Competition Games in Duopoly Settings with Two Stochastic Factors

Dean Paxson and Helena Pinto

Keywords: Real Options, pre-emption, two stochastic processes.

February, 2003

Both authors are from Manchester Business School. Contact author: Helena Pinto, Manchester Business School, Booth Street West, Manchester M15 6PB, United Kingdom, +44(0) 1612756511, hpinto@man.mbs.ac.uk. Dean Paxson, +44(0) 2087892905 or dpaxson@man.mbs.ac.uk.

Acknowledgements: We appreciate the comments of Alan Jones on parts of this paper. Helena Pinto gratefully acknowledges financial support from Fundação para a Ciência e Tecnologia.

ABSTRACT

This paper presents two different real options models, with two stochastic factors considering strategic interactions.

In the first model the profits per unit and the number of units follow two different stochastic paths and, in the second model the returns and the investment cost pursue different paths. For both models we analyse dissimilar games considering that the roles of the players are pre-assigned and also exogenous to the models, always assuming that the first mover has a competitive advantage over the second mover.

Closed form solutions are obtained for the value functions of the first and second mover and for its trigger functions, except for the trigger of the first mover in pre-emptive environments.

The paper analyses the effect of returns, investment cost and uncertainty on the models. Standard results do not always hold: uncertainty can delay the adoption of the first mover.

Although pre-emption affects the leader's trigger it does not seem to influence the entry point of the follower.

1. Introduction

It is now widely accepted that real options can compete with traditional methodologies of valuating investments.

All the investments have the following characteristics: they are at least partially sunk; they cannot be totally predicted because the future is stochastic and most of the times they can be delayed (Dixit and Pindyck 1994). The traditional methodologies have failed to correctly consider those characteristics; while using the framework of financial options to analyse investment decisions, real options arises as a powerful management tool because it includes those factors in the analysis and at the same time forces managers to analyse and provides them a way to quantify all the opportunities associated with the investment in consideration.

One of the problems of using the financial options methodology to analyse investment decisions is that strategic considerations are not important to financial options, because the competitors' actions in capital markets do not influence the value of options, they are; however, significant for investment decisions. Firms are not alone in the market and their decisions influence their competitors and vice versa. In truth even the way that the investor's problem is solved is different. In a pre-emption environment, optimisation techniques are substituted by a stopping game (see Duta and Rustochini (1991) for details).

There is a new trend of literature in real options that considers strategic interactions.

Fudenberg and Tirole (1985) created the foundations of real options in a competitive setting while developing a model of games of timing in which a continuous time version of strategy equilibrium is presented. Following these authors game theory principles have been applied to real options.

Spatt and Sterbenz (1985) consider learning and pre-emption. Smets (1991) considers a strategic setting where firms can act under the fear of pre-emption.

Grenadier (1996) applies the model to the real estate market. The effect of incomplete information is analysed by Lambrecht and Perraudin (1997); strategic competition in Kulatilaka and Perotti (1998); R&D competition in Weeds (2000); network advantages are included in Mason and Weeds (2000); Hope (2000) considers second mover advantages; Smit and Trigeorgis (2001) modelled different investment strategies under contrary quantity or reciprocating price competition; using an "artificial" competitive equilibrium Grenadier (2002) introduces a time to build feature; and Tsekrekos (2003) studies the sensitivity of the leader and follower value function to market share, assumed to be constant after the follower enters. All the previous authors considered one stochastic factor, normally the profit flow, Shackleton and Tsekrekos and Wojakowski (2002) present for the first time a game theoretic approach to real options in a duopoly framework where two stochastic factors are considered. In their paper they look at a market that can only accommodate one active firm and where the idle firm has the option to claim the market. The operating profitability of both firms follows a different diffusion process.

The objective of this paper is to derive two real options models, for a duopoly situation where we have more than one diffusion process and where the roles of the players are defined both exogenously and endogenously[1]. In the first model the two stochastic factors will be the number of units and the profit per unit. In the second model the total profit and the investment cost will follow separate stochastic processes.

Our approach is different from the one of Shakleton and Tsekrekos and Wojakowski (2002) because; firstly, we are looking at a market that allows more than one firm and secondly, we are interested not only in pre-emptive markets but also in non-pre-emptive ones.

[1] We are only interested in situations where the firms act only on their own interest. In Mason and Weeds (2000) we can find a game where companies act co-operatively with the objective of maximising their joint value.

The paper is organised as follows: after the introduction in section 1; we will derive a real options model, in section 2, where the profit per unit and the number of units follow a different stochastic process; in section 3 we will consider that the investment cost and the returns follow a different stochastic process. In section 4 we justify the choice of the two models and in section 5 we conclude.

2- Profit per unit and number of units as stochastic variables

Traditionally in the real options literature the return and the number of units are not disaggregated[2], but it probably makes sense to consider those two variables separately[3]. The path followed by the return is not necessarily the same followed by the number of units sold because the price and the costs can be affected by different factors than the ones that affect the number of units. Due to the different characteristics they can even follow different stochastic process[4].

We start our analyses by considering that both the return per unit and the number of units follow different but possibly correlated geometric Brownian motion.

Let V represent the profit per unit sold and M the number of units of a market.

Both variables have a domain from 0 to infinity; the lower limit implies that a company can not have a negative profit per unit[5] and that it can not sell a negative number of units. Profit, as assumed in the literature, can go to infinity. Our justification for letting the number of units go to infinity, is that the number of units sold can be so big that in the limit approaches infinity.

Let V and M follow a geometric Brownian motion:

[2] Most of the real options models consider that the return follows a certain stochastic process. Nevertheless exceptions exist, for example in one of the models presented inHowell et al. (2001) the randomness exists in the number of units sold.

[3] Examples of authors that considered two stochastic factors in a single setting are: Lee and Paxson (2003); Tunaru and Clark and Viney (2002); Williams (1991) and Quigg (1993).

[4] Notice that by definition the return is a continuous variable but the number of units is a discrete one.

[5] In practice this is not necessarily so but the constrain has to be imposed because of the kind of stochastic process assumed.

$$dV = \mu V dt + \sigma V dz \qquad (1)$$

$$dM = \omega M dt + \alpha M dz \qquad (2)$$

Where:

μ and ω are the expected gain of V and M respectively or, in other words the drift of the Brownian motion;

σ and α are the volatilities.

We will assume that the two variables can be correlated with ρ as the correlation coefficient.

2.1 The Game

In our scenario we have a new market and two firms that are contemplating the option to enter that market[6]. The firm that enters first, from now on defined as the leader, will acquire a first mover advantage, we will assume that the leader will always have a higher share of the market and consequently will sell a higher number of units.

We will analyse equilibrium without pre-emption and with pre-emption. In the without pre-emption setting we will assume that the roles of the leader and the follower will be pre-assigned and the game can result either in a sequential adoption, if the players agree to enter apart, or a simultaneous one, if the players enter together. In the with pre-emption setting the model will define the roles of leader and follower, resulting in a sequential or a simultaneous equilibrium depending on the levels of total profits.

The two state variables follow a Markov process and we will assume that the strategies of both players will also be Markovian and therefore integrating all

[6] We are assuming that the only decisions of the firms is either to enter the market or not and when to do it. Options like the option to exit e.g. Dixit and Pindyck (1994) are not considered.

payoffs relevant factors in the game, those Markov strategies will yield a Nash equilibrium in every proper sub game.

If one player applies Markovian strategies, its rival has as best reaction a Markovian strategy as well. For this reason a Markovian equilibrium will stay an equilibrium when history dependent strategies are also permitted, even if other equilibriums exist (Fubenberg and Tirole (1991)).

2.1.1 With Pre-emption

Following Fudenberg and Tirole (1985) we will describe a game where the roles of the leader and the follower are endogenous.

The two firms are ex-ante symmetrical but asymmetrical post-ante[7]; the leader will always have a competitive advantage over the follower and as so each firm will want to win the position of the leader which creates a pre-emption effect. As usual in these games we will start by defining the follower's value function.

2.1.1.1- The follower's value function

Let $P_0^F(V,M)$ be a portfolio that replicates the value function of an idle follower. This portfolio will be long on an option and short on Δ_1 and Δ_2 units of V and M, respectively. After doing all the substitutions and collecting all the terms, we obtain the partial differential equation for an idle follower:

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 P_0^F}{\partial V^2}\sigma^2 V^2+\frac{1}{2}\frac{\partial^2 P_0^F}{\partial M^2}\alpha^2 M^2+\frac{\partial P_0^F}{\partial M\partial V}MV\sigma\alpha\rho+(r-\mu)V\frac{\partial P_0^F}{\partial V}+\\&(r-\omega)M\frac{\partial P_0^F}{\partial M}-rP_0^F=0\end{aligned}\qquad(3)$$

[7] In Pawlina and Kort (2002) the firms are ex-ante asymmetrical.

Equation (3) is the partial differential equation that explains the movements on the value function of an idle follower and it should be subject to two boundary conditions: The first boundary is the value matching that states that there is a value of $P_0^F(V,M)$ at which the follower will invest. At the point in time where $P_0^F(V,M)$ reaches that value, the value function will equal the present value of the cash flows minus the investment cost[8] (denoted by K):

$$P_0^F(V,M) = \frac{VM}{2r-\omega-\mu} - K \tag{4}$$

The second boundary condition is the smooth pasting that basically states that the derivatives of the two functions presented in (6) have to be equal at that point in time[9]:

$$\frac{\partial P_0^F(V,M)}{\partial M} = \frac{V}{2r-\omega-\mu} \tag{5}$$

To obtain a closed form solution for equation (3) we will use similarity methods. Let $X = VM$ implying that $P_0^F(X) = P_0^F(V,M)$. After all the substitutions[10] (3) can be written as:

$$\frac{1}{2}X^2 P_0^F{}''(X)\left[\sigma^2+\alpha^2+2\sigma\alpha\rho\right] + XP_0^F{}'(X)\left[\sigma\alpha\rho+2r-\mu-\alpha\right] - rP_0^F(X) = 0 \tag{6}$$

[8] Note that K is fixed regardless of M. An example can be a factory that can produce an unlimited number of units, M, where almost all the production costs are variable (e.g. service industry).
[9] Notice that at the point where the value matching condition occurs the idle follower will invest, becoming an active follower. In order for the value function of the follower to be continuous the slopes of the value functions of an active and an idle follower have to be the same.
[10] For details on the computation of (6) see appendix A.

With solution:

$$P_0^F(X) = AX^{\beta_1} + BX^{\beta_2} \tag{7}$$

Where β_1 and β_2 are given by:

$$\beta_{1/2} = \frac{1}{2z^2}\left(-4r + z^2 + 2\omega + 2\mu - 2\alpha\rho\sigma \pm \sqrt{8rz^2 + \left(4r - z^2 - 2\omega - 2\mu + 2\alpha\sigma\rho\right)}\right)$$

And where

$$z^2 = \alpha^2 + \sigma^2 + 2\sigma\alpha\rho .$$

Let $X = VM$ denote the total return. We know that as X increases, the value function of the follower has to increase, implying that in equation (7) B has to be zero[11].

Rewriting our boundary conditions using our substitution $X = VM$ we obtain:

$$P_0^F(X_F) = \frac{X_F}{2r - \omega - \mu} - K \tag{8}$$

Where X_F is the trigger value, meaning the value of X at which the follower should enter the market, and:

$$P_0^F{}'(X_F) = \frac{1}{2r - \omega - \mu} \tag{9}$$

[11] Notice that β_2 is negative, meaning that as X increases the second term in (7) will decrease.

Equations (7), (8) and (9) imply that:

$$X_F = \frac{K(2r - \omega - \mu)\beta_1}{\beta_1 - 1} \tag{10}$$

And:

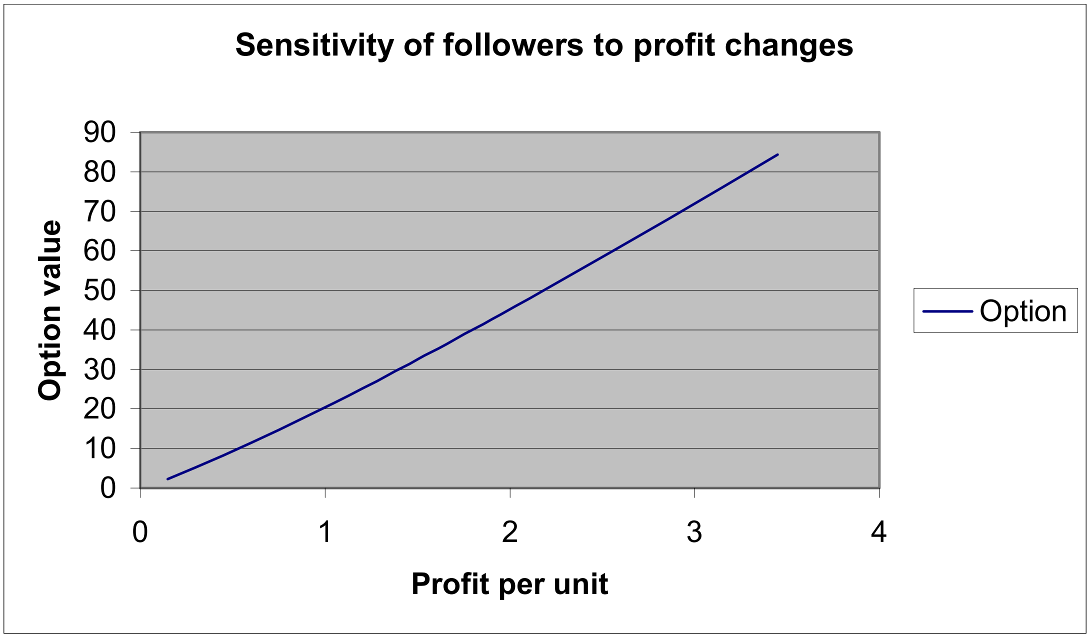
$$A = \frac{K}{\beta_1 - 1} X_F^{-\beta_1} \tag{11}$$

Putting equations (7), (10) and (11) together we obtain the value function of the follower:

$$\begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{X}{X_F}\right)^{\beta_1} & X < X_F \\ \dfrac{X}{2r - \omega - \mu} - K & X \geq X_F \end{cases} \tag{12}$$

Equation (12) describes the value function of the follower before and after the trigger is hit. Before the trigger X_F is hit the follower has not yet entered the market and his value function is described by the option to wait to invest. At the trigger the follower invests and after that his value function is described by the net present value.

In Figure 1 we present the sensitivity of the option to increases in the profit per unit sold:



Fig, 1- The parameters are: r=5%; μ=3%; ω=3%; σ=10%; α=20%; ρ=0; M=3 and V varies from 0.05 until 1.15.

As it was expected, as the return increases, so does the option value to invest because the probability of the follower exercising his option is also increasing.

The sensitivity of the option value to volatility is presented in Figure 2:

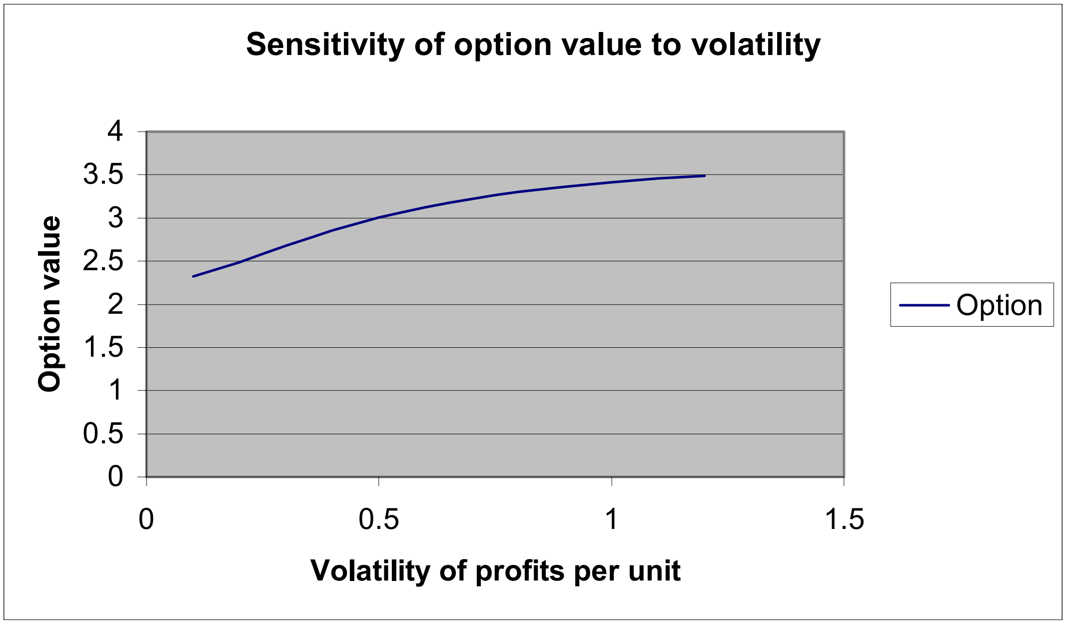


Fig 2- The parameters are: r=5%; μ=3%; ω=3%; α=20%; ρ=0; M=3; V=0.05 and σ varies from 0.1 until 1.2.

As expected the option value increases with increases in volatility, since a call option limits the downsize risk and so increases in volatility enlarge the possibility of gains[12].

According to our model, the right strategy for the follower is stated in proposition 1:

Proposition 1. *The optimal follower strategy conditional on a previous entry by the leader is to invest as soon as the revenues reach X_F, in other words the optimal time for the follower to invest is:*

[12] There are some exceptions to this rule, for example in the Margrabe (1978) exchange option the Vegas are positive if the correlation coefficient is negative or zero, but when the correlation is positive, depending on his magnitude and on the relative volatilities of the two assets, the Vega can be negative. Negative Vegas can be seen in almost all different kinds of correlation options, this happens because the aggregate volatility may decline as the individual volatilities increase with certain correlation coefficients (see Zhang (1997) for details).

$$T_F = \inf\left\{ t \geq 0; X = \frac{K(2r - \omega - \mu)\beta_1}{\beta_1 - 1} \right\}$$

2.1.1.2- The leader's value function

Before the follower decides to enter the leader is alone in the market and so his decision to enter or to wait in order to maximize his value function may seem identical to the decision to invest in a single setting framework. The difference is that the trigger value of the leader can be affected by his fear of losing his position, in other words while knowing that entering the market first he will obtain a competitive advantage[13], the leader can enter the market not at the trigger of the single setting framework but sooner[14].

The trigger function of the leader will now be defined considering the strategic interactions between the leader and the follower. At the moment in time when the trigger is hit both firms will want to enter in order to obtain the leader advantage[15]. Notice that if both firms entered, the game would be one of simultaneous exercise and not sequential as we are intending, so we will assume that when the trigger function is hit the leader will be determined by a toss of a coin. At that point both firms have a 50% chance of being a leader or a follower; the winner will be the leader entering the market and in order to maximize his decision the follower will wait until X_F is hit[16].

[13] We are assuming that the leader will always have a competitive advantage over the follower. Tellis and Golder (1996) present as advantages of the leader product line breath, product line and especially market share.

[14] The leader's option to wait will diminish or may even disappear in a competitive setting.

[15] Notice that the trigger is defined by indifference and not by optimisation techniques.

[16] This "arbitrary" determination is also used e.g. in Grenadier (1996); Weeds (2000) and Tsekrekos (2003). Since the model is defining the roles, and being the two firms identical, the determination of the role has to be defined by a randomness procedure like the outcome of a toss of a coin.

After entering the market the leader has no further actions to take, he will enjoy monopolistic revenues while he is alone and he will share the market with the follower after the latter enters.

Let $\overline{m}$ be a factor that multiplied by M results in the maximum number of units that can be absorbed by the market at a certain moment in time and m the advantage in number of units, the value function of the leader while alone in the market, can be explained by the following equation:

$$E\left[\int_0^{T_F} e^{-r\tau} X\overline{m}d\tau\right] + E\left[e^{-rT_F}\right]\frac{X_F m}{2r-\mu-\omega} - K \tag{13}$$

The first term in the equation represents the revenue that the leader receives while he is alone in the market, the second term is the revenues of the leader at the moment that the follower enters the market.

For simplification purposes let:

$$h(X) = E\left[e^{-rT_F}\right]$$

Note that h(X) is the expectation of the discount term at the risk free rate. Since T_F depends on the realization of X, which follows a geometric Brownian motion; h(X) can be written as:

$$\frac{1}{2}z^2 h''(X)X^2 + \left[\sigma\alpha\rho + 2r - \mu - \omega\right]Xh'(X) - rh(X) = 0 \tag{14}$$

With solution:

$$h(X) = CX^{\beta_1} + DX^{\beta_2} \tag{15}$$

Equation (15) can be subjected to two boundary conditions: as our state variable X approaches the trigger function of the follower X_F the optimal time for the follower to invest is being reached, meaning that the time until the follower invests T_F becomes very small. At the moment it reaches zero, our function h(x) will be 1, so $\lim_{X \to X_F} h(X) = 1$. The other boundary is defined as X goes to zero. If X is going to zero then it will never reach X_F implying that the optimal time for the follower to invest will be delayed. This obviously implies that the time until the follower invests will be delayed, as T_F gets very big h(X) gets very small, in the limit it approaches zero, so $\lim_{X \to 0} h(X) = 0$.

The last condition implies that D in equation (15) must be zero and using the first condition we obtain the following solution for h(X):

$$h(X) = \left(\frac{X}{X_F} \right)^{\beta_1} \qquad (16)$$

Let now g(X) represent the first expectation term in equation (13):

$$g(X) = E\left[\int_0^{T_F} e^{-r\tau} X\overline{m} d\tau \right]$$

Note that g(X) is the expected revenues from the moment the leader enters the market until the follower enters. In other words it represents the revenues obtained by the leader while he is alone in the market. Once again the expectation factor depends on the realization of our state variable X and as so g(X) can be explained by:

$$\frac{1}{2}z^2X^2g''(X)+(\sigma\alpha\rho+2r-\mu-\omega)Xg(X)-rg(X)+X\overline{m}=0 \tag{17}$$

With a general solution:

$$g(X)=EX^{\beta_1}+FX^{\beta_2}+\frac{X\overline{m}}{2r-\omega-\mu} \tag{18}$$

We can subject solution (18) to two boundary conditions: the first one as X goes to zero, as this happens the revenues of the leader have necessarily to decrease and consequently, F in equation (18) has to be zero. The second one can be found when the follower enters the market. As X tends to X_F the excess revenues that the leader receives, as he is alone in the market, will tend to disappear because the follower will enter and the market will be shared and so:

$$EX_F^{\beta_1}+\frac{X_F\overline{m}}{2r-\mu-\omega}=0$$

Putting the two boundaries together we obtain, after some simplifications:

$$g(X)=-\frac{X_F\overline{m}}{2r-\beta-\omega}\left(\frac{X}{X_F}\right)^{\beta_1}+\frac{X\overline{m}}{2r-\beta-\omega} \tag{19}$$

The second term in equation (19) is the solution of the non-homogeneous part of equation (17) and represents the present value of the revenues obtained by the leader while he is alone in the market; the first term represents the negative impact that the entry of the follower has in the value function of the leader.

Substituting (19) and (16) back into equation (13) we obtain after some simplifications:

$$\begin{cases} \left(\dfrac{X}{X_F}\right)^{\beta_1} \dfrac{K\beta_1}{\beta_1 - 1}\left(m - \overline{m}\right) + \dfrac{X\overline{m}}{2r - \mu - \omega} - K & X < X_F \\ \dfrac{Xm}{2r - \mu - \omega} - K & X \geq X_F \end{cases} \tag{20}$$

In figure 3 we plot the sensitivity of the value function of the follower and the leader to changes in profits per unit.

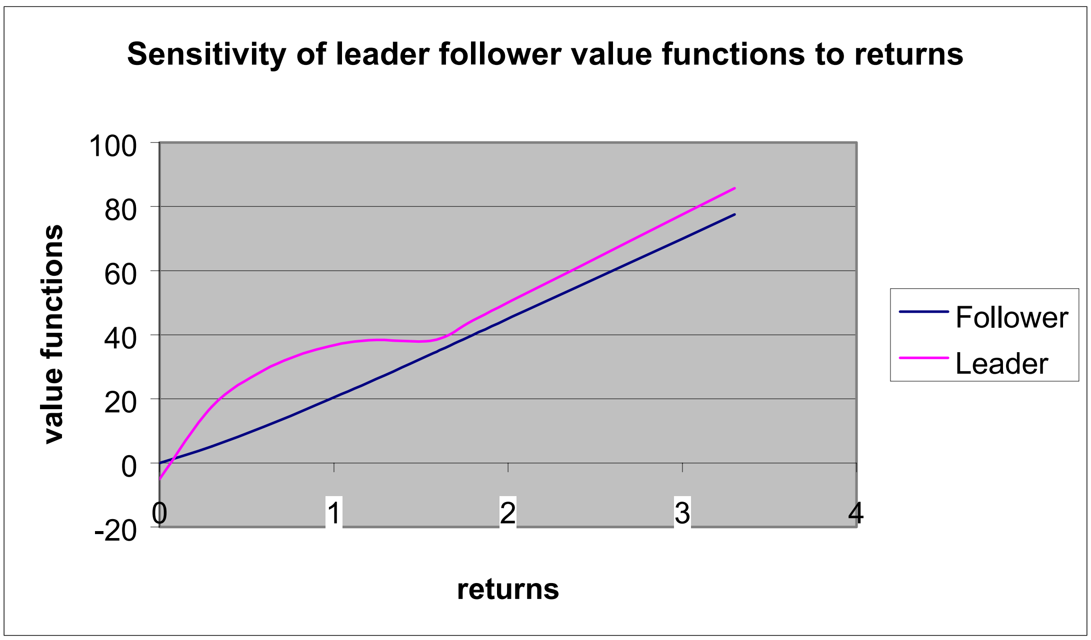


Fig 3 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; M=3; V=0.05; m=1.1; $\overline{m}$ =10; K=5 and V varies from 0 until 1.15.

We can see in figure 3 that the returns of the leader are almost always larger than the ones of the follower, except when the unit profits are very low. Notice that at this point the follower has not yet entered the market, and if the leader was already in the market he would be better off being a follower. We can also notice that when the follower enters the market[17] the two functions get closer, additionally, if there was no competitive advantage the two functions would be exactly the same from X_F onwards. Dixit and Pindyck (1994) describe this as a smooth-like-pasting condition of the present values.

The value function of the leader is more complicated than the one of the follower. It is concave until the moment the follower enters and at that precise moment in time it is discontinuous. This happens because although the total revenues are increasing due to an increase in the revenues per unit, they are also approaching the trigger function of the follower meaning that the negative effect of the entry of the follower increases. After the follower enters the two functions are very similar, with the leader having a higher value function because of a permanent advantage.

Also in figure 3 we can see that there is a point where the two functions meet, before that point a firm would be better of being a follower and after that a leader, this point represents the equalization principle of Fudenberg and Tirole (1985). If until that point a firm is better off being a follower and after it a leader, that point should be the trigger function of the leader[18], X_L.

We cannot obtain a closed form solution for X_L because the resulting function is highly non-linear, but we can obtain numerically the trigger[19]. We can also prove that this new function has a unit root strictly below X_F, implying that the trigger point exists and that it is unique[20].

Proposition 2. *The optimal leader strategy is to invest as soon as the revenues reach X_L, in other words the optimal time for the leader to invest is:*

$$T_L = \inf\{t > 0; X \in [X_L, X_F]\}$$

[17] For these parameters the follower enters the market when V=0.52, that is X=VM=1.57.

[18] Implying that this should be the point where a coin will be tossed and the winner will get the leader position.

[19] This can be done easily using for example the function solver in excel.

[20] See appendix B for details.

2.1.2 Without Pre-emption

In a game without pre-emption the role of the leader and the one of the follower is assigned exogenously. With the roles pre-assigned the adoption point of the leader does not have to be defined by indifference, like it was in the previous section, and can be defined optimally.

Let's start by assuming that the leader and the follower will enter the market at different points in time, so once again we will look at sequential equilibrium (we will consider simultaneous equilibrium later in this section).

2.1.2.1 The follower's value Function

The follower's value function is not affected by the fact that he is in a without pre-emption environment. Note that in the previous section we derived the value function using optimisation arguments. In other words the follower will enter the market, considering that the leader is already there, when he maximizes his value function, at the point in time where it is worthwhile to exercise his option. Exactly the same arguments can be used for the follower's value function when the roles were pre-assigned, and as so the optimal time for the follower to enter the market conditional on the previous entrance of the leader is the moment in time where the state variable reaches:

$$X_F = \frac{K(2r - \omega - \mu)\beta_1}{\beta_1 - 1} \tag{10}$$

And the value function:

$$
\begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{X}{X_F}\right)^{\beta_1} & X < X_F \\ \dfrac{X}{2r - \omega - \mu} - K & X \geq X_F \end{cases} \tag{12}
$$

2.1.2.2 The Leader's Value Function

Since the leader does not have to act under the fear of pre-emption he can enter the market optimally. The value function of the leader can be described by three components: before he enters the market he holds the option to wait to invest, after he enters the market but before the follower enters, he receives "monopolistic like" returns. After the follower enters the market he shares the market with the follower holding a first mover advantage.

The second and the third component are exactly the same as it was derived before; the first one will now be derived.

Without pre-emption the leader can enter the market when it is optimal to do so. His value function before he enters can be explained by equation (6) with solution:

$$
P_0^L(X) = GX^{\beta_1} + HX^{\beta_2} \tag{21}
$$

Where the roots are exactly as defined before.

Since β_2 is negative, we know from previous exposition that H=0. The constant G and the trigger point of the leader is defined by the value matching condition:

$$
P_0^L(X_L) = \left(\frac{X_L}{X_F}\right)^{\beta_1} \frac{K\beta_1}{\beta_1 - 1}\left(m - \overline{m}\right) + \frac{X_L \overline{m}}{2r - \mu - \omega} - K \tag{22}
$$

And the smooth pasting:

$$P_0^{L}{}'(X_L)=\left(\frac{X_L}{X_F}\right)^{\beta_1-1}\frac{K\beta_1^2}{(\beta_1-1)X_F}\left(m-\overline{m}\right)+\frac{\overline{m}}{2r-\mu-\omega} \tag{23}$$

Putting equations (21), (22) and (23) altogether we obtain the trigger function of the leader:

$$X_L=\frac{K(2r-\mu-\omega)}{\overline{m}}\frac{\beta_1}{\beta_1-1} \tag{24}$$

And:

$$G=X_F^{-\beta_1}K\frac{\beta_1}{\beta_1-1}(m-\overline{m})+X_L^{-\beta_1}\frac{K}{\beta_1-1} \tag{25}$$

The value function of the leader is described by:

$$\begin{cases}\left(\dfrac{X}{X_F}\right)^{\beta_1}K\dfrac{\beta_1}{\beta_1-1}(m-\overline{m})+\left(\dfrac{X}{X_L}\right)^{\beta_1}\dfrac{K}{\beta_1-1} & X<X_L\\ \left(\dfrac{X}{X_F}\right)^{\beta_1}\dfrac{K\beta_1}{\beta_1-1}\left(m-\overline{m}\right)+\dfrac{X\overline{m}}{2r-\mu-\omega}-K & \left[X_L;X_F\right]\\ \dfrac{Xm}{2r-\mu-\omega}-K & X>X_F\end{cases} \tag{26}$$

The value function of the leader can be divided in three components. In the first equation we have the value of the option to enter the market, in the second one the monopolistic like revenues and in the third one the market shared with the follower.

In figure 4 we can see the sensitivity of the value function of the leader to increases in profits per unit:

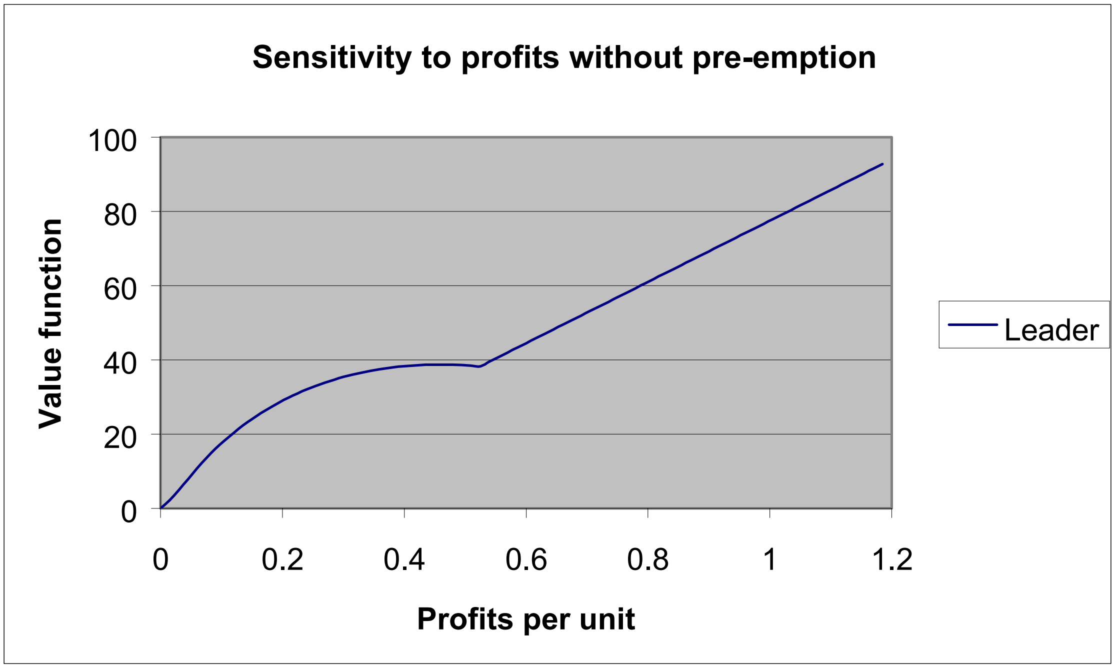


Fig. 4 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; M=3; K=5; m=1.1; $\overline{m}$ =10 and V varies from 0 until 1.185[21].

Before the leader enters, the function behaves like an option, so it is almost a straight line when plotted against profits per unit. After the leader enters the function is concave reflecting not only the increases in value due to increases in the profit per unit but also decreases due to an imminent entrance of the follower. After the follower enters the leader shares the market.

Proposition 3. *The optimal leader strategy is to invest as soon as the revenues reach X_L , in other words the optimal time for the leader to invest in a non-pre-emptive environment is:*

$$T_L = \inf\left\{ t > 0; X = \frac{K(2r - \mu - \omega)}{\overline{m}} \frac{\beta_1}{\beta_1 - 1} \right\}$$

[21] For these parameters the trigger function of the leader is 0.157 and the one of the follower 1.57.

Let's now consider the alternative where the leader and the follower invest simultaneously.

We will assume that if the companies invest simultaneously their number of units sold will be multiplied by a factor m_S that is larger than one but smaller than m. With this framework we are trying to describe a situation where the sequential equilibrium will be better for the leader and the simultaneous one better for the follower[22].

Following the same procedure described above we obtain the trigger and value function of each firm when the investment is simultaneous:

$$X_S = \frac{K(2r-\mu-\omega)}{m_S}\frac{\beta_1}{(\beta_1-1)} \tag{27}$$

$$\begin{cases} \dfrac{K}{\beta_1-1}\left(\dfrac{X}{X_S}\right)^{\beta_1} & X < X_S \\ \dfrac{Xm_S}{2r-\mu-\omega}-K & X \geq X_S \end{cases} \tag{28}$$

The sensitivity of the leader and the follower to increases in profits per unit in simultaneous equilibrium is shown in figure 5:

[22] Note that simultaneous equilibrium always arises after the follower's trigger, in other words if the initial value of the state variable is bigger than X_F and one the firm enters the other one would follow immediately. With this framework we are imposing simultaneous entry despite the initial value of the state variable.

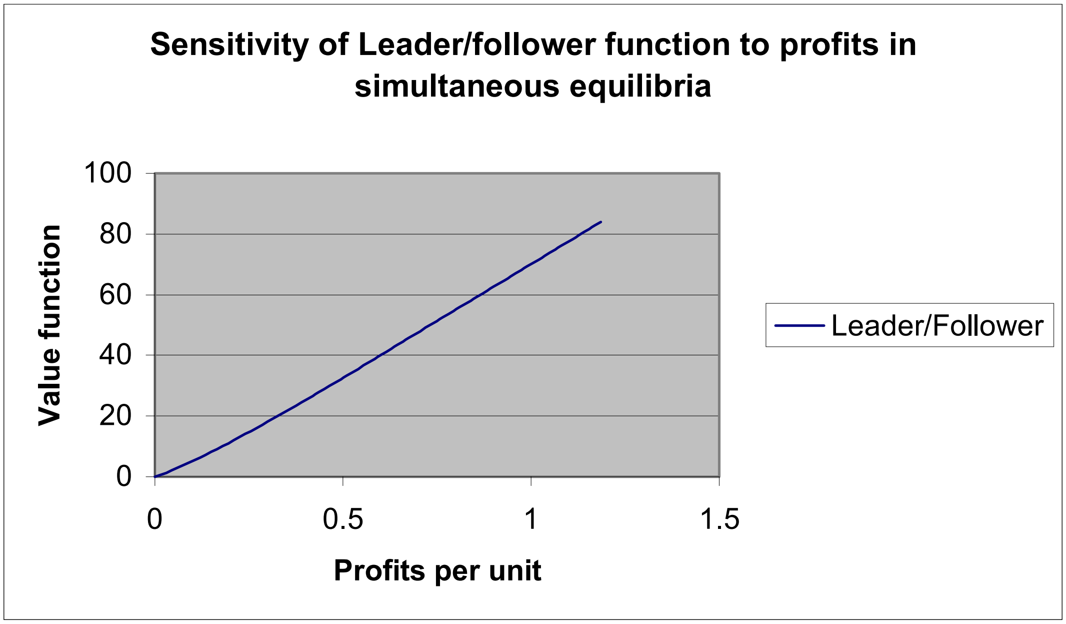


Fig. 5 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; Ms=1.001; K=5; and V varies from 0 until 1.185.

As expected the value function of both the leader and the follower increases with increases in the profits per unit.

Proposition 4. *The optimal strategy in a simultaneous equilibrium setting is to invest as soon as the revenues reach* X_S*, in other words the optimal time for both firms to invest in a non-pre-emptive environment is:*

$$T_S = \inf\left\{ t > 0; X = \frac{K(2r - \mu - \omega)}{m_S} \frac{\beta_1}{\beta_1 - 1} \right\}$$

In figure 6 we present the triggers sensitivity to volatility of the possible outcomes in a without pre-emption environment:

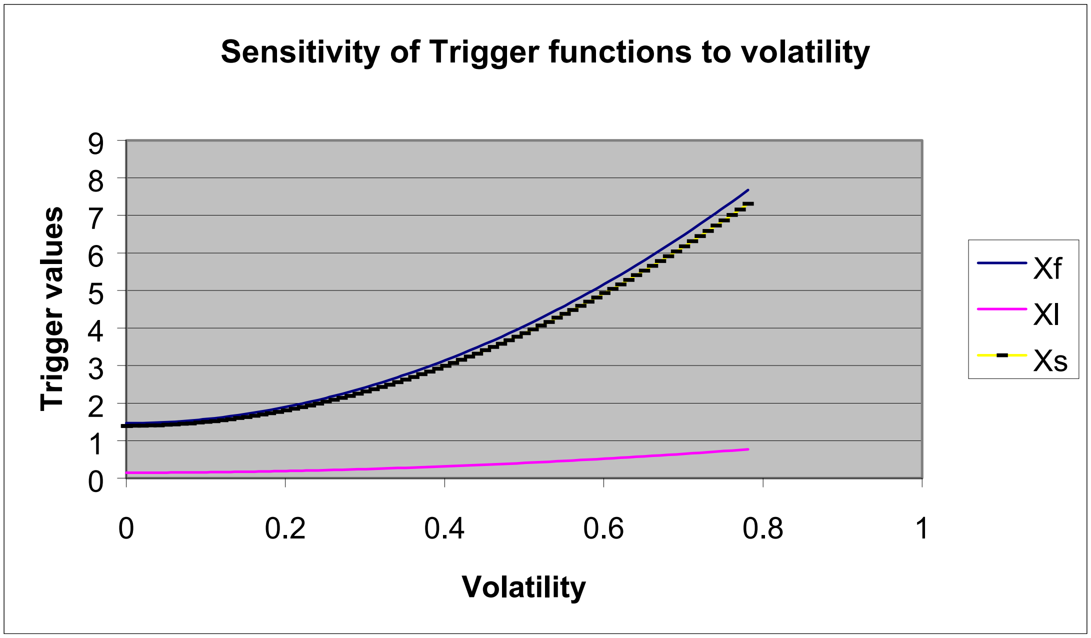


Fig. 6 - The parameters are: μ=3%; ω=3%; α=20%; ρ=0; ms=1.05; K=5; M=3; m=1.1; V=0.05 and σ varies from 0.0001 until 0.781.

The triggers always increase with volatility, but it seems that the triggers are more sensitive to changes at high levels of volatility. What this seems to suggest is that if volatility is very high the investor will want to wait more for the uncertainty to be resolved.

Both the triggers of the follower and of the simultaneous investor are more sensitive to volatility than the trigger of the leader, this may happen because the leader always has a competitive advantage and the trade-off of that advantage compensates the value lost while exercising the option to invest.

The triggers of the follower and of the simultaneous investor behave in a very similar way, this happens because the two functions are indeed very similar, having been derived with the same arguments[23]. The trigger of the simultaneous investor is smaller than the one of the follower due to m_s, since while investing

[23] Notice that the value function of the leader is a lot more complicated, being divided in three different functions.

simultaneously the follower will sell more units than if he was investing alone, he will enter the market sooner.

2.2 Comparing the equilibriums

As described in the previous sections in a pre-emption setting the roles of the leader and the follower are endogenous to the model having each firm a 50% probability of becoming a leader or a follower. The equilibrium outcomes of this setting will depend on the initial level of total profits, X. According to proposition 2 if $X < X_L$ both firms will wait, until X_L is hit.

When X_L is hit one of the firms will enter the market, so if X $\in$ $[X_L, X_F]$ a sequential equilibrium will arise, one firm will enter at X_L and in order to maximise its profits the other one will wait and enter at X_F.

If the state variable X is bigger than X_F, and none of the firms have already entered the market, then a simultaneous equilibrium will arise, if one firm enters the other one will immediately follow in order to maximise its value function.

So, if the total profits of the market are smaller than X_L *do not enter, wait until the trigger is hit. If X* $\in$ $[X_L, X_F]$ *enter the market, you will receive "monopoly like" revenues for a while and you will obtain a competitive advantage that will be important even after your competitor enters, if your competitor entered before you then wait until your total profits equal* X_F. *If X is bigger than* X_F *you should immediately enter the market despite the action of your opponent.*

In a without pre-emption setting either a simultaneous or a sequential equilibrium may also arise but now the actions depend on what it was accorded by the two companies.

Sequential equilibrium in without pre-emption environments may arise for example in markets where one of the companies that is considering to enter is so strong that the other one would prefer to wait and see the reaction of the market.

Simultaneous equilibrium may arise if both companies agree on the advantage of entering together.

In figure 7 the sensitivity of the trigger function to volatility both in environments with and without pre-emption are presented.

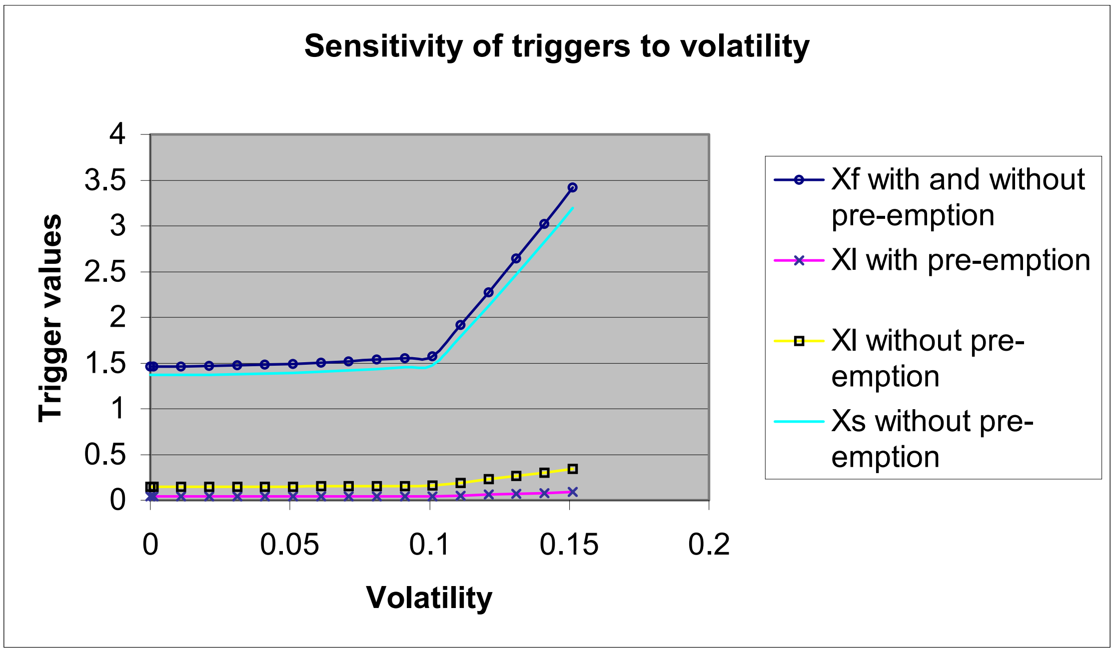


Fig. 7 - The parameters are: μ=3%; ω=3%; α=20%; ρ=0; Ms=1.01; K=5; M=3; m=1.1; V=0.05 and σ varies from 0.0001 until 0.151.

We can see that the trigger function of the follower and the one of the simultaneous investor tend to behave in a similar way and the leader's trigger both in a pre-emption and in an without pre-emption environment tend also to be similar. As expected in a pre-emption environment the trigger of the leader is less sensitive to volatility because, as it was already explained, the trigger was defined with rules of indifference and not with optimisation ones, in other words, in a pre-emption environment the leader acts under the fear of pre-emption[24] not at the

[24] The leader acts sooner to achieve the first mover advantage.

moment that it is optimal to exercise the option to invest but at the moment where it is indifferent to be a leader or a follower.

Pre-emption does not seem to affect the follower as it does the leader. Notice that the trigger of the follower is the same in a pre-emption environment and in a without pre-emption one, meaning that the follower will delay his entry exactly the same amount of time regardless of pre-emption.

3- Profit flow and investment as stochastic variables

Models where both the investment cost and the profit flow follow a stochastic process can be seen in Williams (1991) and Quigg (1993). Quigg (1993) develops a model to price land incorporating the option to wait to develop, the option value being a function of both the developed building and development costs.

We will now develop a similar model to Quigg's (1993) but considering strategic implications.

Let R stand for total return and K for investment, both these variables follow a Geometric Brownian Motion of the kind:

$$dR = \mu R dt + \sigma R dz \qquad (27)$$

And

$$dK = \mu K dt + \sigma K dz \qquad (28)$$

Where:

μ and ω are the expected gain of R and K respectively or, in other words the drift of the Brownian motion;

σ and α are the volatilities and ρ the correlation coefficient.

3.1- The Game

The scenarios will be similar to the ones designed in section 2. We will have two companies considering entering a new market and we will analyse both with and without pre-emption environments.

We will also consider that the leader will always have a competitive advantage m defined previously.

3.1.1- With Pre-emption

As previously in the with pre-emption setting the roles of the leader and the follower will be assigned endogenously. We will start by defining the value function of the follower.

3.1.1.1- The follower's value function

Following the methodology described in section 2 we obtain the partial differential equation of the value function of an idle follower:

$$\frac{1}{2}\frac{\partial^2 P_0^F}{\partial R^2}\sigma^2 R^2 + \frac{1}{2}\frac{\partial^2 P_0^F}{\partial K^2}\alpha^2 K^2 + \frac{\partial P_0^F}{\partial R \partial K} RK\sigma\alpha\rho + (r-\mu)R\frac{\partial P_0^F}{\partial R} + (r-\omega)K\frac{\partial P_0^F}{\partial K} - rP_0^F = 0 \tag{31}$$

This equation should be subject to the usual two boundary conditions; the first one is the value matching, note that at the point where the value of the option equals the present value of the total return minus the investment, the investment is done and consequently all the uncertainty related to the investment cost disappears; the second boundary comes from the smooth pasting condition.

In order to obtain a closed form solution for equation (31), as we did in the previous section, we will use similarity methods.

Let now the substitution be $Y = \frac{R}{K}$ implying that $P_0^F(R,K) = Kp\left(\frac{R}{K}\right) = Kp(Y)$. After all the computations (31) can be written as[25]:

$$\frac{1}{2}Y^2 p''(Y)\gamma^2 + Yp'(Y)[\omega - \mu] - \omega p(Y) = 0 \tag{32}$$

Where:

$$\gamma^2 = \sigma^2 + \alpha^2 - 2\sigma\alpha\rho \tag{33}$$

Similarly to the previous section (32) is of Euler's type with solution:

$$p(Y) = IY^{\beta_1} + JY^{\beta_2} \tag{34}$$

Where β_1 and β_2 are given by:

$$\beta_{1/2} = \frac{1}{2} + \frac{(\mu - \omega)}{\gamma^2} \begin{matrix} + \\ - \end{matrix} \sqrt{\left(\frac{\omega - \mu}{\gamma^2} - \frac{1}{2}\right)^2 + \frac{2\omega}{\gamma^2}} \tag{35}$$

Being $Y = \frac{R}{K}$, we know that if the total return is zero Y will be zero implying that J in equation (34) has to be zero.

---

[25] See appendix C for details.

Rewriting our boundary conditions using our substitution we obtain the value function of the follower after some computations:

$$\begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{Y}{Y_F}\right)^{\beta_1} & Y < Y_F \\ \dfrac{R}{r-\mu} - K & Y \geq Y_F \end{cases} \tag{36}$$

Where the trigger is given by:

$$Y_F = \frac{(r-\mu)\beta_1}{\beta_1 - 1} \tag{37}$$

Proposition 5. *The optimal follower strategy conditional on a previous entry by the leader is to invest as soon as the revenues reach Y_F, in other words the optimal time for the follower to invest in an environment where both the returns and the investment cost follow a geometric Brownian motion is:*

$$T_F = \inf\left\{ t \geq 0; Y = \frac{(r-\mu)\beta_1}{\beta_1 - 1} \right\}$$

3.1.1.2- <u>The leader's value function</u>

We will consider that the leader will always have an advantage over the follower, before the later enters the market the leader receives $R\overline{m}$, where $\overline{m}$ is a factor that multiplied by the return gives the total return of a monopoly. After the follower enters the market the leader will still have an advantage, m, m is once again a multiplicative factor, bigger than one that can represent either bigger market share or higher prices.

The value function of the leader, while alone in the market can be explained by:

$$E\left[\int_0^{T_F} e^{-r\tau} R_\tau \overline{m} - K_\tau d\tau\right] + \int_{T_F}^{\infty} R_\tau m d\tau \qquad (38)$$

Using the same methodology of section 2.1.2, and using the substitution presented in section 3.2.1 we obtain the value function of the leader:

$$\begin{cases} \dfrac{R\overline{m}}{r-\mu} + \left(\dfrac{Y}{Y_F}\right)^{\beta_1} \dfrac{\beta_1 K}{\beta_1 - 1}\left(m - \overline{m}\right) - K & Y \le Y_F \\ \dfrac{Rm}{r-\mu} - K & Y > Y_F \end{cases} \qquad (39)$$

In figure 8 we plotted both the value function of the leader and the follower as total return increases:

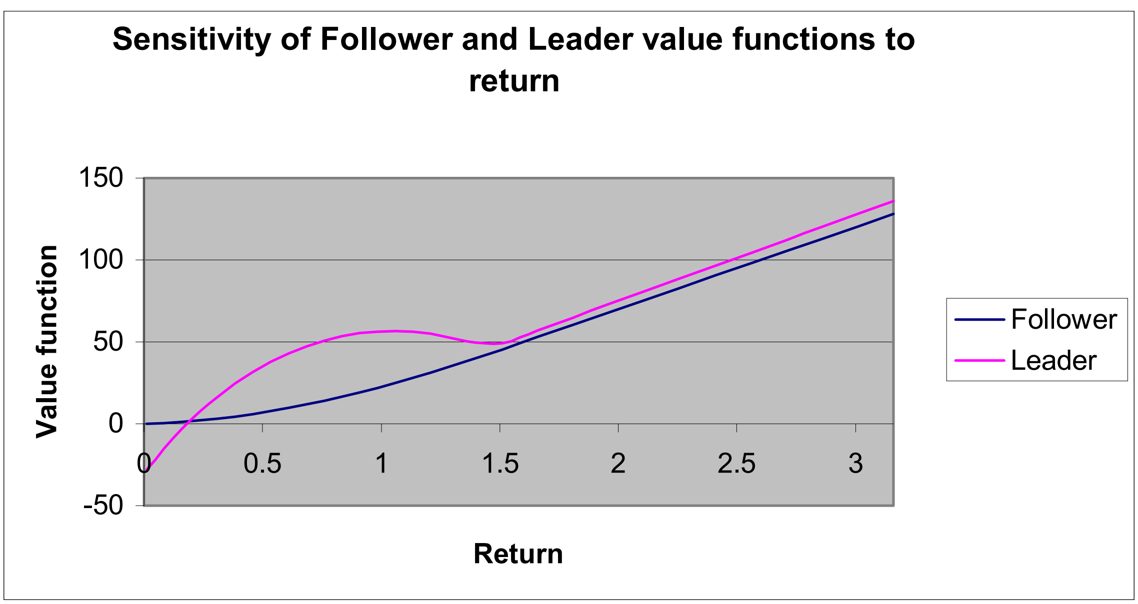


Fig 8 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; K=30; r=0.05; m=1.05; $\overline{m}$ =4 and R varies from 0.01 until 3.16.

Figure 8 is very similar to figure 3. Both the value function of the follower and the one of the leader increase with return. Once again we can see a point where the two functions meet, below that point the firm will be better of being a follower and after that a leader.

Using the methodology presented in appendix B we can prove that the trigger point of the leader is unique. So there is an optimal point for the leader to invest, before that point he would be better of being a follower.

Proposition 6. *The optimal leader strategy is to invest as soon as the revenues reach* Y_L*, in other words the optimal time for the leader to invest is:*

$$T_L = \inf\{t > 0; Y \in [Y_L, Y_F]\}$$

The sensitivity of the value functions to investment cost is plotted in figure 9:

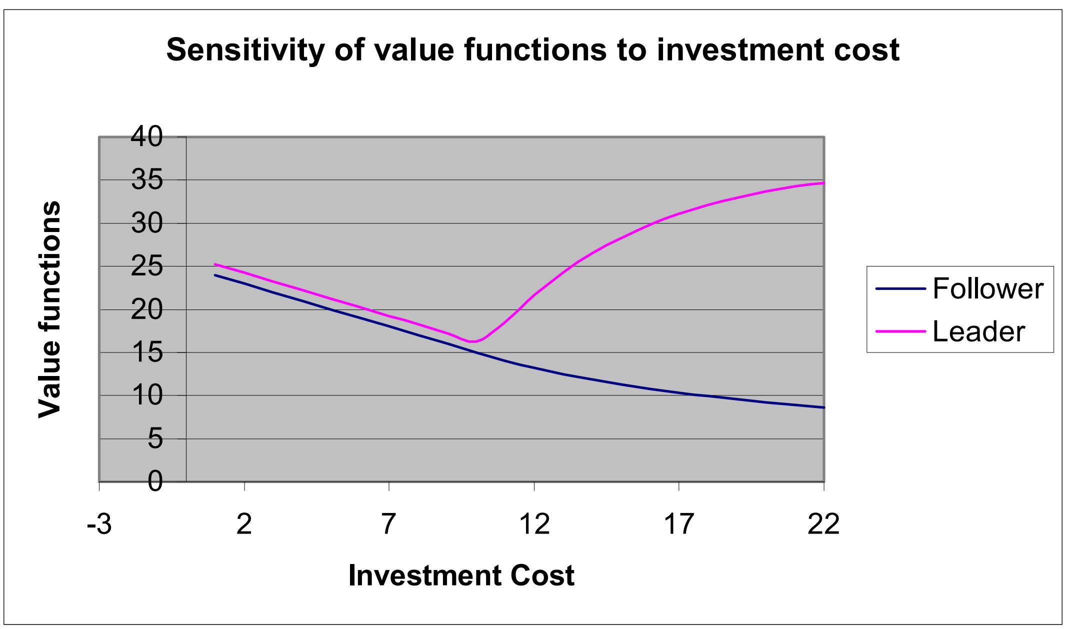


Fig 9 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; R=0.5; r=0.05; m=1.05; $\overline{m}$ =4 and K varies from 1 until 22.

Figure 9 is, as expected, almost a mirror image of figure 8 since the two analysed variables have an opposite effect on the value functions.

The results obtained for the value function of the follower agree with the literature, i.e. as investment cost increases the value function decreases.

The value function of the leader seems to react in a counterintuitive way, it decreases initially and then increases sharply. Notice that while the follower is in the market, in this example the beginning of the graph until the investment cost equals 10, the effect of the investment cost on the leader is similar to the effect on the follower[26]. The function starts then increasing for high values of investment, although this may seem outlandish it can be explained by the positive effect that the high investment cost will have while discouraging the follower from entering the market. In other words, the investment cost, after a certain level, has a double effect on the value function of the leader, it decreases its value function but while stopping the follower's entry allows the leader to receive "monopolistic like" revenues for longer. Looking at the plot of the value function of the leader while the follower is not in the market it looks like the positive effect of increases in investment cost outperforms the negative one.

3.1.2- Without Pre-emption

As it was already described in the without pre-emption environment we are assuming that the roles of the leader and the follower are assigned exogenously, in other words there will be no race to the roles. We will have a sequential and a simultaneous equilibrium, we will start by defining the sequential equilibrium, the simultaneous will be defined later in the section.

3.1.2.1 - The follower's value function

[26] Notice the value function of the leader after the follower enters in equation (39).

The value function of the follower will be exactly the same as it was in the with pre-emption setting, notice that the follower entered optimally in the previous section and he will obviously do the same in a without pre-emption environment. The value function of the follower will then be:

$$\begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{Y}{Y_F}\right)^{\beta_1} & Y < Y_F \\ \dfrac{R}{r-\mu} - K & Y \geq Y_F \end{cases} \tag{36}$$

3.1.2.2- <u>The Leader's value function</u>

In a without pre-emption setting the value function of the leader can be defined optimally.

Before the leader enters the market he owns the option to invest, after entering but before the follower does it he enjoys "monopolistic like" revenues, and after the follower enters he shares the market maintaining a competitive advantage m.

Following the procedure already described we obtained the value function of the leader:

$$\begin{cases} \dfrac{K}{\beta_1 - 1}\left(\dfrac{Y}{Y_L}\right)^{\beta_1} + \dfrac{K\beta}{\beta_1 - 1}(m - \overline{m})\left(\dfrac{Y}{Y_F}\right)^{\beta_1} & Y \leq Y_L \\ \dfrac{R\overline{m}}{r-\mu} + \left(\dfrac{Y}{Y_F}\right)^{\beta_1} \dfrac{\beta_1 K}{\beta_1 - 1}\left(m - \overline{m}\right) - K & Y \in \left[Y_L, Y_F\right] \\ \dfrac{Rm}{r-\mu} - K & Y > Y_F \end{cases} \tag{40}$$

Where the Y_L is given by:

$$Y_L = \frac{r-\mu}{\overline{m}} \frac{\beta_1}{\beta_1 - 1} \tag{41}$$

In figure 10 we plot the value functions against changes in returns:

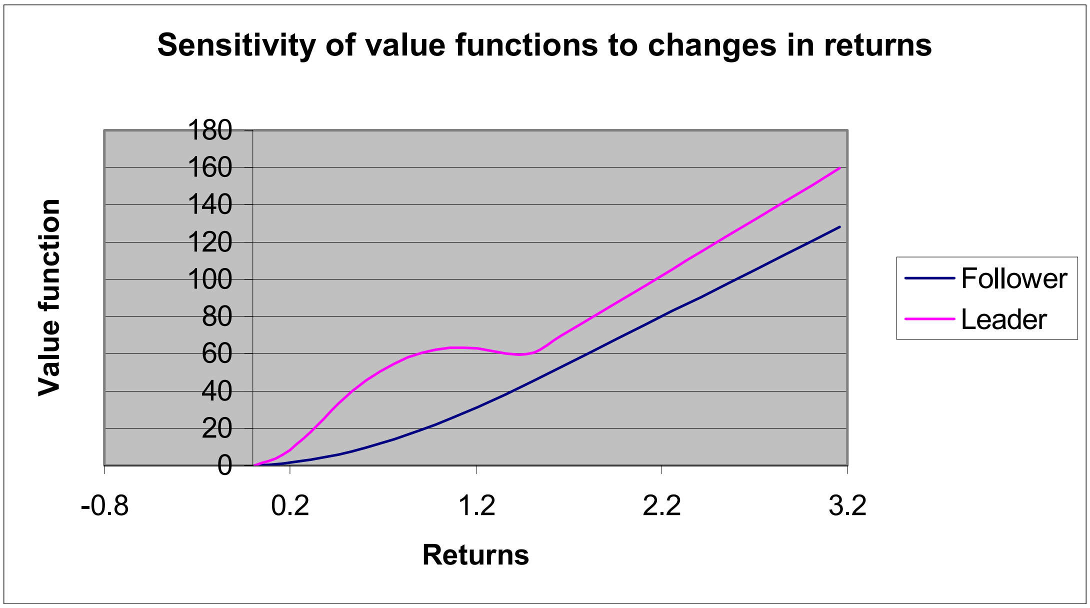


Fig 10 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; K=30; r=0.05; m=1.2; $\overline{m}$ =4 and R varies from 0.01 until 3.16.

Both value functions increase with returns. Comparing with figure 8 we can see that the value function of the leader always has a positive value in this case, this happens because the value function is now defined with optimisation principles.

The plot of the value functions against investment cost is presented in figure 11:

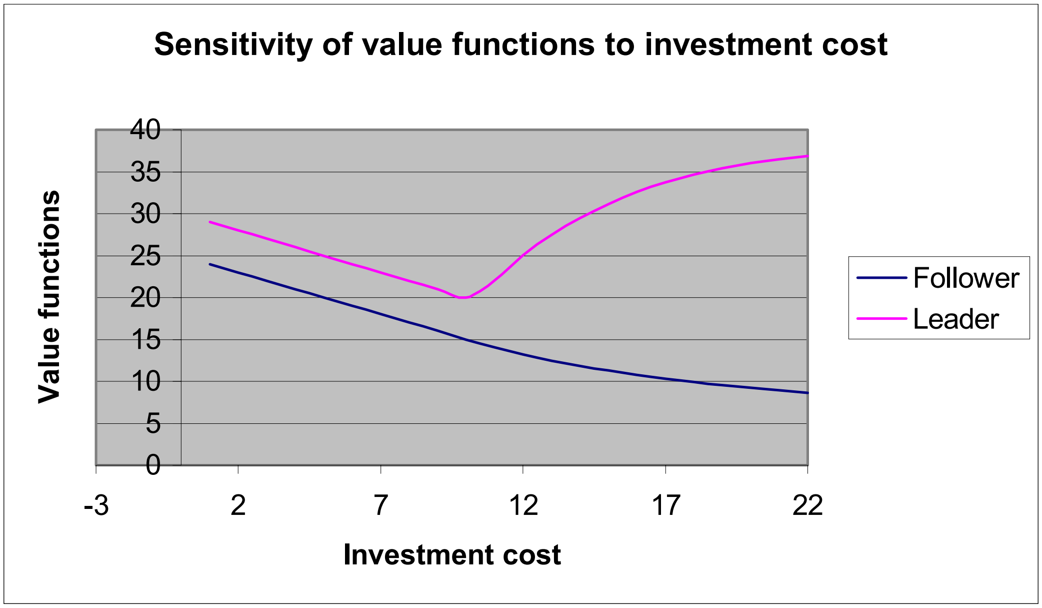


Fig 11 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; R=0.5; r=0.05; m=1.2; $\overline{m}$ =4 and K varies from 1 until 22.

Once again figure 11 behaves like a mirror image of figure 10.

Proposition 7. *The optimal leader strategy is to invest as soon as the revenues reach* Y_L *, in other words the optimal time for the leader to invest in a non-pre-emptive environment where both the return and the investment cost follow a geometric Brownian motion is:*

$$T_L = \inf\left\{t > 0; Y = \frac{r-\mu}{\overline{m}} \frac{\beta_1}{\beta_1 - 1}\right\}$$

When both firms agree to enter at the same time their value function will be[27]:

[27] Notice that as we did for the previous model we are assuming that if the firms enter simultaneously their return will multiplied by a factor m_S that in the lower limit can be one but can never be bigger than m.

$$\begin{cases} \dfrac{K}{\beta-1}\left(\dfrac{Y}{Y_S}\right)^{\beta_1} & Y < Y_S \\ \dfrac{Rm_S}{r-\mu} - K & Y \geq Y_S \end{cases} \tag{42}$$

Where Y_S is given by:

$$Y_S = \frac{r-\mu}{m_S}\frac{\beta_1}{\beta_1 - 1} \tag{43}$$

Figure 12 presents the value functions reaction to changes in investment cost:

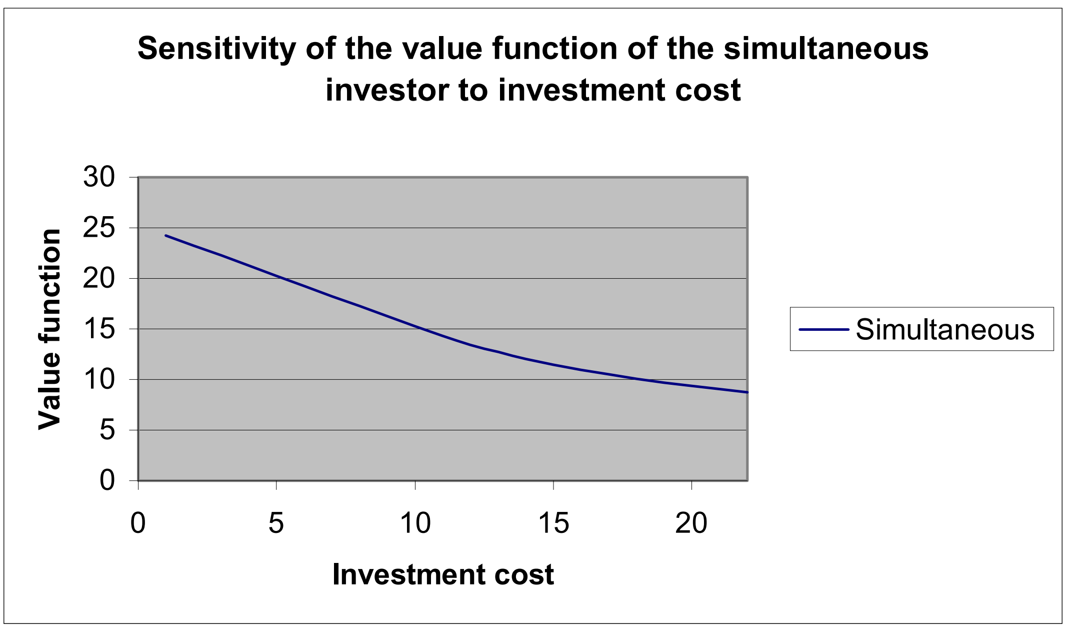


Fig 12 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; R=0.5; r=0.05; m_S=1.01 and K varies from 1 until 22.

When investment is simultaneous the value function of both firms is very similar to the follower's value function, having been derived with the same arguments and consequently, the value function decreases almost linearly as investment cost increases.

Proposition 8. *The optimal strategy in a simultaneous equilibrium setting is to invest as soon as the revenues reach Y_S, in other words the optimal time for both firms to invest in a non-pre-emptive environment is:*

$$T_S = \inf\left\{ t > 0; Y = \frac{(r-\mu)}{m_S} \frac{\beta_1}{\beta_1 - 1} \right\}$$

3.2 <u>Comparing the equilibriums</u>

Having obtained the value functions of both the leader and the follower and the trigger functions we can now design the strategies for the different games. Those strategies will be based on the propositions stated in this section.

Either in game where both players want to grab the position of the leader or on the one where the players agree on the position, the actions should be defined by the initial value of the state variable Y, in other words the moves of the players should be defined according to the initial value of the ratio total return to investment cost.

In a setting with pre-emption if the initial value of Y is smaller than Y_L both firms should wait until Y_L is hit. As soon as Y hits Y_L one of the firms should enter and the other one has, as best action, to wait until Y_F is hit[28].

Two possible equilibriums may arise, if $Y < Y_F$ the outcome will be a sequential equilibrium. The firm that wins the game will be the leader and the other one will only enter the market in it is optimal. If $Y > Y_F$ and none of the firms is in the market, they would enter immediately and a simultaneous equilibrium will arise.

[28] Propositions 5 and 6.

In a without pre-emption setting we can also have a sequential or a simultaneous equilibrium, depending on what it was agreed between the players. Propositions 7 and 8 present the optimal entering time for the two possible equilibriums.

In figure 13 we present the sensitivity of our trigger functions to volatility:

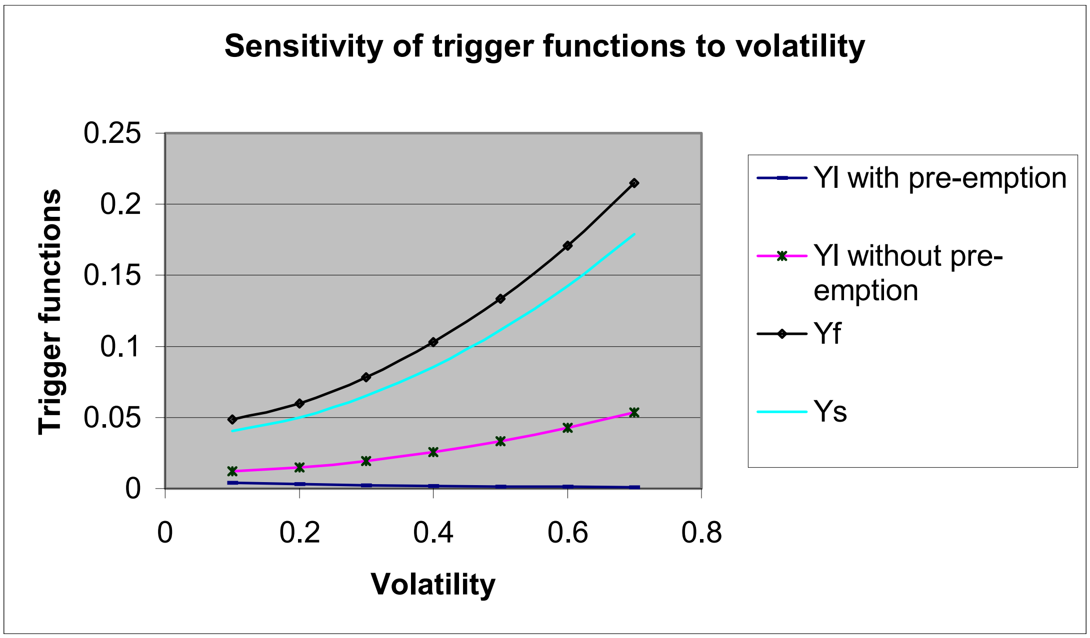


Fig 13 - The parameters are: σ =0.1; μ=3%; ω=3%; α=20%; ρ=0; R=0.5; r=0.05; m_S=1.2 and m=1.5 and K varies from 1 until 22.

In figure 13 we can see that as expected the triggers increase with volatility with one exception, the trigger of the leader in an environment with pre-emption.

The trigger of the pre-emptive leader is almost indifferent to volatility, counter intuitively it decreases slightly. This odd result can probably be explained due to pre-emption effects and can be also be found e.g. in Mason and Weeds (2000). A possible explanation is that since this trigger is obtained by indifference, and not optimally, the trigger of the follower is influencing it[29]. In other words since the trigger of the follower influences the trigger of the leader and the former increases with volatility that can induce a decrease in the leader's trigger as volatility increases.

4– Different models for different situations

Although the main principles of the derived models are very similar, the models are indeed different and should be applied to different situations.

In the first model, derived in section 2, the variables elected to follow two different stochastic processes are the profits per unit and the number of units.

While doing this we are considering that those are the variables that we expect to be more effected by volatility and therefore this model should be applied by companies which business focuses on high volumes of production, for example internet providers or mobile phone companies.

In the second model, derived in section 3, the total profit and the investment cost follow different paths.

Companies that are more susceptible to apply these models are the ones to whom the investment cost has a high probability of changing with time. In our opinion a natural selection are R&D firms. The high uncertainty concerning the time to develop a product in R&D companies results in high levels of uncertainty concerning returns and investment costs, and as so, our model provides a possible approach to determine the value of the investments of those firms.

[29] Note that the trigger is obtained by: $\frac{R\overline{m}}{r-\mu}+\left(\frac{Y_L}{Y_F}\right)^{\beta_1}\frac{\beta_1 K}{\beta_1-1}\left(m-\overline{m}\right)-K-\frac{K}{\beta_1-1}\left(\frac{Y_L}{Y_F}\right)^{\beta_1}=0$, where Y_L is the unknown.

5- Conclusion

Real options provide a powerful framework to investment decisions, however the application of financial options valuation methodologies to real investments has to consider strategic implications.

We derive and analyse two models both in an environment in which the players fight for their position and where their roles are pre-assigned. In these two models we have two stochastic factors: in the first one the profits per unit and the number of units follow a different stochastic process and in the second one the returns and the investment cost follow different stochastic paths.

We obtained closed form solutions for the value functions of the leader and the follower in both models and for the different designed games. Analytical solutions for the trigger functions of the leader and the follower were also obtained, with the exception of the leader's trigger in pre-emption environments.

The sensitivity of several parameters is studied. Almost all parameters behave according to what is expected from the literature, the only exception is the trigger of the leader in the model where both the investment cost and the returns are stochastic, which counter-intuitively decreases with increases in volatility.

In future work we would like to allow the number of units of the first model, to follow not a geometric Brownian motion but a different stochastic process. As it was pointed out while we were introducing the first model, the number of units has very different characteristics from the profits per unit, it would be interesting to consider a stochastic process for this variable that better explains its movements. In this paper the players are symmetrical ex-ante and asymmetrical ex-post, of further interest would also be to permit for asymmetries of the players ex-ante.

References

Dixit, A. and Pindyck, R. (1994), *Investment under Uncertainty*, Princeton, NJ: Princeton University Press.

Dutta, P. and Rutstichini, A. (1995), "Equilibria in Stochastic Games," *Journal of Economic Theory*, 67, 1-39.

Fudenberg, D. and Tirole, J. (1985), "Pre-emption and Rent Equalisation in the Adoption of New Technology," *Review of Economic Studies*, 52, 383-401.

Fudenberg, D. and Tirole, D. (1991), *Game Theory*, The MIT Press. Cambridge Massachusetts.

Grenadier, S. (1996), "The Strategic Exercise of Options: Development Cascades and Overbuilding in Real Estate Markets," *Journal of Finance*, 51, 1653-1679.

Grenadier, S (2002), "Option Exercise Game: An Application to the Equilibrium Investment Strategies of Firms," *The Review of Financial Studies*, 15, Nº 3, 691-721.

Hope, H. (2000), "Second-mover Advantages in the Strategic Adoption of New Technology under Uncertainty," *International Journal of Industrial Organization,* 18, 315-338.

Howell, S. et al (2001) *Real options – Evaluating Corporate Investment Opportunities in a Dynamic World*, edited by Prentice Hall, London.

Kulatilaka, N. and Perotti, E.C. (1998), "Strategic Growth Options," *Management Science*, 44, 1021-1031.

Lambrecht, B. and Perraudin, W.R. (1997), "Real Options and Pre-emption," *Working paper*, Cambridge University.

Lee, J. and Paxson, D. (2003), “Approximate Valuation of Real R&D American Sequential Exchange Options,” *in Real R&D Options*, edited by Dean Paxson.

Margrabe, W. (1978), “The Value of an Option to Exchange one Asset for Another,” *Journal of Finance*, vol. 33: 1, pp. 177-186.

Mason, R. and Weeds, H. (2000), “Networks, Options and Pre-emption,” *Working paper*, Cambridge University.

Pawlina, G. and Kort, P (2002), “Real Options in an Asymmetric Duopoly: Who Benefits from your Competitive Disadvantage,” *Working paper presented in the European Finance Association.* Berlin (2002).

Quigg, L. (1993), “Empirical Testing of Real Option-Pricing Models,” *The Journal of Finance*, vol.4, nº. 2, pp. 621-640.

Shackleton, M and Tsekrekos, A. and Wojakowski, R. (2002), “Strategic Entry and Market Leadership in a Two-Player Real Options Game,” *Working paper presented in the Real Options Conference (2002).*

Smets, F.R. (1991), “Exporting versus foreign direct investment: the effect of uncertainty, irreversibilities and strategic interactions,” *Working paper*, Yale University.

Smit, H and Trigeorgis, L. (2001), “Flexibility and Commitment in Strategic Investments,” *in Real Options and Investment under Uncertainty*, edited by Eduardo Schwartz and Lenos Trigeorgis.

Spatt, C.S. and Sterbenz, F.P. (1985), “Learning, pre-emption and the degree of rivalry,” *Rand Journal of Economics*, 16, 84-92.

Tellis, G. and Golder, P. (1996), "First to Market, First to Fail? Real Causes of Enduring Market Leadership," *Sloan Management Review*, Winter, 65-75.

Tsekrekos, A. E. (2003), "The Effect of First-mover Advantages on the Strategic Exercise of Real Options," *in Real R&D Options*, edited by Dean Paxson.

Tunaru, R. and Clark, E. and Viney, H (2002) "Valuation of football players," *Working paper presented in the Real Options Conference (2002).*

Weeds, H. (2000), "Strategic Delay in a Real Options Model of R&D Competition," *Working paper*, Warwick University.

Williams, J. (1991), "Real Estate Development as an Option," *Journal of Real Estate Finance and Economics*, vol. 4, pp. 191-208.

Zhang, P. (1997), *Exotic Options*, World Scientific. Hong Kong.

Appendix A

Let equation (3) represent the value function of an idle follower:

$$\frac{1}{2}\frac{\partial^2 P_0^F}{\partial V^2}\sigma^2 V^2 + \frac{1}{2}\frac{\partial^2 P_0^F}{\partial M^2}\alpha^2 M^2 + \frac{\partial P_0^F}{\partial M \partial V} MV\sigma\alpha\rho + (r-\mu)V\frac{\partial P_0^F}{\partial V} + (r-\omega)M\frac{\partial P_0^F}{\partial M} - rP_0^F = 0 \quad (3)$$

As it was already said previously in order to obtain a closed form solution for this equation we have to use similarity methods.

Let $X = VM$ implying that:

$$P_0^F(X) = P_0^F(V, M)$$

$$\frac{\partial P_0^F(V,M)}{\partial M} = \frac{\partial P_0^F(X)}{\partial X} V$$

$$\frac{\partial P_0^F(V,M)}{\partial V} = \frac{\partial P_0^F(X)}{\partial X} M$$

$$\frac{\partial^2 P_0^F(V,M)}{\partial M^2} = \frac{\partial^2 P_0^F(X)}{\partial X^2} V^2$$

$$\frac{\partial^2 P_0^F(V,M)}{\partial V^2} = \frac{\partial^2 P_0^F(X)}{\partial X^2} M^2$$

$$\frac{\partial^2 P_0^F(V,M)}{\partial MV} = \frac{\partial^2 P_0^F(X)}{\partial X^2} VM + \frac{\partial P_0^F(X)}{\partial X}$$

Substituting back in equation (3) we obtain equation (6):

$$\frac{1}{2} X^2 P_0^F{''}(X)\left[\sigma^2 + \alpha^2 + 2\sigma\alpha\rho\right] + XP_0^F{'}(X)\left[\sigma\alpha\rho + 2r - \mu - \alpha\right] - rP_0^F(X) = 0$$

<u>Appendix B</u>

We will prove now the uniqueness of the leader's trigger point in the model derived in section 2.

If we evaluate V(X), the value function of both the leader and the follower, at X=0 we obtain:

$V(X) = -K \quad < 0$

And evaluating V(X) at X_F :

$$V(X_F) = \frac{K\beta(m-1)}{\beta - 1} + K > 0$$

And as so, from $(0, X_F)$ there must exist at least one root. If now the function V(X) is strictly concave over $(0, X_F)$ the root X_L is unique. The second derivative of V(X) is:

$$\frac{K(m-\overline{m})\left(\dfrac{X}{X_F}\right)^{\beta-2}\beta^2}{X_F^2} - \frac{K\left(\dfrac{X}{X_F}\right)^{\beta-2}\beta}{X_F} < 0$$

Thus the root is unique, with V(X)<0 for X $\in \left(0, X_L\right)$ and V(X)>0 for X $\in \left(X_L, X_F\right)$.

So we have demonstrated that there is a single point X_L where the functions of the leader and the follower have the same value. Below that point the firm would be better off being a follower and after that point being a leader.

<u>Appendix C</u>

Let equation (31) represent the value function of an idle follower:

$$\begin{aligned}&\frac{1}{2}\frac{\partial^2 P_0^F}{\partial R^2}\sigma^2 R^2 + \frac{1}{2}\frac{\partial^2 P_0^F}{\partial K^2}\alpha^2 K^2 + \frac{\partial P_0^F}{\partial K \partial R} RK\sigma\alpha\rho + (r-\mu)R\frac{\partial P_0^F}{\partial R} + \\ &(r-\omega)K\frac{\partial P_0^F}{\partial K} - rP_0^F = 0\end{aligned} \tag{31}$$

As it was already said previously in order to obtain a closed form solution for this equation we have to use similarity methods.

Let $Y = \frac{R}{K}$ and:

$$P_0^F(R,K) = Kp\left(\frac{R}{K}\right) = Kp(Y)$$

$$\frac{\partial P_0^F(R,K)}{\partial R} = \frac{\partial p(Y)}{\partial Y}$$

$$\frac{\partial P_0^F(R,K)}{\partial K} = p(Y) - \frac{R}{K}\frac{\partial p(Y)}{\partial Y}$$

$$\frac{\partial^2 P_0^F(R,K)}{\partial R^2} = \frac{\partial^2 p(Y)}{\partial Y^2}\frac{1}{K}$$

$$\frac{\partial^2 P_0^F(R,K)}{\partial K^2} = \frac{\partial^2 p(Y)}{\partial Y^2}\frac{R^2}{K^3}$$

$$\frac{\partial^2 P_0^F(R,K)}{\partial R \partial K} = -\frac{\partial^2 p(Y)}{\partial Y^2}\frac{R}{K^2}$$

Substituting back in equation (31) we obtain equation (32):

$$\frac{1}{2}Y^2 p''(Y)\gamma^2 + Yp'(Y)[\omega - \mu] - \omega p(Y) = 0$$