$

When n=3 the investment opportunity matures at the 3^{rd} entrance in the market, and its value, $F_3(V)$, is governed by an appropriate version of (6), bounded as previously. Although a bit more complicated, a solution exists, taking the form:

$$F_3(V) = \begin{cases} A_3V^{\beta_1}+b_1(V)V^{\beta_1}+b_2(V)V^{\beta_2} & for\ V < V_1^* \\ \\ B_3V^{\beta_1}+b_1(V_1^*)V^{\beta_1}+b_2(V_1^*)V^{\beta_2}+ & \\ \bar{b_1}(V)V^{\beta_1}+\bar{b_2}(V)V^{\beta_2}-\bar{b_1}(V_1^*)V^{\beta_1}-\bar{b_2}(V_1^*)V^{\beta_2} & for\ V \in [V_1^*, V_2^*[\\ \\ C_3V^{\beta_1}+b_1(V_1^*)V^{\beta_1}+b_2(V_1^*)V^{\beta_2}+ & \\ \bar{b_1}(V_2^*)V^{\beta_1}+\bar{b_2}(V_2^*)V^{\beta_2}-\bar{b_1}(V_1^*)V^{\beta_1}-\bar{b_2}(V_1^*)V^{\beta_2}+ & \\ \lambda\omega(-\frac{V}{(\beta_1-1)(\beta_2-1)}+\frac{K}{\beta_1\beta_2})-\bar{a_1}(V_2^*)V^{\beta_1}-\bar{a_2}(V_2^*)V^{\beta_2} & for\ V \in [V_2^*, V_3^*[\\ \\ V-K & for\ V \geq V_3^* \end{cases} \tag{29}$$

where:

$$b_1(x) = -\left(\frac{\lambda^2\omega^2A_1}{(\beta_1-\beta_2)^3}+\frac{\lambda\omega A_2}{\beta_1-\beta_2}\right)log(x)+\frac{\lambda^2\omega^2A_1}{2(\beta_1-\beta_2)^2}log^2(x)$$

$$b_2(x) = \left(\frac{2\lambda^2\omega^2A_1}{(\beta_1-\beta_2)^4}+\frac{\lambda\omega A_2}{(\beta_1-\beta_2)^2}\right)x^{\beta_1-\beta_2}-\frac{\lambda^2\omega^2A_1}{(\beta_1-\beta_2)^3}log(x)x^{\beta_1-\beta_2}$$

$$\bar{b_1}(x) = \frac{\lambda\omega\left[log(x)\left[-\bar{a_1}(V_1^*)+a_1(V_1^*)+B_2\right]+\frac{x^{-\beta_1+\beta_2}(\bar{a_2}(V_1^*)-a_2(V_1^*))}{\beta_1-\beta_2}\right]}{-\beta_1+\beta_2}+\frac{\lambda^2\omega^2\left(\frac{x}{(\beta_1-1)^2(\beta_2-1)}-\frac{K}{\beta_1^2\beta_2}\right)}{-\beta_1+\beta_2}x^{-\beta_1}$$

$$\bar{b_2}(x) = \frac{\lambda\omega\left[log(x)\left[-\bar{a_2}(V_1^*)+a_2(V_1^*)\right]+\frac{x^{\beta_1-\beta_2}(-\bar{a_1}(V_1^*)+a_1(V_1^*)+B_2)}{\beta_1-\beta_2}\right]}{\beta_1-\beta_2}+\frac{\lambda^2\omega^2\left(\frac{x}{(\beta_1-1)(\beta_2-1)^2}-\frac{K}{\beta_1\beta_2^2}\right)}{\beta_1-\beta_2}x^{-\beta_2}$$

and, now, x is either V, V_1^*, or V_2^*

$$A_3 = \frac{1-\left(\frac{2\lambda^2\omega^2 A_1}{(\beta_1-\beta_2)^3}+\frac{\lambda\omega A_2}{\beta_2-\beta_1}\right)V_3^{*\beta_1-1}-\beta_1\left(\frac{2\lambda^2\omega^2 A_1}{(\beta_1-\beta_2)^3}+\frac{\lambda\omega A_2}{\beta_2-\beta_1}\right)log(V_3^*)V_3^{*\beta_1-1}}{\beta_1 V_3^{*\beta_1-1}}+$$

$$+\frac{-\beta_1\left(\frac{2\lambda^2\omega^2 A_1}{(\beta_1-\beta_2)^4}+\frac{\lambda\omega A_2}{(\beta_2-\beta_1)^2}\right)V_3^{*\beta_1-1}-\frac{\beta_1\lambda^2\omega^2 A_1}{2(\beta_1-\beta_2)^2}log^2(V_3^*)V_3^{*\beta_1-1}-\frac{\lambda^2\omega^2 A_1}{(\beta_1-\beta_2)^2}log(V_3^*)V_3^{*\beta_1-1}}{\beta_1 V_3^{*\beta_1-1}}$$

$$B_3 = \frac{1-\beta_1\left[b_1(V_1^*)-\bar{b_1}(V_1^*)\right]V_3^{*\beta_1-1}-\beta_2\left[b_2(V_1^*)-\bar{b_2}(V_1^*)\right]V_3^{*\beta_2-1}}{\beta_1 V_3^{*\beta_1-1}}+$$

$$+\frac{\lambda\omega\left[\frac{\lambda\omega(-\beta_1+\beta_2)}{(\beta_1-1)^2(\beta_2-1)^2}+V_3^{*\beta_1-1}a_1(V_1^*)\left(log(V_3^*)\beta_1+\frac{\beta_2}{\beta_2-\beta_1}\right)+\frac{V_3^{*\beta_1-1}\bar{a_1}(V_1^*)\left(\beta_2+log(V_3^*)\beta_1(-\beta_1+\beta_2)\right)}{\beta_1-\beta_2}\right]}{\beta_2-\beta_1}+$$

$$+\frac{\frac{\lambda\omega\left[-B_2 V_3^{*\beta_1}\left(\beta_2+log(V_3^*)\beta_1(-\beta_1+\beta_2)\right)+V_3^{*\beta_2}\left(\bar{a_2}(V_1^*)-a_2(V_1^*)\right)\left(-log(V_3^*)\beta_2^2+\beta_1\left(1+log(V_3^*)\beta_2\right)\right)\right]}{V(\beta_1-\beta_2)}}{\beta_2-\beta_1}$$

$$C_3 = \frac{1-\beta_1\left[b_1(V_1^*)+\bar{b_1}(V_2^*)-\bar{b_1}(V^*1)-a_1(V_2^*)\right]V_3^{*\beta_1-1}}{\beta_1 V_3^{*\beta_1-1}}+$$

$$+\frac{-\beta_1\left[b_2(V_1^*)+\bar{b_2}(V_2^*)-\bar{b_2}(V_1^*)-a_2(V_2^*)\right]V_3^{*\beta_1-1}+\frac{\lambda\omega}{(\beta_1-1)(\beta_2-1)}}{\beta_1 V_3^{*\beta_1-1}}$$

The trigger value V_3^* is the solution to the equation:

$$V_3^*+(\beta_1-\beta_2)\left[b_2(V_1^*)+\bar{b_2}(V_2^*)-\bar{b_2}(V_1^*)-a_2(V_2^*)\right]V_3^{*\beta_2}+$$
$$+\lambda\omega\left(-\frac{V_3^*}{\beta_2-1}+\frac{K}{\beta_2}\right)-\beta_1(V_3^*-K)=0 \qquad (30)$$

The general solution for n=N, takes the form

$$F_N(V) = \begin{cases} A_N V^{\beta_1} + [\int f_{1,1}(V)dV]V^{\beta_1} + [\int f_{1,2}(V)dV]V^{\beta_2} & for\ V < V_1^* \\ \\ B_N V^{\beta_1} + [\int_0^{V_1^*} f_{1,1}(V)dV]V^{\beta_1} + [\int_0^{V_1^*} f_{1,2}(V)dV]V^{\beta_2} + \\ [\int_{V_1^*}^{V} f_{2,1}(V)dV]V^{\beta_1} + [\int_{V_1^*}^{V} f_{2,2}(V)dV]V^{\beta_2} & for\ V \in [V_1^*, V_2^*[\\ \\ C_N V^{\beta_1} + [\int_0^{V_1^*} f_{1,1}(V)dV]V^{\beta_1} + [\int_0^{V_1^*} f_{1,2}(V)dV]V^{\beta_2} + \\ [\int_{V_1^*}^{V_2^*} f_{2,1}(V)dV]V^{\beta_1} + [\int_{V_1^*}^{V_2^*} f_{2,2}(V)dV]V^{\beta_2} + \\ [\int_{V_2^*}^{V} f_{3,1}(V)dV]V^{\beta_1} + [\int_{V_2^*}^{V} f_{3,2}(V)dV]V^{\beta_2} & for\ V \in [V_2^*, V_3^*[\\ \\ D_N V^{\beta_1} + [\int_0^{V_1^*} f_{1,1}(V)dV]V^{\beta_1} + [\int_0^{V_1^*} f_{1,2}(V)dV]V^{\beta_2} + \\ [\int_{V_1^*}^{V_2^*} f_{2,1}(V)dV]V^{\beta_1} + [\int_{V_1^*}^{V_2^*} f_{2,2}(V)dV]V^{\beta_2} + \\ [\int_{V_2^*}^{V_3^*} f_{3,1}(V)dV]V^{\beta_1} + [\int_{V_2^*}^{V_3^*} f_{3,2}(V)dV]V^{\beta_2} + \\ [\int_{V_3^*}^{V} f_{4,1}(V)dV]V^{\beta_1} + [\int_{V_3^*}^{V} f_{4,2}(V)dV]V^{\beta_2} & for\ V \in [V_3^*, V_4^*[\\ \\ (...) \\ \\ \psi_N V^{\beta_1} + [\int_0^{V_1^*} f_{1,1}(V)dV]V^{\beta_1} + [\int_0^{V_1^*} f_{1,2}(V)dV]V^{\beta_2} + \\ [\int_{V_1^*}^{V_2^*} f_{2,1}(V)dV]V^{\beta_1} + [\int_{V_1^*}^{V_2^*} f_{2,2}(V)dV]V^{\beta_2} + \\ [\int_{V_2^*}^{V_3^*} f_{3,1}(V)dV]V^{\beta_1} + [\int_{V_2^*}^{V_3^*} f_{3,2}(V)dV]V^{\beta_2} + \\ [\int_{V_3^*}^{V_4^*} f_{4,1}(V)dV]V^{\beta_1} + [\int_{V_3^*}^{V_4^*} f_{4,2}(V)dV]V^{\beta_2} + \\ ...+ \\ [\int_{V_{N-1}^*}^{V} f_{N,1}(V)dV]V^{\beta_1} + [\int_{V_{N-1}^*}^{V} f_{N,2}(V)dV]V^{\beta_2} & for\ V \in [V_{N-1}^*, V_N^*[\\ \\ V - K & for\ V \geq V_N^* \end{cases} \tag{31}$$

where A_N, B_N, D_N,..., ψ_N are constants to be determined, as well as V_N^*, using the boundaries (as previously), and $f_x(V)$ are properly chosen functions of V.

3 Numerical Example

In this section we present a hypothetical example, in order to implement the methodology. Let V=100; K=80; λ=0,40; 0,25; 0,15; 0,05; 0,01; 0,001; σ=0,25; r=0,03 and δ=0,06. Simulating the available "‘places"’ in the market from 1 to 3, the values for the investment opportunity and for the optimal timing are as follows:

	N=1	N=2	N=3	PAO*
λ=0,40	20,017	23,289	24,468	27,072
λ=0,25	20,442	23,994	25,235	27,072
λ=0,15	21,353	24,893	25,999	27,072
λ=0,05	23,805	26,399	26,888	27,072
λ=0,01	26,157	27,021	27,068	27,072
λ=0,001	26,971	27,071	27,072	27,072
PAO*	27,072	27,072	27,072	

Table 1: The Real Options Value for several probabilities of entrance and available "places" in the market (* PAO stands for Perpetual American Options).

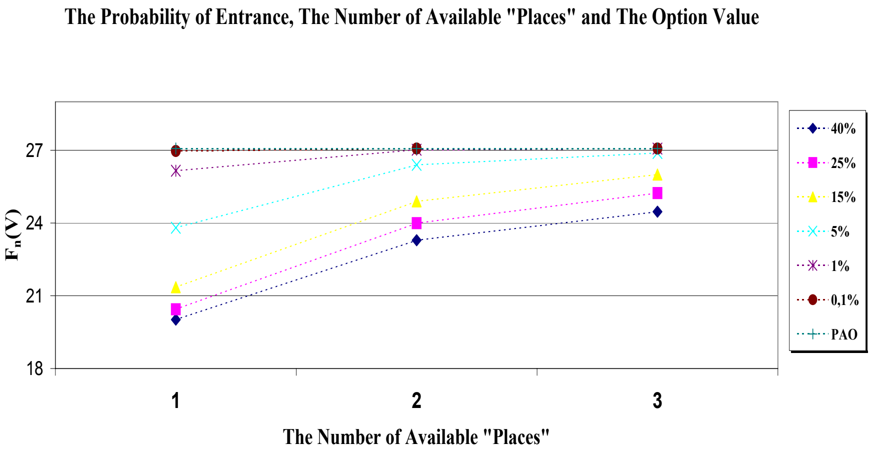

Figure 1: The probability of entrance, the number of available "places", and the IO's value.

	N=1	N=2	N=3	PAO*
λ=0,40	100,960	122,094	128,329	138,574
λ=0,25	105,562	126,477	131,934	138,574
λ=0,15	111,100	130,632	134,919	138,574
λ=0,05	123,097	136,303	137,979	138,574
λ=0,01	134,171	138,406	138,562	138,574
λ=0,001	138,085	138,572	138,574	138,574
PAO*	138,574	138,574	138,574	

Table 2: The Optimal Timing to invest for several probabilities of entrance and available "places" in the market (* PAO stands for Perpetual American Options)

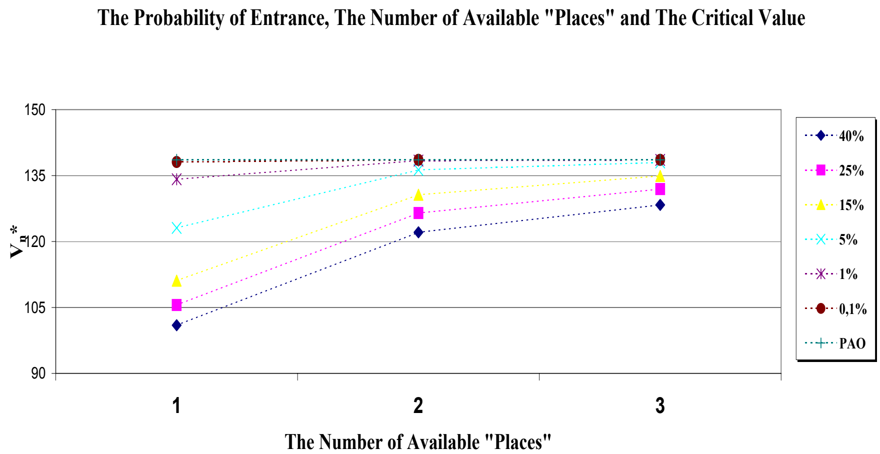

Figure 2: The probability of entrance, the number of available "places", and the critical value.

Table 1 and Figure 1, presented above, show the value of an investment opportunity, simulating the number of available "places" in the market, and the competitor's probability of entrance. The results show, as expected, that the value of the investment opportunity decreases as the probability of a competitor's entrance increases or (and) the number of available places in the market decreases. That's not surprising because competitors' entrances reduce the chances for investing, so the IO should be less valuable. On contrary, the value of the IO tends to the value of a Perpetual American Options (PAO) as $N \to +\infty$, or (and) $\lambda \to 0$. The convergence to the value of the PAO is slower for higher values of λ.

Similar conclusions can be taken about the optimal timing (see Table 2 and Figure 2). The critical value decreases as the number of available "places" in the market decreases (that's what we would expect, because the *random maturity* is less distant), or (and) decreases as the probability of entrance increases (both due to the lower opportunity cost from deferring). As above, the trigger value tends to the trigger value of a American exchange options (PAO) as $N \to +\infty$, or (and) $\lambda \to 0$.

4 Conclusions and Future Research

This paper investigates the value and timing of investment opportunities (IO) that are *shared* by several companies, in a competitive environment. By assuming that the market can accommodate a finite number of firms (N), which are assumed to enter the market stochastically, we define that the IO matures with the entrance of the last "admitted" company, meaning that, at that moment, the option to invest disappears. Since companies can invest anytime up to, but not including, the *random* maturity date, it's easy to see the importance of the determination of the optimal timing to invest in this context.

An example showed that the value of the IO, and its trigger value for investing, decreases as the available "places" in the market decreases, or (and) as the probability of a competitor entrance increases; also shows that the IO's value and the optimal timing tend to the value and to the optimal timing of a perpetual American option as N tends to infinity or (and) the probability of a competitor entrance tends to zero.

At the moment, we are working on three extensions to the basic model, so that it will be even closer to reality, as follows:

(i) Incorporate the uncertainties of the investment cost (K), allowing it to evolve stochastically.

(ii) Relax the assumption that the market is totally "homogeneous" -see assumption (v)- since the first entrants may have some advantages against the others (by choosing the best places/locations, for example);

(ii) Finally, the last extension will incorporate the possibility of both entrances and exits from the market (N following a birth and death process). In fact, an installed firm may decide (or be "forced") to leave the market, creating a new available "place".

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