The Optimal Value of Waiting to Invest

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Abstract

We extend McDonald and Siegel's (1986) model to the case where the expected rate of return of a project cannot be observed but is known to be either low or high. Waiting and observing the realizations of the value of the project provides information to the investor who can update her beliefs about the true value of the expected return. Moreover, the investor can purchase some additional information which allows her to control the learning speed of her beliefs. Investor's beliefs follow a martingale and the optimal investment trigger depends on the degree of optimism. The optimal amount of information purchased is found to be increasing in payoffs and the variance of beliefs but decreasing in the project volatility. Overall, the opportunity to purchase information enhances the option value of waiting to invest, thus delays investment.

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1. INTRODUCTION

McDonald and Siegel (1986) were among the first to study the implications of irreversibility on the timing of investment decisions under uncertainty. Since then, an extensive literature in real options has emphasized the benefits from delaying an irreversible investment. When the payoffs of an irreversible investment are stochastic, the investor has an option and when investing she chooses to kill her option. This implies that at the optimal date for investing the present discounted value of future cash-flows exceeds the investment cost by the option value, the marginal benefits of investing being equal to the marginal cost of investing *and* giving up the option. For more details, the reader can refer to the seminal book, Investment under Uncertainty, by Dixit and Pindyck (1994) which represents a comprehensive review on real options.

The traditional way of introducing uncertainty is to assume that the value of the project (or some other economic indicator) follows a given (stochastic) law of motion *known* by the potential investor. Hence, uncertainty arises from shocks that affect the value of the project. Irreversibility implies that the optimal decision is to wait until the value of the project hits a fixed threshold (investment trigger) which is higher than the cost of investing. However, in many real life investment opportunities, the characteristics of a project are hardly known with perfect accuracy. Applications include investments in new and unfamiliar markets (joint ventures), research and development, new start-up companies. Sometimes it is possible to gather information about the project for instance by hiring the services of a consultancy. The objective of this paper is to introduce incomplete information into the traditional real option framework and to allow the investor to costly control her learning speed about some non-observable characteristics of a project.

1.1. Related Literature

This paper is at the crossroad of two literatures: Real options and the value of information. McDonald and Siegel (1986) point out that a firm should not invest as soon as the value of the project equals the cost of investing I but postpone its investment decision until the value exceeds a threshold that can actually be much larger than I. They call this effect "The value of waiting to invest". In this paper, we give an active role to the investor and aim at characterizing *the optimal value of waiting to invest* as well as the optimal strategy for purchasing information. Some of the central issues of this paper are related to the work by Bernanke (1983) who highlights that only unfavorable outcomes actually matter for the decision to undertake or postpone an investment. In other words, the distribution of payoffs is truncated and actually, only the left tale of the distribution is to be considered. He calls this effect the "bad news principle of irreversible investment". In order to allow uncertainty to be periodically renewed, he develops a discrete time and finite horizon model using a cartel framework where investment decisions depend on the arrival of information. He shows that a short-run cycle in investment may occur, as agents choose to wait for some new information leading to a trough in investment and latter decide to use the stored-up investment capacity, creating a boom in investment. Most of the papers in the literature examine the case where the investor is passive in the sense that she cannot undertake some action while waiting. For instance, Venezia (1983) considers a firm that can sell an asset whose mean value is unknown, but the value of the asset is observable. Given some initial normally distributed priors about the mean, the posterior distribution is obtained using Bayes' rules. Venezia obtains that a Bayesian manager has more incentive to keep the asset than a manager who does not revise her beliefs using new arriving information. Thus, the former has a higher reservation price than the latter. Bernardo and Chowdhry (2002) consider the case of a firm that can learn about its own resources and decide to either to exit the market (i), scale up its existing business (ii) or diversify its activities (iii) in an irreversible fashion. They show that the firm chooses options (i) or (ii) if it accesses its resources to be low enough. Conversely the firm chooses option (iii) when resources are thought to be high enough. In between, the firm keeps on experimenting. Roche (2002) investigates the case where the investor does not have access to an additional source of information which corresponds in this paper to a purchasing strategy where $\alpha \equiv 0$. He obtains that the option value of a Bayesian investor can be above or below the option value of a non-updating investor, depending on the beliefs and the value of the project. However, for a given belief, the non-updating investor always requires a higher wedge than her Bayesian fellow to trigger investment. Demers (1991) considers a risk neutral firm that is uncertain about the state of demand and updates its beliefs using Bayes' rule. He shows that irreversibility and anticipation of receiving information signal in the future while eventually learning the true state of nature lead to a more cautious investment behavior than under complete information. Cukierman (1980) investigates how a risk neutral firm selects projects among several investment opportunities. Increased uncertainty causes a decrease in the current level of investment by making it more profitable to wait longer for more information before choosing an investment project.

Recently, scholars have focused their attention to endogenous acquisition of information. Massa (2002) develops a general equilibrium model of asset pricing in which some uninformed agents can purchase outside market information and then choose their optimal portfolio-consumption rules. In this framework, agents have normally distributed beliefs about the non-observable dividend growth rate. Massa investigates the impact of the introduction of financial innovation and shows that the effects on the amount of information purchased and the resulting asset allocation depend on the market informational structure (high versus low informational asymmetry) which dynamically evolves across time. In this article, we characterize the value of the information and focus on the impact of optimism - defined as the beliefs that the investment project is "good", on an irreversible investment decision. Our information background is a continuous-time model of Bayesian learning a la Bolton and Harris (1993) where the decision maker knows that the non-observable parameter is a constant that can only be two possible values. In their paper, Bolton and Harris (1999) derive the shadow value of experimenting. In our paper, this shadow value is interpreted as the value of time of waiting to learn. Using the a similar information environment, Moscarini and Smith (2001) consider the case of a decision maker who can buy some information to improve the precision of a signal before undertaking some action. They show that the optimal experimentation level increases with a project's expected payoff. In the model proposed in this paper, we combine McDonald and Siegel (1986) and Moscarini and Smith (2001) frameworks. Therefore, the problem studied here has two dimensions: the value of the project and the beliefs. The payoffs of exerting the option are those of McDonald and Siegel (1986); Beliefs and the cost of information are similar to Moscarini and Smith (2001). Finally, martingale techniques used in this paper are similar to these presented in Cuoco and Zapatero (2000). They allow us to deal with the change of beliefs within a dynamic framework to derive some key properties about the option value and the optimal investment trigger frontier.

1.2. Results

The main contribution of the paper is to clarify the effects of learning and uncertainty on irreversible investment decisions in presence of incomplete information. We consider the case of a risk neutral investor who is discounting future at a constant rate r and knows that the expected return of the project μ can only take two values, h in which case the project is good or l, in which case the project is bad, with $0 < l < h < r$. At time t, we denote by $p(t)$ the probability (beliefs) that the project is good, i.e., $\mu = h$. By waiting, the investor can observe the realizations of the value V of the project and has a costly access to an additional source of information. The investor can thus control her learning speed and update her beliefs about the nature (good or bad) of the potential investment.

We first prove the existence of a solution to the optimal control program by transforming the original investor's program into an equivalent program that satisfies some sufficient conditions for existence as exposed in Fleming and Soner (1993). Then we derive some analytical properties for the option value and the optimal trigger investment frontier. In particular, this latter lies above the one that prevails when no additional information can be purchased. Hence, the access to an additional signal enhances the value of waiting to invest, thus delaying investment. We rely on numerical simulations to investigate the impact of beliefs on the optimal amount of information purchased. Since the Bellman equation of the program is non-linear, we use an iterative method that allows us to easily incorporate the free boundary condition. The optimal amount of information purchased is found to be increasing in payoffs and in the degree of ignorance (variance of beliefs). Finally, the volatility of the project has two opposite effects on the option value. The direct effect, as in the case of complete information, is to increase the value of waiting; the indirect one is to reduce the learning speed and therefore, to decrease the value of waiting is. Overall, numerical simulations show that the direct effect dominates the indirect one. The indirect effect also suggests that since a higher project volatility damages learning, incentives to purchase information should be reduced, which is confirmed by the numerical simulations undertaken.

The paper is organized as follows. Section 2 describes the economic setting and provides some analytical results on the option value and the optimal investment trigger frontier. Section 3 displays some numerical simulations and in particular we analyze the optimal amount of information purchased. Section 4 concludes. Proofs of all results are collected in the appendix.

2. THE ECONOMIC SETTING

We consider a standard irreversible investment problem. Time is continuous; a firm has to choose optimally the timing of its investment under uncertainty and decide the optimal amount of information to purchase to improve its knowledge about the project. The main innovation of the paper lies in the fact that the average growth rate (drift) of the project is unknown and waiting provides some information whose amount can be costly controlled by the decision maker.

2.1. Investment Opportunity and Information Structure

Uncertainty is modeled by a probability space $(\Omega, \mathcal{F}, P^w)$ on which is defined a two dimensional (standard) Brownian motion $w = (w_1, w_2)$. A state of nature ω is an element of Ω. $\mathcal{F}$ denotes the tribe of subsets of Ω that are events over which the probability measure P^w is assigned.

A risk neutral investor has to choose when to invest into a project whose value V fluctuates across time according to a geometric Brownian motion

$$dV(t) = V(t)\left(\mu dt + \sigma dw_1(t)\right),$$

where $dw_1(t)$ is the increment of a standard Wiener process, μ represents the average growth rate of the value of the project and σ captures the magnitude of the uncertainty. The parameter μ is *unknown* to the investor but the latter knows that μ can only take two values, h (high) or l (low). The investment is irreversible with cost $I > 0$ and the risk-free rate $r > 0$ is a constant. For technical reasons, we assume that $r > h > l > 0$.

Even though an investor does not observe the true value for μ, she can observe the value of the project V which is a noisy signal and therefore infer the true value for the drift. In addition, she has access to a costly signal A whose law of motion is given by

$$dA(t) = A(t)\left(\lambda dt + \Sigma_1\sqrt{\frac{\alpha(t)}{1+\alpha(t)}}dw_1(t) + \Sigma_2\sqrt{\frac{1}{1+\alpha(t)}}dw_2(t)\right),$$

where $\Sigma_1 > 0$, $\Sigma_2 > 0$ and λ are known parameters and $dw_2(t)$ is the increment of a standard Wiener process independent from $dw_1(t)$. α $(t \geq 0)$ in $\mathbb{R}_+$ represents the quantity of information the investor can buy. The higher α, the higher the correlation between the signal A and the project V, thus the more informative the signal is. The cost function c is assumed to be increasing ($c' > 0$), strictly convex ($c'' > 0$), twice-continuously differentiable with

$$c(0) = c'(0) = 0, \quad \lim_{\alpha\to\infty} c'(\alpha) = \infty.$$

In the sequel, we use

$$c(\alpha)=\frac{\alpha^{1+n}}{1+n},\ n>0.$$

Let $\mathcal{F}_t$ be the σ-algebra generated by the observations of the value of the project and the signal, $\{V(s),A(s);$ for some strategy $\alpha(s),0\leq s\leq t,\}$ and augmented. At time t, the investor's information set is $\mathcal{F}_t$. The filtration $\mathbb{F}=\{\mathcal{F}_t,\ t\in\mathbb{R}_+\}$ is the information structure and satisfies the usual conditions (increasing, right-continuous, augmented). At time t, the control $\alpha=\alpha_t(\omega)(t\geq 0)$ is restricted to be a progressively measurable function with respect to $\mathcal{F}_t$ and depends on the past histories $\{V(s),A(s);$ for some strategy $\alpha(s),0\leq s\leq t,\}$. Owing to the Markovian structure of the problem, the control α is actually a function of the state variables (V,p). At time t let $p(t)$ be the probability or the investor's belief that μ equal h, .i.e., $p(t)=\Pr(\mu=h\mid\mathcal{F}_t)$. The evolution across time of the posterior probability p is given by the following lemma.

Lemma 2.1. *The law of motion of the posterior belief P is given by*

$$dp(s)=\frac{h-l}{\sigma}(1-p(s))p(s)\left(d\overline{w}_1(s)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(s)\right),$$

where

$$\begin{aligned}
d\overline{w}_1(s)&=dw_1(s)+\frac{1}{\sigma}\left(\mu-(p(s)h+(1-p(s))l)\right)ds\\
d\overline{w}_2(s)&=dw_2(s)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2\sigma}\left(\mu-(p(s)h+(1-p(s))l)\right)ds,
\end{aligned}$$

are the increment of two independent (standard) Wiener processes under the probability measure P, relative to the filtration $\mathbb{F}$.

Proof. See appendix 1. ■

Changes in beliefs are increasing in the wedge $h-l$: when the two average growth rates differ significantly more information can be obtained. The qualify of the information can be divided in two parts. On the one hand, when the project has a high variance σ of the project reduces the information that can be extracted from the observations of the realizations of V. On the other hand, the greater the amount of information α purchased, the larger the changes in beliefs. When the investor is almost certain of the value of μ (p close to 0 or 1), little information can be extracted and therefore beliefs do not change much. Finally, note that p is a martingale under $\mathbb{F}$ so on average, the investor's belief does not change.

Let P_h and P_l be the probability measures corresponding to the process V when $\mu=h$ and $\mu=l$, respectively. For $\mu\in\{l,h\}$, define two new processes $\gamma^\alpha_{p,\mu}$ and $\xi^\alpha_{p,\mu}$ such that

$$\gamma^\alpha_{p,\mu}=\left(\frac{\mu-(ph+(1-p)l)}{\sigma},-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2\sigma}\left(\mu-(ph+(1-p)l)\right)\right)$$

and

$$\xi^{\alpha}_{p,\mu}(t) = \exp\left(-\int_0^t \gamma^{\alpha}_{p,\mu}(s)^{\top} dw(s) - \frac{1}{2}\int_0^t \left|\gamma^{\alpha}_{p,\mu}(s)\right|^2 ds \right).$$

$\xi^{\alpha}_{p,\mu}$ is the density process of the Randon-Nikodym derivative[1] of P with respect to P_μ, i.e.,

$$\xi^{\alpha}_{p,\mu}(t) = \frac{dP(t)}{dP_\mu(t)}.$$

It can be shown that when $\mu = h$, then $\xi^{\alpha}_{p,h}(t) = \frac{1}{p(t)}$ and when $\mu = l$, then $\xi^{\alpha}_{p,l}(t) = \frac{1}{1-p(t)}$. Finally, define $\phi^{\alpha}_{p,\mu}(t) = \xi^{\alpha}_{p,\mu}(t) - 1$ and it can be easily checked that

$$\begin{aligned} d\phi^{\alpha}_{p,h}(t) &= -\frac{h-l}{\sigma}\phi^{\alpha}_{p,h}(t)\left(dw_1(t) - \sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}dw_2(t)\right) \\ d\phi^{\alpha}_{p,l}(t) &= \frac{h-l}{\sigma}\phi^{\alpha}_{p,l}(t)\left(dw_1(t) - \sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}dw_2(t)\right). \end{aligned}$$

$\phi^{\alpha}_{p,h}$ and $\phi^{\alpha}_{p,l}$ are therefore geometric Brownian motions and thus easy to deal with. Finally, in the sequel we will use the following identity: If $E^P[. \mid \mathcal{F}_t]$ denotes the conditional expectation at time t under the investor's beliefs P, for all adapted process X, we have

$$E^P[X(s) \mid \mathcal{F}_t] = p(t)E[X(s) \mid \mathcal{F}_t,\ \mu = h] + (1-p(t))E[X(s) \mid \mathcal{F}_t,\ \mu = l]$$

Before describing the program of an investor's making use of the information, we examine the benchmark case of a non-Bayesian investor who never changes her initial beliefs.

2.2. Benchmark Case: Complete Information

We start by recalling the complete information case where $p = 0$ or $p = 1$. This case has been studied extensively in the literature (see for instance Dixit and Pindyck, 1994 chapter 6, p 180-185.). We briefly recall some of the major results.

For $\mu \in \{l, h\}$, let β_μ be the positive roots of the quadratic

$$\frac{\sigma^2}{2}x^2 + \left(\mu - \frac{\sigma^2}{2}\right)x - r = 0.$$

Notice that $\beta_\mu > 1$ since $r > h > l$. When μ is known and equal to l (respectively h), then $p = 0$ (respectively $p = 1$). The option value is given by

$$\begin{aligned} F^{\mu}(V) &= A_\mu V^{\beta_\mu} \text{ for } V \le V^*_\mu \\ &= V - I \text{ for } V \ge V^*_\mu \end{aligned}$$

with

$$V^*_\mu = \frac{\beta_\mu}{\beta_\mu - 1}I$$

[1] If A is a $m \times n$ matrix, the notation $|A|$ refers to the norm of A with $|A| = \sqrt{Tr(A^{\top}A)}$.

and

$$A_\mu = \frac{1}{\beta_\mu}(V_\mu^*)^{1-\beta_\mu}.$$

In the next section, we describe the investor's program and derive some properties about the value function and the optimal trigger investment frontier.

2.3. Investor's Problem

At time $t = 0$, given the observations of the value of the project V and the signal A her beliefs p, an investor has to choose an optimal stopping time τ and an optimal purchasing strategy α in order to maximize the benefits of investing, i.e.,

$$F(V,p) = \sup_{\alpha\geq 0, \tau \geq 0} E^P \left[- \int_0^\tau c(\alpha(t))e^{-rt}dt + (V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \right] \tag{2.1}$$

$$\begin{aligned}
\text{s.t. } dV(s) &= V(s)\left((p(s)h + (1-p(s))l)ds + \sigma d\overline{w}_p(s)\right) \\
dp(s) &= \frac{h-l}{\sigma}(1-p(s))p(s)\left(d\overline{w}_1(s) - \sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(s)\right) \\
V(0) &= V \text{ and } p(0) = p.
\end{aligned}$$

2.3.1. Existence and characterization of a solution

Due to the bi-dimensional nature of the problem, we cannot follow Moscarini and Smith (2001) to show existence. Details of the existence of a solution to program (2.1) are displayed in appendix 2. The proof relies on optimal control theory as exposed in Fleming and Soner (1993). We first transform the original program into an equivalent program using a change of probability measure. Then, we check that new program satisfies some sufficient conditions for existence. The investor has two distinct decisions to take at each moment: she can wait and acquire more information or she can exert her option. More precisely, we can define the inaction region IR as

$$IR = \{(t,V,p); F(V,p) > V - I\}.$$

As proved in appendix 3, the inaction region has the following shape

$$IR = \{(t,V,p); 0 < V < V^*(p)\},$$

where V^* is a function of p to be characterized in the sequel. Note that IR is non-empty since for all p in $[0,1]$, $(0,p)$ belongs to IR. Hence, for any (V,p) inside the inaction region IR, the Hamilton-Jacobi-Bellman (HJB) equation is

$$rF(V,p)dt = \sup_{\alpha \geq 0} E^P\left[dF(V,p) \mid \mathcal{F}_t\right] - c(\alpha)dt.$$

Using Ito lemma leads to the following expression for the HJB equation

$$\begin{aligned} rF(V,p) &= \sup_{\alpha\geq 0}\left\{V(ph+(1-p)l)F_1(V,p)+\frac{\sigma^2}{2}V^2F_{11}(V,p)+V(h-l)p(1-p)F_{12}(V,p)\right.\\ &\quad\left.+\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2\left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right)F_{22}(V,p)-c(\alpha)\right\}. \end{aligned}$$

The initial condition is $F(0,p)=0$ for all p and the value-matching and smooth pasting (free boundary) conditions are

$$\begin{aligned} F(V^*(p),p) &= V^*(p)-I \\ \nabla F(V^*(p),p) &= (1,0), \end{aligned}$$

where $V^*(p)$ denotes the investment trigger value given the investor's beliefs p and $\nabla F = (F_1,F_2)$ is the gradient of F.

2.3.2. Interpretation of the Value Function

As usual, the return of investing an amount $F(V,p)$ into a safe asset must be equal to the optimal expected capital gain from waiting (since no dividend is paid). The interpretation of the terms of the HJB goes as follows. The first two terms are the usual ones (given a fixed value for p) and represent the optimal expected change in the option value as V varies. Appearing in the last term, $\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2\left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right)$ is a measure of informativeness, and $\frac{1}{r}F_{22}(V,p)$ is the shadow price of information. Given a strategy α, the net gain from waiting is

$$\frac{1}{2r}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2\left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right)F_{22}(V,p)-\frac{c(\alpha)}{r}.$$

In particular, if $h-l$ is small, σ is large or p is close to 0 or 1, the gain from waiting is small. On the contrary, the informativeness is maximal when $p=\frac{1}{2}$, i.e., when the investor is very confused about the true value of the drift μ and increases with the amount of information purchased α. The first order condition in the HJB is

$$c'(\alpha^*)=\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2(p(1-p))^2\frac{\Sigma_1^2}{\Sigma_2^2}F_{22}(V,p),$$

and the SOC is met since $c''>0$. It is obvious that when the investor knows the true value of drift, she will not purchase any information so

$$\alpha^*(V,0)=\alpha^*(V,1)=0 \text{ for all } V.$$

The median term in the equation $V(h-l)p(1-p)F_{12}(V,p)$ represents the correlation between the project and the beliefs. The sign of the cross derivative F_{12} is somewhat difficult to predict. Nevertheless, one can note that in the case where the drift μ is known, the marginal value of the option is decreasing in μ. When p increases, this somehow corresponds to a rise in the perceived value of the drift. This intuitive reasoning leads us to think that F_{12} must be negative.

One can realize that the magnitude of the uncertainty σ now plays an ambiguous role. On the one hand, an increase in σ rises the option value as in the classical case. On the other hand, when σ increases, less information can be extracted from the observations of V and therefore, it lowers the option value by decreasing the amount of informativeness.

Example When the cost function is given by $c(\alpha) = \frac{1}{n+1}\alpha^{n+1}$, $n > 0$, the optimal condition is

$$(\alpha^*)^n = \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2 \left(\frac{\Sigma_1}{\Sigma_2}\right)^2 F_{22}(V,p).$$

Hence

$$\alpha^* = \left(\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(\frac{\Sigma_1}{\Sigma_2}\right)^2 (p(1-p))^2 F_{22}(V,p)\right)^{\frac{1}{n}}.$$

For a fixed shadow price of information $\frac{1}{r}F_{22}(V,p)$, we see that the optimal amount of information purchased is hump shape and maximal at $p = \frac{1}{2}$ when there is a lot to be learned. Numerical simulations will confirm this intuition. The corresponding HJB is

$$\begin{aligned} rF(V,p) &= V(ph+(1-p)l)F_1(V,p) + \frac{\sigma^2}{2}V^2F_{11}(V,p) + V(h-l)p(1-p)F_{12}(V,p) \\ &\quad + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2 F_{22}(V,p)\left(1 + \frac{n\left(\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2\left(\frac{\Sigma_1}{\Sigma_2}\right)^2 (p(1-p))^2 F_{22}(V,p)\right)^{\frac{1}{n}}}{n+1}\right). \end{aligned}$$

2.3.3. Some properties of the Option Value and investment trigger frontier

In this paragraph, we derive some useful properties about the option value F and the optimal investment trigger frontier V^*. The proofs are reported in appendix 2.

Property 1 F is strictly increasing and convex in its first argument and $F(V,0) \leq F(V,p) \leq F(V,1)$. This implies that for $V \geq V^*(1)$, $F(V,p) = V - I$, for all $p \in [0,1]$. It follows that given p, $V^*(p)$ is uniquely defined with $V^*(0) \leq V^*(p) \leq V^*(1)$, for all $p \in [0,1]$.

Property 2 If at some date t, $p'(t) > p(t)$, then for all $s \geq t$, $p'(s) \geq p(s)$ for some strategy α: if one investor is more optimistic than a second investor, she will always remain more optimistic provided that the two investors follow the same strategy α.

Property 3 F is non-decreasing and strictly convex in its second argument; Optimism increases the option value and information is always valuable.

Property 4 The optimal investment trigger frontier V^* is non-decreasing in p. An optimistic investor requires a higher trigger value as she thinks that her option value is higher.
Proof. See appendix 3. ■

From property 1, we can conclude that the optimal stopping time τ^* is less than the stopping time τ' it takes the process $V_0 \exp\left((l - \frac{\sigma^2}{2})t + \sigma\overline{w}_1(t)\right)$ to reach $V^*(1)$ starting from $V_0 < V^*(1)$. Since $E[\tau'] = \frac{\ln \frac{V^*(1)}{V_0}}{l - \frac{\sigma^2}{2}} < \infty$, it follows that $E[\tau^*] < \infty$, so $P(\tau^* < \infty) = 1$.

One particular strategy is $\alpha \equiv 0$ as studied in Roche (2002). Obviously this strategy may not be optimal so for all (V, p) in $\mathbb{R}_+ \times [0, 1]$

$$F(V, p) \geq F(V, p; \alpha \equiv 0).$$

This implies that the investment trigger frontier V^* must be above the one that prevails when no additional information can be purchased. Allowing the investor to collect information enhances her option value and therefore postpones the investment decision. In addition, it is easy to see that the more expensive it is to acquire information, the lower is the option value.

In the next section, we display numerical simulations about the option value and the optimal purchasing strategy choosing as a benchmark the case when no additional information is available.

3. COMPARATIVE STATICS AND NUMERICAL SIMULATIONS

In this section, we choose the following analytical expression for the cost function

$$c(\alpha) = \frac{1}{n+1}\alpha^{n+1}, \ n > 0.$$

3.1. Benchmark Case: No Purchasing $\alpha \equiv 0$

In this case, the HJB equation is linear. We use a finite difference approach to compute numerically the option value F. For (V, p) in $[0, V^*(1)] \times [0, 1]$, we discretize the HJB equation writing $V = i\Delta V$ and $p = j\Delta p$ for $(i, j) \in [1, N_V] \times [1, N_p]$. Then, by re-indexation, $k = (i-1)N_p + j$, we convert the problem into solving a $N = N_V \times N_p$ linear system of the type

$$AF = B,$$

where A is a $N \times N$ square matrix, B is a $N \times 1$ vector incorporating the boundary conditions $F(0, p)$, $F(V, 0)$, $F(V, 1)$ and $F(V^*(1), p) = V^*(1) - I$. The free boundary condition is dealt with by using successive over-relaxations (SOR), where at each iteration, we check that the value obtained for the option value is above the corresponding payoff of exerting the option. If not, we replace the computed value by the corresponding payoff. One drawback with this method is that it requires to solve a linear system whose size grows very quickly with the degree of precision desired. The main advantage is that we obtain *all* the values for F. Results are presented in table I.

TABLE I

Option Value F when $\alpha \equiv 0$

V	0	0.34	0.68	1.02	1.36	1.7	2.04	2.38	2.72	3.06	3.4	3.74	4.075
$p=0$	0	0.003	0.023	0.124	0.340	0.670*	1.004	1.338	1.672	2.006	2.340	2.674	3.075
$p=\frac{1}{4}$	0	0.028	0.090	0.204	0.396	0.677	1.004*	1.338	1.672	2.006	2.340	2.674	3.075
$p=\frac{1}{2}$	0	0.055	0.151	0.292	0.487	0.738	1.025	1.338*	1.672	2.006	2.340	2.674	3.075
$p=\frac{3}{4}$	0	0.082	0.214	0.382	0.587	0.828	1.095	1.383	1.687	2.006*	2.340	2.674	3.075
$p=1$	0	0.112	0.280	0.479	0.701	0.943	1.200	1.472	1.757	2.054	2.362	2.680	3.075*

$r = 0.8,\ h = 0.6,\ l = 0.2,\ \sigma = 0.15,\ I = 1$

Table I confirms our theoretical results and reveals that the effects of beliefs can be significant on the option value F. The star indicates the immediate value above or equal to the investment threshold. As shown numerically in Roche (2002), the optimal trigger investment frontier is convex in p.

3.2. Option Value and Optimal Purchasing Strategy α

There are two main difficulties associated with solving numerically the HJB equation. As in the benchmark case, the free boundary is part of the problem. In addition, the HJB is non-linear. However, note that for a given value of α the HJB is linear and equal to

$$\begin{aligned} rF(V,p) &= -c(\alpha) + V(ph+(1-p)l)F_1(V,p) + \frac{\sigma^2}{2}V^2F_{11}(V,p) \\ &\quad + V(h-l)p(1-p)F_{12}(V,p) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 (p(1-p))^2\left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right)F_{22}(V,p). \end{aligned}$$

We solve numerically the PDE using an iterative method. First, we start with an initial guess $\alpha^0 \equiv 0$. Then using the methodology described in the benchmark case section, we solve numerically the equation obtaining a value function F^{α^0}. Second, the next value for

α is set to be equal to the optimal amount of information purchased

$$\alpha^1(V,p) = \left(\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(\frac{\Sigma_1}{\Sigma_2}\right)^2 (p(1-p))^2 F_{22}^{\alpha^0}(V,p)\right)^{\frac{1}{n}}.$$

Again, we solve numerically the corresponding linear HJB when $\alpha = \alpha^1$ and obtain a new value function F^{α^1}. We repeat the procedure N times until

$$\left|\alpha^N - \alpha^{N-1}\right| < \varepsilon,$$

for some $\varepsilon > 0$ arbitrary small. Numerical simulations indicate that this algorithm converges. Results are presented in tables II and III.

TABLE II

Option Value F

V	0	0.34	0.68	1.02	1.36	1.7	2.04	2.38	2.72	3.06	3.4	3.74	4.075
$p=0$	0	0.003	0.023	0.124	0.340	0.670*	1.004	1.338	1.672	2.006	2.340	2.674	3.075
$p=\frac{1}{4}$	0	0.028	0.097	0.226	0.422	0.692	1.037*	1.397	1.716	2.036	2.395	2.755	3.075
$p=\frac{1}{2}$	0	0.055	0.158	0.314	0.51	0.754	1.062	1.398*	1.716	2.036	2.395	2.755	3.075
$p=\frac{3}{4}$	0	0.082	0.220	0.404	0.605	0.837	1.123	1.431	1.724	2.036*	2.395	2.755	3.075
$p=1$	0	0.112	0.280	0.479	0.701	0.943	1.200	1.472	1.757	2.054	2.362	2.680	3.075*

$r = 0.8,\ h = 0.6,\ l = 0.2,\ \sigma = 0.15,\ \frac{\Sigma_1^2}{\Sigma_2^2} = 1.5,\ I = 1,\ n = 2$

Indeed both the option value and the optimal investment trigger frontier are greater than the one corresponding to no information purchased. More simulations (not displayed here) indicate that the optimal trigger frontier is also convex in p.

3.3. Optimal Purchasing Strategy α

Numerical simulations about the optimal purchasing strategy are displayed in Table III.

TABLE III

Optimal Purchasing Strategy α^*

	p	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
$V = 0.4$	α^*	0	0.005	0.021	0.034	0.042	0.044	0.037	0.026	0.013	0.007	0
$V = 1.2$	α^*	0	0.153	0.162	0.161	0.157	0.147	0.135	0.120	0.101	0.073	0
$V = 2.4$	α^*	0	0	0	0	0	0.513	0.456	0.417	0.324	0.228	0

$r = 0.8,\ h = 0.6,\ l = 0.2,\ \sigma = 0.15,\ \frac{\Sigma_1^2}{\Sigma_2^2} = 1.5,\ I = 1,\ n = 2$

We notice that the optimal strategy α^* is increasing in the value of the project V and hump shape in beliefs p. Actually to be more precise, further simulations (not displayed here) shows that when V is getting close to the optimal investment frontier, α^* starts to decrease. It follows that the shadow price of information $\frac{1}{r}F_{22}$ is also increasing in the value of the project providing that V is not too close to the V^*. When the value of the project does not increase significantly, the investor revises her beliefs downward. At the same time, she is not willing to buy much information: her learning speed slows down. On the contrary, when the value of the project increases quickly, the investor believes that μ is likely to be high; she is more keen on purchasing information and therefore learns more quickly.

Another interesting feature of the optimal purchasing strategy α as a function of belief p is its increasing skewness to the left as the value of the project rises. Given a belief p, when V is getting close to the optimal investment frontier, the investor is willing to purchase relatively more information since she knows her decision of investing is arriving and it is irreversible.

Note that we require the option value F to be continuously differentiable (value matching and smooth pasting conditions) and we have not impose any condition stronger condition. In particular, the optimal purchasing strategy α^* is a function of F_{22} that might have a jump at the curve V^*: given p, if $V > V^*(p)$, then $\alpha^*(V, p) = 0$ but α^* may be positive for V slightly below $V^*(p)$.

3.3.1. Effects of the Project Volatility σ^2

As in the case where no additional information is available (Roche 2002), simulations (not displayed here) show that a higher volatility σ^2 overall enhances the option value of waiting. Too be more specific, the direct effect of the volatility of the project is as in the complete information case to increase the option value. The indirect effect lies in the learning component of the option: a higher volatility means a noisier signal V and thus damages learning. The indirect effect negatively impacts the option value. Overall, the direct effect outweights the indirect one. In addition, ceteris paribus, the dynamics of the beliefs reveal that a higher σ reduces the learning speed via the term $\frac{h-l}{\sigma}$ thus incentives to purchase information. Consequently the optimal strategy α^* decreases with σ.

4. CONCLUSION

We used a very simple model of irreversible investment to explore the implications of incomplete information when the investor can control at a cost the amount of information received. Observing the realizations of the project over time as well as having access to an additional signal provide some information about the true value of the average growth rate of the project. As in the case where no action can be taken, optimistic investors tend to have a higher option value to wait and therefore, choose to postpone their decision of investing. Having the opportunity to buy information enhances the option value and thus reinforces delaying the decision to invest with respect to the case where the agent has no access to outside sources of information. From a methodological point of view, the Hamilton Jacobi Bellman equation of the program is non-linear with a free boundary to be determined. We propose an original algorithm to solve it which can be used in other similar optimal control problems. In particular, numerical simulations show that the amount of information purchased is increasing in the value of the project, hump shape in beliefs and decreasing in the volatility of the project.

5. APPENDIX

5.1. APPENDIX 1

Derivation of the filtering problem: We follow the approach of Bolton and Harris (1999)

Proof. Observing V and A is equivalent to observing $x = \frac{\ln V}{\sigma}$ and $y = \frac{\sqrt{1+\alpha}\sigma \ln A - \sqrt{\alpha}\Sigma_1 \ln V}{\Sigma_2 \sigma}$. Using Ito's lemma, it is easy to check that

$$\begin{aligned} dx(t) &= \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dt + dw_1(t) \\ dy(t) &= \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m - \sqrt{\alpha}\Sigma_1\left(\mu - \frac{\sigma^2}{2}\right)\right) dt + dw_2(t) \end{aligned}$$

with $m = \lambda\sigma - \frac{\alpha\Sigma_1^2}{2(1+\alpha)} - \frac{\Sigma_2^2}{2(1+\alpha)}$. Applying Bayes 'rule, we have

$$p(t+dt) = \frac{p(t)H(h)}{p(t)H(h) + (1-p(t))H(l)}$$

where

$$\begin{aligned} F(\mu) &= \frac{1}{\sqrt{2\pi dt}} \exp(-\frac{(dx(t) - \frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dt)^2}{2dt}) \\ G(\mu) &= \frac{1}{\sqrt{2\pi dt}} \exp\left(-\frac{(dy(t) - \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m - \sqrt{\alpha}\Sigma_1\left(\mu - \frac{\sigma^2}{2}\right)\right) dt)^2}{2dt}\right) \end{aligned}$$

and $H(\mu) = F(\mu)G(\mu)$ is the probability of observing $(dx(t), dy(t))$. Hence

$$dp(t) = \frac{(1-p(t))p(t)(\widetilde{H}(h) - \widetilde{H}(l))}{p(t)\widetilde{H}(h) + (1-p(t))\widetilde{H}(l)},$$

where

$$\begin{aligned} \widetilde{H}(\mu) = \exp\Bigg(&\frac{1}{\sigma}\left(\mu - \frac{\sigma^2}{2}\right) dx(t) - \frac{1}{2\sigma^2}\left(\mu - \frac{\sigma^2}{2}\right)^2 dt \\ &+ \frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m - \sqrt{\alpha}\Sigma_1(\mu - \frac{\sigma^2}{2})\right) dy(t) \\ &- \frac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+\alpha}m - \sqrt{\alpha}\Sigma_1(\mu - \frac{\sigma^2}{2})\right)^2 dt\Bigg) \end{aligned}$$

It follows

$$\begin{aligned}
\widetilde{H}(\mu) &= 1+\left(\frac{1}{\sigma}\left(\mu-\frac{\sigma^2}{2}\right)dx(t)+\frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(\mu-\frac{\sigma^2}{2})\right)dy(t)\right.\\
&\quad-\left(\frac{1}{2\sigma^2}\left(\mu-\frac{\sigma^2}{2}\right)^2+\frac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(\mu-\frac{\sigma^2}{2})\right)^2\right)dt\\
&\quad+\tfrac{1}{2}\left(\tfrac{1}{\sigma}\left(\mu-\tfrac{\sigma^2}{2}\right)dx(t)+\tfrac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(\mu-\tfrac{\sigma^2}{2})\right)dy(t)\right.\\
&\quad\left.-\left(\tfrac{1}{2\sigma^2}\left(\mu-\tfrac{\sigma^2}{2}\right)^2+\tfrac{1}{2\Sigma_2^2\sigma^2}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(\mu-\tfrac{\sigma^2}{2})\right)^2\right)dt\right)^2\\
&= 1+\frac{1}{\sigma}\left(\mu-\frac{\sigma^2}{2}\right)dx(t)+\frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(\mu-\frac{\sigma^2}{2})\right)dy(t)
\end{aligned}$$

where we have suppressed the terms of degree $dt^{\frac{3}{2}}$ and higher and use the fact that $(dx(t))^2 = (dx(t))^2 = 1$ and $dx(t)dy(t) = 0$. Therefore

$$\begin{aligned}
dp(t) &= \frac{\frac{h-l}{\sigma}(1-p(t))p(t)\left(dx(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}dy(t)\right)}{1+\frac{1}{\sigma}(p(t)(h-l)+l)-\frac{\sigma^2}{2})dx(t)+\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(p(t)(h-l)+l-\frac{\sigma^2}{2})\right)\frac{dy(t)}{\Sigma_2\sigma}}\\
&= \frac{h-l}{\sigma}(1-p(t))p(t)\left(dx(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}dy(t)-\frac{1}{\sigma}(p(t)h+(1-p(t))l)-\frac{\sigma^2}{2})dx(t)\right.\\
&\quad\left.-\frac{1}{\Sigma_2\sigma}\left(\sqrt{1+\alpha}m-\sqrt{\alpha}\Sigma_1(p(t)h+(1-p(t))l-\frac{\sigma^2}{2})\right)dy(t)\right)\\
&= \frac{h-l}{\sigma}(1-p(t))p(t)\left[dw_1(t)+\frac{1}{\sigma}\left(\mu-(p(t)h+(1-p(t))l\right))dt\right.\\
&\quad\left.-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}\left(dw_2(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2\sigma}\left(\mu-(p(t)h+(1-p(t))l\right))dt\right)\right]\\
&= \frac{h-l}{\sigma}(1-p(t))p(t)\left(d\overline{w}_1(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(t)\right)
\end{aligned}$$

where

$$\begin{aligned}
d\overline{w}_1(t) &= dw_1(t)+\frac{1}{\sigma}\left(\mu-(p(t)h+(1-p(t))l\right))dt\\
d\overline{w}_2(t) &= dw_2(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2\sigma}\left(\mu-(p(t)h+(1-p(t))l\right))dt
\end{aligned}$$

are (standard) Brownian motions under the investor belief P. Then

$$\begin{aligned}
dV(t) &= V(t)\left((p(t)h+(1-p(t))l)dt+\sigma d\overline{w}_1(t)\right)\\
dp(t) &= \frac{h-l}{\sigma}(1-p(t))p(t)\left(d\overline{w}_1(t)-\sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}d\overline{w}_2(t)\right)\\
dA(t) &= A(t)\left(\lambda dt+\Sigma_1\sqrt{\frac{\alpha}{1+\alpha}}d\overline{w}_1(t)+\Sigma_2\sqrt{\frac{1}{1+\alpha}}d\overline{w}_2(t)\right).
\end{aligned}$$

When $\mu = l$, define $\phi = \frac{p}{1-p}$. Using Ito lemma leads to

$$\begin{aligned} d\phi(t) &= \frac{dp(t)}{(1-p(t))^2} + \left(\frac{h-l}{\sigma}\right)^2 \frac{p^2(t)(1-p(t))^2}{(1-p(t))^3}\left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right)dt \\ &= \frac{h-l}{\sigma}\phi(t)\left(dw_1(t) - \sqrt{\alpha}\frac{\Sigma_1}{\Sigma_2}dw_2(t)\right) \end{aligned}$$

$$G(V,\phi) = (1+\phi)F(V, \frac{\phi}{1+\phi})$$

$$\begin{aligned} F(V,p) &= \sup_{\alpha\geq 0,\tau\geq 0} E^P\left[-\int_0^\tau c(\alpha(t))e^{-rt}dt + (V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\ &= \sup_{\alpha\geq 0,\tau\geq 0} \frac{1}{\xi_{p,l}(0)}E^l\left[\xi_{p,l}(\tau)\left(-\int_0^\tau c(\alpha(t))e^{-rt}dt + (V(\tau)-I)e^{-r\tau}\right) \mid \mathcal{F}_0\right] \\ &= \sup_{\alpha\geq 0,\tau\geq 0} \frac{1}{\xi_{p,l}(0)}E^l\left[E^l\left[\xi_{p,l}(\tau) \mid \mathcal{F}_t\right]\left(-\int_0^\tau c(\alpha(t))e^{-rt}dt\right) + \xi_{p,l}(\tau)(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\ &= \sup_{\alpha\geq 0,\tau\geq 0} \frac{1}{\xi_{p,l}(0)}E^l\left[\left(-\int_0^\tau E^l\left[\xi_{p,l}(\tau) \mid \mathcal{F}_t\right]c(\alpha(t))e^{-rt}dt\right) + \xi_{p,l}(\tau)(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\ &= \sup_{\alpha\geq 0,\tau\geq 0} \frac{1}{\xi_{p,l}(0)}E^l\left[-\int_0^\tau \xi_{p,l}(t)c(\alpha(t))e^{-rt}dt + \xi_{p,l}(\tau)(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \\ &\quad \text{(because } \xi_{p,l} \text{ is a martingale)} \\ &= \sup_{\alpha\geq 0,\tau\geq 0} \frac{1}{1+\phi}E^l\left[-\int_0^\tau (1+\phi(t))c(\alpha(t))e^{-rt}dt + (1+\phi(\tau))(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right] \end{aligned}$$

Hence

$$G(V,\phi) = \sup_{\alpha\geq 0,\tau\geq 0} E^l\left[-\int_0^\tau (1+\phi(t))c(\alpha(t))e^{-rt}dt + (1+\phi(\tau))(V(\tau)-I)e^{-r\tau} \mid \mathcal{F}_0\right]$$

■

5.2. APPENDIX 2

Existence of a solution

The main difficulty is that the control space is not compact. A progressively measurable control process α is said to be admissible if

$$E\left[\int_t^{t_1} |\alpha(s)|^m \, ds \mid \mathcal{F}_t\right] < \infty, \text{ for all } m = 1, 2, ...$$

Let A denotes the set of admissible progressively measurable control processes. We first show that it is enough to restrict the set of control processes of our problem to A. Denoting by $F(V,0)$ (respectively $F(V,1)$) the option value when the drift μ is equal to l (respectively h), it is clear that for all (V,p), we have

$$F(V,0) \leq F(V,p) \leq F(V,1).$$

Showing existence of F is equivalent to show existence of

$$G(V,\phi) = (1+\phi)F(V, \frac{\phi}{1+\phi})$$

Using the change of variables

$$\begin{aligned} x &= \ln V \\ y &= \ln \phi \end{aligned}$$

and writing $H(x,y) = G(e^x, e^y)$, we are left with showing the existence of

$$H(x,y) = \sup_{\alpha \geq 0, \tau \geq 0} E^l \left[-\int_0^\tau (1+e^{y(s)})c(\alpha(s))e^{-rs}ds + (1+e^{y(\tau)})(e^{x(\tau)} - I)e^{-r\tau} \mid \mathcal{F}_0 \right]$$

$$\begin{aligned} \text{s.t. } dx(t) &= (l - \frac{\sigma^2}{2})dt + \sigma dw_1(t) \\ dy(t) &= -\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(1+\alpha(t)\frac{\Sigma_1^2}{\Sigma_2^2}\right) dt + \frac{h-l}{\sigma}\left(dw_1(t) - \sqrt{\alpha(t)}\frac{\Sigma_1}{\Sigma_2}dw_2(t)\right) \end{aligned}$$

The corresponding HJB equation is

$$\begin{aligned} rH(x,y) &= \sup_{\alpha \geq 0} \left\{ -(1+e^y)c(\alpha) + (l - \frac{\sigma^2}{2})H_1(x,y) + \frac{\sigma^2}{2}H_{11}(x,y) + (h-l)H_{12}(x,y) \right. \\ & \left. - \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right) H_2(x,y) + \frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right) H_{22}(x,y) \right\}. \end{aligned}$$

We need to check the existence conditions as in Fleming and Soner (F.S.) (1993), chapter 4, p 171. We denote by $X = (x,y)$ and

$$\begin{aligned} f(X,\alpha) &= \left\{ \begin{array}{c} l - \frac{\sigma^2}{2} \\ -\frac{1}{2}\left(\frac{h-l}{\sigma}\right)^2 \left(1+\alpha\frac{\Sigma_1^2}{\Sigma_2^2}\right) \end{array} \right. \\ \Sigma(X,\alpha) &= \left[\begin{array}{cc} \sigma & 0 \\ \frac{h-l}{\sigma} & -\frac{h-l}{\sigma}\left(\frac{\sqrt{\alpha}\Sigma_1}{\Sigma_2}\right) \end{array} \right] \end{aligned}$$

It is easy to verify condition (5.2) and (5.3) in F.S. since

$$\begin{aligned} |f_X| &\leq C, \ |\Sigma_X| \leq C, \\ |f(X,\alpha)| &\leq C(1+|X|+|\alpha|), \ |\Sigma(X,\alpha)| \leq C(1+|X|+|\alpha|) \end{aligned}$$

for some constant C. Actually, we can strengthen these growth conditions since we have

$$\begin{aligned} |f_X| &\leq C, \ |\Sigma_X| \leq C, \\ |f(X,\alpha)| &\leq C(1+|\alpha|), \ |\Sigma(X,\alpha)| \leq C(1+|\alpha|). \end{aligned}$$

Since f and Σ are independent from X, the growth condition on the reward function $g(x,y,t)$ to be checked is

$$|g(x,y,t)| \leq C(1+\left(e^{2x}+e^{2y}\right)^{\frac{k}{2}})$$

for some k and constant C. Since $g(x,y,t) = (1+e^y)(e^x - I)e^{-rt}$, the condition is clearly satisfied. Finally

$$\begin{aligned} |L(x,y,t)| &= \left|(1+e^y)c(\alpha)e^{-rs}\right| \\ &\leq C(1+|\alpha|^k + \left(e^{2x}+e^{2y}\right)^{\frac{k}{2}}), \end{aligned}$$

so condition (2.7) in F.S. is also satisfied. Let A_n denote the set of admissible control process $\alpha \geq \frac{1}{n}$. In this case, the matrix $\Sigma(X,\alpha)$ is positive definite so there exists a constant $c(n) > 0$ such that for all vector Y in $\mathbb{R}^2$

$$Y^T \Sigma(X,\alpha) Y \geq c(n)\,|Y|^2 .$$

This implies that the HJB is uniformly parabolic and therefore there is a unique solution to the problem (F.S. p 162). Finally, since for all $(x,y) \in \mathbb{R}^2_+$

$$\max\left(A_l e^{\beta_l x}, e^x - I\right) \leq H(x,y)$$

This shows that we only need to consider the stopping times τ such that

$$E^l\left[-\int_0^\tau (1+e^{y(s)})c(\alpha(s))e^{-rs}ds \mid \mathcal{F}_0\right] > -\infty$$

or equivalently

$$E^l\left[\int_0^\tau (1+e^{y(s)})c(\alpha(s))e^{-rs}ds \mid \mathcal{F}_0\right] < \infty.$$

Hence for all $t_1 > t > 0$, we have

$$E^l\left[\int_t^{t_1} c(\alpha(s))ds \mid \mathcal{F}_0\right] < \infty$$

which is condition 2.3 in F.E. p 159. Let

$$\begin{aligned} \tau^+ &= \{\inf_{t\geq 0} V_h(t) = V_1^*\} \\ \tau^- &= \{\inf_{t\geq 0} V_l(t) = V_1^*\} \end{aligned}$$

Since $l - \frac{\sigma^2}{2} > 0$, we have $P(\tau^+ < \infty) = P(\tau^- < \infty) = 1$. Then, set $O_1 = \{V : 0 \leq V \leq V_1^*\}$, $O_0 = \{V : 0 \leq V \leq V_0^*\}$. Since $V_l \leq V \leq V_h$ and for all (V,p), $F(V,0) \leq F(V,p) \leq F(V,1)$, the optimal exit cylinder O must be such that $O_0 \subseteq O \subseteq O_1$ and the candidates τ for the optimal stopping time must satisfy

$$\tau^+ \leq \tau \leq \tau^-.$$

Therefore it must be the case that $P(\tau < \infty)$. It follows that conditions of Theorem 5.1. p 172 in F.S. hold. In particular,

$$H^n(x,y) = \sup_{\alpha \geq \frac{1}{n}, \tau^n \geq 0} E^l \left[-\int_0^{\tau^n} (1 + e^{y^n(s)}) c(\alpha(s)) e^{-rs} ds + (1 + e^{y^n(\tau^n)})(e^{x^n(\tau^n)} - I) e^{-r\tau^n} \mid \mathcal{F}_0 \right]$$

is well defined. For any couple (x,y), $H^n(x,y)$ is an increasing sequence in n which is bounded by $(1+e^y)F(e^x,1)$, so by the monotone convergence theorem, it has a limit. This limit is

$$H(x,y) = \sup_{\alpha \geq 0, \tau \geq 0} E^l \left[-\int_0^{\tau} (1 + e^{y(s)}) c(\alpha(s)) e^{-rs} ds + (1 + e^{y(\tau)})(e^{x(\tau)} - I) e^{-r\tau} \mid \mathcal{F}_0 \right].$$

The proof is complete. ■

5.3. APPENDIX 3

Proof. We want to show that given p_0, if V_0 is in IR, then $W_0 < V_0$ is also in IR. Assume V_0 is in IR. Let (τ_{W_0}, α^*) be the optimal stopping time and purchasing strategy when the process V starts at W_0. Writing $V(t) = V_0 K(\alpha, t)$ with $K(\alpha,t) = \exp\left(-\int_0^t \left(p(s)h + (1-p(s))l - \frac{1}{2}\sigma^2\right) ds + \int_0^t \sigma d\tilde{}\right.$
it follows that

$$\begin{aligned} F(V_0,p_0) - F(W_0,p_0) &\leq E^P\left[-\int_0^{\tau_{W_0}} c(\alpha^*(t)) e^{-rt} dt + (V_0 K(\alpha^*, \tau_{W_0}) - I) e^{-r\tau_{W_0}} \mid \mathcal{F}_0 \right] \\ &\quad - E^P\left[-\int_0^{\tau_{W_0}} c(\alpha^*(t)) e^{-rt} dt + (W_0 K(\alpha^*, \tau_{W_0}) - I) e^{-r\tau_{W_0}} \mid \mathcal{F}_0 \right] \\ &\leq E^P\left[(V_0 - W_0) K(\alpha^*, \tau_{W_0}) e^{-r\tau_{W_0}} \mid \mathcal{F}_0 \right]. \end{aligned}$$

Thus

$$\begin{aligned} \frac{F(V_0,p_0) - F(W_0,p_0)}{V_0 - W_0} &\leq E^P\left[K(\alpha^*, \tau_{W_0}) e^{-r\tau_{W_0}} \mid \mathcal{F}_0 \right] \\ &\leq 1 \end{aligned}$$

Given the fact that the stochastic process $t \mapsto K(\alpha^*(t), t) e^{-r\tau}$ is a supermartingale under P and $K(\alpha^*(0), 0) e^{-r0} = 1$, using the optional sampling theorem, we conclude that for all (V,p) in $\mathbb{R}_+ \times [0,1]$

$$F_1(V,p) \leq 1. \tag{5.1}$$

Now, assume that for p_0 given, V_0 is in IR. Let $W_0 < V_0$. Relationship (**??**) implies that $F(V_0, p_0) - F(W_0, p_0) \leq V_0 - W_0$, so

$$F(W_0, p_0) \geq F(V_0, p_0) - V_0 + W_0$$

Since V_0 is in IR, then $F(V_0, p_0) > V_0 - I$, which implies

$$F(W_0, p_0) > W_0 - I,$$

so indeed W_0 is in IR. ■

5.4. APPENDIX 4

Proof of Property 1.

Proof. By definition

$$F(V, p) = \sup_{\alpha \geq 0, \tau \geq 0} E^P \left[- \int_0^\tau c(\alpha(t)) e^{-rt} dt + (V(\tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right]$$

where

$$\begin{aligned} V(t) &= V_0 \exp\left(\int_0^t (p(s)h + (1 - p(s))l - \frac{\sigma^2}{2}) ds + \sigma \overline{w}_1(t) \right) \\ &= V_0 K(\alpha, t). \end{aligned} \tag{5.2}$$

Note that for *some given strategy* α, $V_0 K(\alpha, t) < V_0' K(\alpha, t)$ for $V_0 < V_0'$. Since the optimal strategy for a process V starting at V_0 is in particular an admissible strategy for a V starting at V_0', it follows easily that if $V_0 < V_0'$, then $F(V_0, p) < F(V_0', p)$ for all p in $[0, 1]$. Now, we want to show that F is strictly convex in V, that if $\lambda \in (0, 1)$, V_1 and V_2 positive and distinct, for all $p \in [0, 1]$. Moreover note that

$$V_0 \exp\left((l - \frac{\sigma^2}{2}) t + \sigma \overline{w}_1(t) \right) \leq V(t) \leq V_0 \exp\left((h - \frac{\sigma^2}{2}) t + \sigma \overline{w}_1(t) \right),$$

which is independent from α so it follows that $F(V, 0) \leq F(V, p) \leq F(V, 1)$. In addition,

$$\begin{aligned} F(\lambda V_{10} + (1 - \lambda) V_{20}, p) &= \sup_{\alpha \geq 0, \tau \geq 0} E^P \Big[-(\int_0^\tau c(\alpha(t)) e^{-rt} dt \\ &\quad + ((\lambda V_{10} + (1 - \lambda) V_{20}) K(\alpha, \tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \Big] \\ &\leq \sup_{\alpha \geq 0, \tau \geq 0} E^P \left[- \int_0^\tau c(\alpha(t)) e^{-rt} dt + \lambda V_{10} K(\alpha, \tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\ &\quad + \sup_{\alpha \geq 0, \tau \geq 0} E^P \left[- \int_0^\tau c(\alpha(t)) e^{-rt} dt + (1 - \lambda) V_{20} K(\alpha, \tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\ &\leq \lambda F(V_{10}, p) + (1 - \lambda) F(V_{20}, p), \end{aligned}$$

which proves that F is convex in its first argument. ■

Proof of Property 2.

Proof. For a given strategy α, the law of motion of the beliefs is given. Let p and p' be two *Markovian processes* following the *same* law of motion. If at some date θ, $p'(\theta) = p(\theta)$, then we have $p' = p$ for all dates $s \geq \theta$. It follows that $p' \geq p$ for all $t \geq \tau$. ■

Proof of Property 3.

Step 1: For all $V > 0$, F is non decreasing in p.

Consider two initial values (V_0, p_0) and (V_0, p'_0) with $p_0 < p'_0$.Consider α^* the optimal strategy when beliefs starts at p_0. Since $p_0 < p'_0$, given this strategy α for all $t \geq 0$, given property 2 $p(t) \leq p'(t)$ and therefore given relationship (5.2), for all the value $t \geq 0$, the value of the project is higher for investor starting with beliefs p'_0 than the value of the project for investor starting with beliefs p_0. Hence

$$\begin{aligned}
F(V_0, p_0) &= \sup_{\tau \geq 0} E^P \left[-\int_0^\tau c(\alpha^*(t)) e^{-rt} dt + (V(\tau) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\
&= \sup_{\tau \geq 0} E^P \left[-\int_0^\tau c(\alpha^*(t)) e^{-rt} dt \right. \\
&\quad \left. + V_0 \exp\left(\int_0^\tau (p(s)h + (1-p(s))l - \frac{\sigma^2}{2}) ds + \sigma \overline{w}_1(t) \right) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\
&\leq \sup_{\tau \geq 0} E^P \left[-\int_0^\tau c(\alpha^*(t)) e^{-rt} dt \right. \\
&\quad \left. + V_0 \exp\left(\int_0^\tau (p'(s)h + (1-p'(s))l - \frac{\sigma^2}{2}) ds + \sigma \overline{w}_1(t) \right) - I) e^{-r\tau} \mid \mathcal{F}_0 \right] \\
&\leq F(V_0, p'_0)
\end{aligned}$$

since α^* is also an admissible (not necessary optimal) strategy when beliefs start at p'_0.

Step 2: For all $V > 0$, F is strictly convex in p.

It is easy to check that

$$G''(V, \phi) = \frac{1}{(1+\phi)^3} F''(V, \frac{\phi}{1+\phi}) = (1-p)^3 F''(V, p).$$

It follows that F is convex in p if and only if G is convex in ϕ. Now, recall that

$$\begin{aligned}
\phi(t) &= \phi_0 \exp\left(\int_0^t -\frac{1}{2} \left(\frac{h-l}{\sigma} \right)^2 \left(1 + \alpha(s) \frac{\Sigma_1^2}{\Sigma_2^2} \right) ds \right. \\
&\quad \left. + \int_0^t \frac{h-l}{\sigma} \left(dw_1(s) - \sqrt{\alpha(s)} \frac{\Sigma_1}{\Sigma_2} dw_2(s) \right) \right) \\
&= \phi_0 Z(\alpha, t)
\end{aligned}$$

Therefore

$$\begin{aligned}
G(V, \lambda\phi_{10} + (1-\lambda)\phi_{20}) &= \sup_{\alpha \geq 0, \tau \geq 0} E^l \Big[-\int_0^\tau c(\alpha(t))(1 + (\lambda\phi_{10} + (1-\lambda)\phi_{20})Z(\alpha, t))e^{-rt}dt \\
&\quad + (1 + (\lambda\phi_{10} + (1-\lambda)\phi_{20})Z(\alpha, \tau))V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \Big] \\
&\leq \lambda \sup_{\alpha \geq 0, \tau \geq 0} E^l \Big[-\int_0^\tau c(\alpha(t))(1 + \phi_{10}Z(\alpha, t))e^{-rt}dt \\
&\quad + (1 + \phi_{10}Z(\alpha, \tau))V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \Big] \\
&\quad + (1-\lambda)E^l \Big[-\int_0^\tau c(\alpha(t))(1 + \phi_{20}Z(\alpha, t))e^{-rt}dt \\
&\quad + (1 + \phi_{20}Z(\alpha, \tau))V(\tau) - I)e^{-r\tau} \mid \mathcal{F}_0 \Big] \\
&\leq \lambda G(V, \phi_{10}) + (1-\lambda)G(V, \phi_{20}),
\end{aligned}$$

which proves that G and thus F are convex in their second argument. ■

Proof of Property 4.

Proof. Since for any $V > 0$ and $p' \geq p$, $F(V, p') \geq F(V, p)$, using the value matching condition, we have $F(V^*(p), p') \geq V^*(p) - I$. Thus, it follows easily that for $p' \geq p$, $V^*(p') \geq V^*(p)$. ■

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