Real Options Lesson: Learn Before You Act

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Abstract

We study the interaction between learning (reduction of parameter uncertainty under conditions of incomplete information) and direct value-enhancing (control) actions before irreversible investment decisions are made. This framework allows the study of interactions between marketing research and advertisement (or product redesign or repositioning), basic research or exploration actions and product attribute or quality enhancing actions, etc. The framework also allows the analysis of optimal timing of such actions, optimal timing of introduction of pilot projects, early development of the complete project and abandonment options. We provide analytic formulas for compound-growth (pilot project) options with embedded control and learning actions under the assumption that project value follows either diffusion or a jump diffusion process. For complex multistage problems with path dependent actions, we develop a numerical lattice based model. We apply the models to the case of new product development and we illustrate the importance of engaging in learning actions prior to value-enhancing actions and development.

I. Introduction

Apart from the project value estimated through the traditional Net Present Value (NPV) method, many investment projects have additional value that arises from managerial flexibility to wait and react under uncertainty. This option value of waiting has been studied extensively in the real options literature (see for example McDonald and Siegel, 1986). In the present paper we also incorporate another source of value arising from active management actions to enhance project value by learning or control actions. Learning options prior to investment include investments in marketing research, R&D or exploration activities and pilot projects. These actions reduce uncertainty about the true potential of a project enabling management to have valuable information before irreversible investment is undertaken. The firm may also engage in direct value-enhancing (control) actions like advertising or efforts to improve the attributes or the quality of a product. These actions are targeting to an increase in project value albeit with a random outcome.

We develop analytic formulas for compound-growth options with embedded learning and attribute improvement control actions. Our analytic model includes Geske (1979) as a special case. We similarly provide formulas for the case where the underlying asset follows jump diffusion with multiple sources of jumps. The analytic formulas are then employed to show how learning and control actions affect the value of an investment opportunity and what determines the final choice between alternative costly controls or learning actions with different characteristics. The numerical results are interpreted in the context of new product development showing the importance of marketing research, attribute or quality improvement actions, advertisement, building pilot projects, etc.

Real life investment problems include multiple stages decisions with the potential for early development, optimal timing of actions, and interactions between learning and control actions. For these reasons we extend the analysis by developing a numerical model that can be used for the evaluation of such complex cases with path dependencies. The numerical model can accommodate both the pure diffusion and jump diffusion assumption regarding project value. Numerical results are provided that are interpreted in the context of new product development showing the importance and optimal sequence of marketing research, attribute or quality improvement actions, advertisement, building pilot projects, etc.

II. The value of learning and control: Analytic formulas

In this section we provide analytic solutions for European options and compound-growth options with embedded learning or control actions. The results are given for both pure diffusion and jump diffusion processes with multiple classes of jumps. Specifically under a pure diffusion process with $i = 1,2,...,N_A$ optional learning or control actions, project value follows a risk neutral process of the form:

$$\frac{dS}{S} = (r-\delta)dt + \sigma dz + k_i dq_i \qquad (1)$$

Under jump diffusion process we also allow for N_j independent jump classes (sources of jumps), so the stochastic process is described by:

$$\frac{dS}{S} = (r-\delta-\sum_{j=1}^{N_j}\lambda_j \bar{k}_j)dt + \sigma dz + k_i dq_i + \sum_{j=1}^{N_j} k_j d\pi_j \qquad (2)$$

Parameter *r* denotes the risk free rate, δ is the opportunity cost of waiting (foregone profit flows for not developing early), σ is the standard deviation of the rate of change of the state variable, *dz* is an increment to a standard Wiener process describing the exogenous uncertainty of the state variable, k_i is a random variable that represents the effect on project returns of a control or learning action, dq_i is a control variable that takes the value one if the control is activated and zero if not. For the jump diffusion case we have additionally the impact k_j of j=1,2,…,N_j jumps and $d\pi_j$ denotes a Poisson process with frequency of arrival λ_j per year. We assume that controls and jumps have firm specific risks which are uncorrelated with the market portfolio and thus not priced.

For practical reasons we assume that the effect of control actions and jumps are log-normally distributed. Each control or learning action has impact $1+k_i$ that follows:

$$1+k_i \sim \log N\left(\exp(\gamma_i), \exp(\gamma_i)(\exp(\sigma_i^2)-1)^{0.5}\right) \qquad (3)$$

The assumption of log-normally distributed controls is adopted for convenience only, since it allows non-negative asset values, and also, conditional on control activation asset values retain the distributional properties. The characteristics of randomly arriving jumps are similarly defined. We will use (γ_i, σ_i) to denote characteristics of control or learning actions and (γ_j, σ_j^2) to denote the characteristics of randomly arriving jumps, with j = 1,2,…,N_j jump classes. We use $\gamma_i > 0$ to describe efforts to enhance value with random outcome and $\gamma_i < 0$ for control actions to reduce costs. The special case of $\gamma_i = 0$ with $\sigma_i^2 > 0$ is methodologically only similar to the control case. Conceptually it differs and it is used to capture learning (elimination or reduction of parameter uncertainty) about the true project value.

We then use risk neutral pricing as established in Constantinides (1978), Harrison and Pliska (1981), and Cox, Ingersoll, and Ross (1985). We also incorporate δ as an opportunity cost of waiting that should be deducted from the equilibrium-required rate of return (see McDonald and Siegel, 1984). The variable δ may also be used to model exogenous competitive erosion to the project's cash flows (e.g., Childs and Triantis, 1999, and Trigeorgis, 1996, ch.9). Consistent with Merton (1976) for the jump diffusion case we also subtract the expected impact of jumps from the drift so that we keep the equilibrium returns unchanged. The risk neutral distribution of S at time T conditional on the activation of a control for the pure diffusion is given by:

$$\ln\left(\frac{S_T}{S_t}\,|\,i\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2)(T-t)+\gamma_i,\sigma^2(T-t)+\sigma_i^2\right) \tag{4}$$

The distribution conditional on no activation of control is found by setting $\gamma_i = \sigma_i^2 = 0$.

The risk neutral distribution of S at T conditional on the activation of a control for the jump diffusion conditional on the realization $n = \{n_1, n_2, ..., n_{N_j}\}$ of jumps is given by:

$$\ln\left(\frac{S_T}{S_t}\,|\,i, n=\{n_1,n_2,...,n_{N_j}\}\right) \sim N\left((r-\delta-\frac{1}{2}\sigma^2-\sum_{j=1}^{N_j}\lambda_j \bar{k}_j)(T-t)+\gamma_i+\sum_{j=1}^{N_i} n_j\gamma_j,\sigma^2(T-t)+\sigma_i^2+\sum_{j=1}^{N_j} n_j\sigma_j^2\right)$$
(5)

Using the risk neutral expectation approach we now price compound-growth options. We consider the general case of a compound-growth option with two sequential controls, (optionally) activated at $t=0$ and/or at the intermediate date $t=t_1$. The first control k_0 has mean impact and variance of impact characteristics (γ_0, σ_0) and can be activated at $t=0$ at a cost X_0, and the second control (k_1) has distributional characteristics (γ_1, σ_1) and can be activated at $t=t_1$ at a cost X_1. When the compound call option is exercised, option holder gets the option to acquire S for X_2, plus cash equal to a fraction of S equal to mS. The parameter m denotes the cash flows derived through the built up of a pilot project at t_1. The value of the compound option conditional on the activation of control k_0 at $t=0$ is given by:

$$C(.\,|\,k_0) = Se^{-\delta t_2+\gamma_0+\gamma_1}N(a_1,b_1,\rho) - X_2e^{-rt_2}N(a_2,b_2,\rho) + mSe^{-\delta t_2+\gamma_0}N(a_1) - X_1e^{-rt_1}N(a_2) \tag{6}$$

where

$$a_1 = \frac{\ln(S/S^*)+(r-\delta+0.5\sigma^2)t+\gamma_0+0.5\sigma_0^2}{(\sigma^2t_1+\sigma_0^2)^{0.5}},\ a_2 = a_1-(\sigma^2t_1+\sigma_0^2)^{0.5}$$

$$b_1 = \frac{\ln(S/X_2)+(r-\delta)t_2+(\gamma_0+\gamma_1)+0.5\sigma^2t_2+0.5(\sigma_0^2+\sigma_1^2)}{(\sigma^2t_2+\sigma_0^2+\sigma_1^2)^{1/2}},$$

$$b_2 = b_1 - (\sigma^2 t_2 + \sigma_0^2 + \sigma_1^2)^{1/2}$$

$$\rho = \sqrt{\frac{(\sigma^2 t_1 + \sigma_0^2)}{(\sigma^2 t_2 + \sigma_0^2 + \sigma_1^2)}}$$

The value of the option assuming k_0 is not activated at $t = 0$ is denoted by C and is given by setting $\gamma_0 = \sigma_0 = 0$. The unconditional option value of the project with the two embedded optional controls equals $\max\big(C(.\,|\,k_0) - X_0, C\big)$. The compound call option of Geske (1979) is a provided as a special case by setting $\gamma_0 = \sigma_0 = \gamma_1 = \sigma_1 = 0$, $X_0 = 0$, and $m = 0$. The results can be easily extended to the multiperiod sequential case, providing Carr (1988) as a special case.

As in Geske, the critical value S^* is found by solving numerically the equation:

$$S e^{-\delta(t_2 - t_1) + \gamma_1} N(d_1) - X_2 e^{-r(t_2 - t_1)} N(d_2) - X_1 = 0$$

$$d_1 = \frac{\ln(S / X_2) + (r - \delta)(t_2 - t_1) + \gamma_1 + 0.5\sigma^2 (t_2 - t_1) + 0.5\sigma_1^2}{[\sigma^2 t_2 + \sigma_1^2]^{1/2}}$$

$$d_2 = d_1 - [\sigma^2 t_2 + \sigma_1^2]^{1/2}$$

In the case where project value follows jump diffusion with $j = 1,2,\ldots N_j$ sources of jumps with impact γ_j and volatility σ_j, and like before there exist two controls at $t = 0$ and $t = t_1$, the compound-growth option conditional on activation of control action at $t = 0$ is given by:

$$C(.\,|\,k_0) = \sum_{n(h_1)_1=0}^{\infty} \cdots \sum_{n(h_1)_{N_j}}^{\infty} \sum_{n(h_2)_1=0}^{\infty} \cdots \sum_{n(h_2)_{N_j}}^{\infty} \{ p\big(n(h_1)_1, \ldots, n(h_1)_{N_j}\big) p\big(n(h_2)_1, \ldots, n(h_2)_{N_j}\big)$$
$$[S e^{-\delta t_2 \sum_{j=1}^{N_j} (\lambda_j \bar{k}_j t_2) + \sum_{j=1}^{N_j} (n(h_1) + n(h_2))\gamma_j + (\gamma_0 + \gamma_1)} N(a_{1n(h_1)}, b_{1n(h_2)}, \rho_{n(h_1),n(h_2)}) - X_2 e^{-rt_2} N(a_{2n(h_1)}, b_{2n(h_2)}, \rho_{n(h_1),n(h_2)})]\}$$
$$+ \sum_{n(h_1)_1=0}^{\infty} \cdots \sum_{n(h_1)_{N_j}}^{\infty} \{ p\big(n(h_1)_1, \ldots, n(h_1)_{N_j}\big) [m S e^{-\delta t_2 - \sum_{j=1}^{N_j} (\lambda_j \bar{k}_j t_1) + \sum_{j=1}^{N_j} n(h_1)\gamma_j + \gamma_0} N(a_{1n(h_1)}) - X_1 e^{-rt_1} N(a_{2n(h_1)})]\}$$

(7)

where

$$a_{1n(1)} = \frac{\ln(S/S^*) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) + 0.5\sigma^2)t_1 + \sum_{j=1}^{N_j} n(h_1)_j \gamma_j + \gamma_0 + 0.5\sum_{j=1}^{N_j} n(h_1)_j \sigma_j^2 + 0.5\sigma_0^2}{\left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(h_1)_j \sigma_j^2 + \sigma_0^2\right)^{0.5}}$$

$$a_{2n(h_1)} = a_{1n(h_1)} - \left(\sigma^2 t_1 + \sum_{j=1}^{N_j} n(h_1)_j \sigma_j^2 + \sigma_0^2\right)^{0.5}$$

$$b_{1n(2)} = \frac{\begin{array}{l}\ln(S/X_2) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) + 0.5\sigma^2)t_2 + (\gamma_0 + \gamma_1) \\ + \sum_{j=1}^{N_j}\left(n(h_1)_j + n(h_2)_j\right)\gamma_j + 0.5\sum_{j=1}^{N_j}\left(n(h_1)_j + n(h_2)_j\right)\sigma_j^2 + 0.5(\sigma_0^2 + \sigma_1^2)\end{array}}{\left(\sigma^2 t_2 + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j} n(h_2)_j \sigma_j^2\right)^{0.5}}$$

$$b_{2n(h_2)} = b_{1n(h_2)} - \left(\sigma^2 t_2 + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j} n(h_2)_j \sigma_j^2\right)^{0.5}$$

$$\rho_{n(1),n(2)} = \sqrt{\frac{\left(\sigma^2 t_1 + \sigma_0^2 + \sum_{j=1}^{N_j} n(h_1)_j \sigma_j^2\right)}{\left(\sigma^2 t_2 + \sigma_0^2 + \sigma_1^2 + \sum_{j=1}^{N_j}\left(n(h_1)_j + n(h_2)_j\right)\sigma_j^2\right)}}$$

$$p(n(h)_1, ..., n(h)_{N_j}) = \prod_{j=1}^{N_j}[e^{-(\lambda_j h)}(\lambda_j h)^{n(h)_j} / n(h)_j!],$$

for $h = h_1$ and $h = h_2$, $h_1 = t_1$, $h_2 = t_2 - t_1$, $j = (1,2,...,N_j)$

In the case of jump-diffusion we weight the value of the compound option with the probabilities of occurrence of all combinations of jumps that can be realized until t_1, $n(h_1) = \left(n(h_1)_1, .., n(h_1)_{N_j}\right)$, and those realized from t_1 to t_2, $n(h_2) = \left(n(h_2)_1, .., n(h_2)_{N_j}\right)$.

Like before the critical value S^* is found by solving numerically the equation:

$$\sum_{n(h_2)_1}^{\infty} \cdots \sum_{n(h_2)_{N_j}}^{\infty} \{ p\left(n(h_2)_1, n(h_2)_2, \ldots, n(h_2)_{N_j}\right)$$

$$[Se^{-\delta(t_2-t_1)-\sum_{j=1}^{N_j}(\lambda_j \bar{k}_j (t_2-t_1))+\sum_{j=1}^{N_j}(n(h_2)_j \gamma_j)+\gamma_1} N(d_{1n}) - X_2 e^{-rt_2} N(d_{2n})]\} - X_1 = 0$$

$$d_{1n(h_2)} = \frac{\ln(S/X_2) + (r - \delta - \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j) - 0.5\sigma^2)(t_2 - t_1) + \sum_{j=1}^{N_j} n(h_2)_j \gamma_j + \gamma_1 + 0.5\sum_{j=1}^{N_j} n(h_2)_j \sigma_j^2 + 0.5\sigma_1^2}{[\sigma^2(t_2 - t_1) + \sum_{j=1}^{N_j} n(h_2)_j \sigma_j^2 + \sigma_1^2]^{1/2}}$$

$$d_{2n(h_2)} = d_{1n(h_2)} - [\sigma^2(t_2 - t_1) + \sum_{j=1}^{N_j} n(h_2)_j \sigma_j^2 + \sigma_1^2]^{1/2}$$

The analytic solutions for the European compound-growth call on put, put on call and put on put are similarly derived and provided in the appendix. Similarly with Carr (1988), we can allow the exercise price X to be stochastic and follow a geometric Brownian motion process, as long as all costs related to option exercise or activation of learning and control actions are constant fractions of the same stochastic asset X. This is feasible due to the multiplicative nature and the lognormal distribution of both, the randomly arriving jumps, and the learning and control actions.

We now discuss the optimal choice of controls. We note that there are two issues regarding the choice of controls, first the intertemporal choice for the same control (optimal timing) and second the choice between controls with different characteristics. The decision for the optimal timing of learning and control actions should weight the value and the opportunity cost of waiting against the value of early development. The analytic formulas for the compound option do not capture the effect of early exercise, an effect we will consider in the next section using a numerical model that captures also the optimal timing of learning and control. Regarding the second issue we shall investigate it further by concentrating on the joint effect of mean impact and volatility of controls. The choice between a control or learning action with certain characteristics over another control or learning action is made by a comparison of the option´s conditional value after subtracting the extra cost. Figure 1 shows the marginal effect of control parameters γ and σ for different levels of S (i.e. the degree of moneyness of the option) for a simple call option. The figure shows that the impact of the variables is significant for the levels of S that are at the money. The effect of γ starts to be important at the region that the option is at the money and continues to grow in importance as S gets higher, while σ_k is most important for the at the money range.

[Insert figures 1 and 2]

Figure 2 elaborates on the comparison of option values with embedded actions at the second exercise point t_1 of the compound option, by comparing the value of a simple call option as a function of S for two possible controls. The first is a control action with positive impact and volatility and high cost and the second is a learning action. We observe that the process of choosing between control actions is highly sensitive to the cost and the distributional characteristics of actions, and optimal decisions may differ at different levels of S. Figure 3 completes the analysis of choice between control actions by showing the joint effect of the mean impact and volatility of control options for a compound option.

[Insert figure 3]

The results show that for a given volatility of control, the effect of mean impact on the value of compound option grows significantly. For a given mean impact, the effect of increasing the volatility of control on the value of the compound option is less significant. Panels a, b, and c show that as the option gets more in the money, the marginal benefits of increasing the mean impact or volatility of control are reduced.

III. A sequential numerical model with interacting learning and control actions

Now we consider an extended version of the investment problem discussed earlier to allow for multiple stages, multiple interacting learning and control actions with path depedency, growth options, abandonment options and early development in a unified framework. We discuss a numerical method that can be used to evaluate these complex cases.

Assume that the option to invest expires at maturity T, the terminal date where the firm should decide whether to pay X and acquire the value S of the project or abandon. Like before assume that we have N_c control actions available and for the case of jump diffusion process that we also have N_j sources of randomly arriving jumps. Control actions can be either learning or actions to enhance value and can be activated sequentially at the discrete decision points. In the more general case that we will consider the control actions can be interacting, so that the sequence of actions will affect the distributional characteristics of learning and control actions. Note that randomly arriving jumps are completely exogenous and occur with yearly frequencies λ_j. We will also allow for early exercise of development option thus creating a semi-American option setting.

Furthermore consider a discretization of the time to the option maturity T to N_s equally spaced decision points, with $N_s = \{1,2,3...\}$. We use $t = 0, \frac{T}{N_s}, \frac{2T}{N_s}, ..., \frac{(N_s - 1)T}{N_s}$ to denote the corresponding time of actions with $N_s + 1$ to be the terminal decision point at time T. We denote m_t the mode or decision of control or inaction (idleness). The set of actions is as follows: A control (or learning) action is denoted by C_i so that the set of controls is $\{C_1, C_{2,} ..., C_{N_c}\}$. The initial starting point is defined as "wait" (*W*). This will be used for other periods before any control actions are activated to denote the state of idleness. The idle mode after a control is denoted by W_i so in general we will also have $\{W_1,\ W_2, \ldots, W_{N_c}\}$. The latter set of modes is used to keep information on the prior actions that have already been activated. Finally we have two terminal boundary conditions, an early exercise of development mode (*EE*) and the abandonment mode potentially for capital recovery (*A*) or zero. We emphasize that modes {*EE*, *A*} are absorbing states. At exercise mode (*EE*) the firm gets $S - X$ while in the *A* mode the firm recovers a percentage α of past investment in controls. We also allow for the option to acquire a fraction *m* of the project value before final development (pilot project).

We assume that controls and jumps are log-normally distributed with distributions specified in (3).We further consider that the distributional characteristics of these control actions depend on the sequence in which they are activated, conditional on all relevant information for the previous actions. For this reason we will use when needed the more general specification $\gamma(h,i)$ and $\sigma(h,i)$ for the description of the impact and volatility of control, conditional on the previous state *h*.

With activation of action $m_t = C_i$ at *t*, log-returns for the pure diffusion process will follow:

$$\ln\left(\frac{S_{t+\Delta t}}{S_t} \mid h\right) \sim N\left((r - \delta - \frac{1}{2}\sigma^2)\Delta t + \gamma(h,i), \sigma^2 \Delta t + \sigma^2(h,i)\right) \tag{8}$$

For the jump diffusion case the conditional on the realization of $n = \{n_1, n_2, ..., n_{N_j}\}$ jumps thus log-returns follow:

$$\ln\left(\frac{S_T}{S_t} \mid h, n = \{n_1, n_2, .. n_{N_j}\}\right) \sim N\left((r - \delta - \frac{1}{2}\sigma^2)T + \gamma(h,i) + \sum_{j=1}^{N_i} n_j \gamma_j, \sigma^2 T + \sigma^2(h,i) + \sum_{j=1}^{N_j} n_j \sigma_j^2\right) \tag{9}$$

In the cases where no control is activated, $m_t = \{W, W_i\}$, we have $\gamma(h,i) = \sigma^2(h,i) = 0$ regardless of the previous action *h*.

The information regarding the expected impact and volatility of controls will be determined by the sequence (path) in which the controls are being activated. The use of this approach can be illustrated by a specific example. Interpreting C_2 to be an expensive advertising campaign it is reasonable to assume a different (potentially higher) impact for this action if it is activated directly from *W*, denoting this impact $\gamma(W,C_2)$, rather than if another campaign C_1 was already in place, making the new action alone, $\gamma(C_1,C_2)$, a less effective one (although the complete sequence might be more effective than any individual action). Note also that we may allow the cost of each control action to be path dependent. We define the information regarding the path dependency of controls' costs in a matrix where we define the costs $x(h,i)$ that need to be paid for switching from decision h to i.

For example if we interpret C_1 to be an accelerated control strategy of high impact then $x(W,C_1)$ denotes it's cost. If additionally we assume that $\{C_2,C_3\}$ are the first and second of two sequential investments in controls with outcome comparable to that of the first control, then total costs might differ from $x(W,C_1)$ due for example to learning by doing (we will consider such a case in the numerical results), etc. An additional use of switching costs is to disable some actions or a specific action sequence. For example, to disable control action C_4 directly from the initial wait mode we can set $x(W,C_4) = inf$.

When studying problems of path-dependency with many alternative courses of action, it is important to check the *logical-consistency* (or *economic-consistency*) of the switching matrix for costs, impact, and volatility of actions. If for example C_1 and the sequence $\{C_2,C_3\}$ are mutually exclusive alternatives, we should compare $X(W,C_1)$ with $X(W,C_2) + X(C_2,C_3)$. If $X(W,C_1) > X(W,C_2) + X(C_2,C_3)$ it implies cost efficiencies achieved due to learning by doing. The opposite, $X(W,C_1) < X(W,C_2) + X(C_2,C_3)$, would imply scale efficiencies. We must similarly investigate the switching matrix for the impact, and for the volatility of controls.

The numerical solution framework

We allow decisions to be made sequentially at Δt (assumed for simplicity equal) fintervals. We define $V^{m_t}(.)$ the conditional payoff the firm gets under decision $m_t = i$. This payoff is a function of the level of cash flows S at that decision point, the characteristics of available controls, the development cost X, the switching (path-dependent) control costs $X(h,i)$, the recovery rate α for the case of abandonment options, pilot project with cash flows a constant fraction m of S, etc. There is a superset M that includes all information about admissible actions, action sequences, their distributional characteristics, and the values for all option parameters. At each time t, there is a (stochastic) subset M_t^- that describes the history of actions up to time t, and a (similarly stochastic) subset M_t^+ that defines the remaining admissible actions and relevant parameter values.

More specifically, we wish to maximize the value of the investment by making the optimal pre-investment learning/exploration and/or control actions:

$$V^*(S_t,t \mid M,M_t^+,M_t^-) = \max_{M^+}\{V^{m_t}\} \tag{10}$$

We have the following cases for $V^{m_t}(.)$:

$$V^{m_t}(S_t,t \mid M,M_t^+,M_t^-) =$$

$$e^{(-r\Delta t)} E_t \left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-) \right] + m(S_t, M, M_t^+, M_t^-) S_t - X(m_{t-\Delta t}, m_t) \quad \text{(11a)}$$

for $m_t \in \{C_1, C_2, ..., C_{N_c}\}$,

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = S_t - X \quad \text{(11b)}$$

for $m_t \in \{EE\}$,

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = \alpha(S_t, M, M_t^+, M_t^-) \quad \text{(11c)}$$

for $m_t \in \{A\}$,

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) = e^{(-r\Delta t)} E_t \left[V^*(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-) \right] \quad \text{(11d)}$$

for $m_t \in \{W_1, W_2, ..., W_{N_c}\}$.

Finally, at the last decision point $N_s + 1$ at $t = T$, the optimal values are given by the terminal condition:

$$V^{m_T}(S_T, T \mid M, M_T^+, M_T^-) = \max(S_T - X, a(S_T, M, M_T^+, M_T^-)) \quad \text{(11e)}$$

We can see that equations (11) incorporate path dependent costs (and of course path-dependent impact and volatility of impact), early development options and abandonment options to recover a fraction α of the total investments in controls and pilot projects. Expectation when $m_t \in \{C_1, C_2, ..., C_{N_c}\}$ is taken with respect to the distribution of log-returns that depend on the specification chosen for the exogenous process including the impact of controls; and for the case of no control with $m_t \in \{W, W_1, W_2, ..., W_{N_c}\}$, expectation is taken excluding the the impact of controls. Note further that for $m_t \in \{EE, A\}$ the expectation operator returns zero (these are terminal/absorbing states with no feasible continuation of decisions).

In the case of jump diffusion, equations (11b), (11c), and (11e) stay the same, and we have the following adjustments to equations (11a) and (11d):

$$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) =$$
$$e^{(-r\Delta t)} \left[\sum_{n_1=0}^{\infty} ... \sum_{n_{N_j}}^{\infty} \left[p(n_1, ..., n_{N_j}) E_t \left[V^* \left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-, n = (n_1, n_2, ..., n_{N_j}) \right) \right] \right] \right]$$
$$+ m(S_t, M, M_t^+, M_t^-) S_t - X(m_{t-\Delta t}, m_t)$$

for $m_t \in \{C_1, C_2, ..., C_{N_c}\}$, (11'a)

$V^{m_t}(S_t, t \mid M, M_t^+, M_t^-) =$

$$e^{(-r\Delta t)}\left[\sum_{n_1=0}^{\infty}\cdots\sum_{n_{N_j}}^{\infty} p(n_1,\ldots,n_{N_j})\left[E_t\left[V^*\left(S_{t+\Delta t}, t+\Delta t \mid S_t, M, M_t^+, M_t^-, n=(n_1,n_2,\ldots,n_{N_j})\right)\right]\right]\right]$$

(11'd)

We note that for the jump diffusion case the expectations should be taken over all possible realizations of jumps, weighted by the probability of occurrence as the term $\left[\sum_{n_1=0}^{\infty}\cdots\sum_{n_{N_j}}^{\infty} p(n_1,\ldots,n_{N_j})\right]$ demonstrates.

In order to find project value at $t = 0$, we should note that value functions in equation (11) should be evaluated for each decision mode, at each decision point in time and for each state of the underlying asset *S*. Due to the presence of path dependency, $V^{m_i^*}$ cannot be evaluated in the usual backward solution method of dynamic programming. Instead, we must take into account all alternative combinations of actions and paths of the state-variable. We thus implement a forward-backward looking algorithm of exhaustive search (alternatively, see Hull and White, 1993, or Thompson, 1995), and the optimal decision will determine today's option value. For the jump diffusion case the number of calculations that need to be performed for multistage problems can be very large and thus available computational power puts a restriction on the stages that can be evaluated when jump frequencies are high. For many applications, it is reasonable to assume low frequency of jumps (truly "rare" events), reducing thus the number of computational requirements substantially to a manageable degree.

In order to evaluate the expectation operator defined in equations (13) we need a discretized state-space and we thus use a numerical lattice scheme. From equation (10) and (11) the underlying asset *S* has a lognormal distribution between decision points. We approximate this distribution between steps with a binomial lattice with N_{sub} number of steps, with total number of steps N equal to $N_s\, N_{sub}$. The conditional volatilities $v^2(m_t, m_{t+\Delta t})$ between decision points for the pure diffusion are:

$$v^2(m_t, m_{t+\Delta t}) = \sigma^2 \frac{T_{sub}}{N_{sub}} + \frac{\sigma^2(m_t, m_{t+\Delta t})}{N_s}, \tag{15}$$

for $m_t \in \{C_1, C_2, \ldots, C_{N_c}\}$

while for the jump diffusion conditional on the realization of $n = (n_1, n_2, \ldots, n_{N_j})$ jumps we have:

$$v^2(m_t, m_{t+\Delta t} \mid n = (n_1, n_2, ..., n_{N_j})) = \sigma^2 \frac{T_{sub}}{N_{sub}} + \frac{\sigma^2(m_t, m_{t+\Delta t})}{N_{sub}} + \frac{1}{N_{sub}} \sum_{j=1}^{N_j} n_j \sigma_j^2$$

(15')

The specification in (15) allocates the volatility of control actions and jumps to N_{sub} points for a total uncertainty of $\sigma^2(m_t, m_{t+\Delta t})$ and $\sum_{j=1}^{N_j} n_j \sigma_j^2$. When controls are not activated we just set in (15) the volatility of controls to zero.

Furthermore we use the following up and down moves for the lattice between stages:

$$u(m_t, m_{t+\Delta t}) = \exp(v(m_t, m_{t+\Delta t})), \qquad d = \frac{1}{u(m_t, m_{t+\Delta t})}$$

Finally the probabilities for an up and down move (pure diffusion case) for $m_t \in \{C_1, C_2, ..., C_{N_c}\}$ are:

$$p_u(m_t, m_{t+\Delta t}) = \frac{\exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma(m_t, m_{t+\Delta t})}{N_{sub}}\right) - d(m_t, m_{t+\Delta t})}{u(m_t, m_{t+\Delta t}) - d(m_t, m_{t+\Delta t})},$$

$$p_d(m_t, m_{t+\Delta t}) = 1 - p_u(m_t, m_{t+\Delta t})$$

and for the jump-diffusion case:

$$p_u(m_t, m_{t+\Delta t}) = \frac{\exp\left((r-\delta)\frac{T_{sub}}{N_{sub}} + \frac{\gamma(m_t, m_{t+\Delta t})}{N_{sub}} + \frac{1}{N_{sub}}\sum_{j=1}^{N_j} n_j \gamma_j\right) - d(m_t, m_{t+\Delta t})}{u(m_t, m_{t+\Delta t}) - d(m_t, m_{t+\Delta t})},$$

$$p_d(m_t, m_{t+\Delta t}) = 1 - p_u(m_t, m_{t+\Delta t})$$

while for $m_t \in \{W, W_1, ..., W_{N_c}\}$ we set the γ and σ parameters of controls to zero.

With this specification between decision points for the sub-lattice construction we are able to incorporate the asset price and embedded control actions and evaluate the expectation in equations (13).

In the next section we test the accuracy of the numerical model with the analytic solutions provided in the section II. Then, we discuss the importance of options to learn and enhance value by analyzing the new product development case.

IV. Numerical results and applications

Our first set of numerical results compares the analytic and numerical methods for the compound option. We then give numerical results for more complex multistage cases with path dependent learning and control actions. The cases demonstrate stages for new product development with options to conduct marketing research, improve quality attributes, perform pilot projects, early development, etc. Our emphasis is on the importance of learning actions prior to development that increase the effectiveness of the value-enhancing controls.

Table 1 shows the comparison between the analytic and lattice based numerical model for the case of a compound-growth option with learning.

[Insert Table 1]

At the intermediate date T_1, besides the value of the option to invest at the terminal date, the firm may also acquire a fraction m of the project value (a pilot project). In our problem specification, the firm cannot take the investment option unless it pays X_1, which we interpret as the cost of getting the pilot project cash flows plus learning for the final project. The first panel provides results for the case where no growth option is available (only learning) and the second panel considers the case with an option to acquire a fraction $m = 0.2$ of the project value, plus learning. We can see that the numerical model provides a very good approximation to the analytic formulas in both cases. Focusing on the first panel we note that the case of zero volatility of control and zero impact reflects the case of the compound option of Geske (1979). The results show that when the mean impact of controls and the volatility are positive the value of learning options embedded in investment options can be extremely important. Taking for example the case where $S =$ 100 we see that compared to the case of a simple compound option with no learning, the value of the compound option with a learning potential (volatility) of 0.1 increases by more than 50%, while a learning potential (volatility) of 0.2 increases value by 242%. In the second panel we see that the availability of growth options besides learning can further enhance project values. It captures the realistic case where a pilot project provides learning. Overall, the results indicate that project value can be substantially underestimated if learning, control, and other project attributes like growth options are neglected. If we interpret the learning action as marketing research, the higher the uncertainty that marketing research will resolve keeping for a given cost the more likely that it will be performed. In the next section we investigate more complex investment decision scenarios in the context of new product development.

The new product development case

In this section we employ the numerical model and we discuss the case of new product development by incorporating more complex and realistic features than in the previous cases. First, we take the scenario where $T_2 = 5$ and $\sigma_k = 0.3$ as base case and we extend it in several dimensions while maintaining only two decision points (at t =0 and $t = T_1$). Then we will extend the framework adding more decision points and more path-dependencies between actions.

The first two columns of Table 2 provide the project´s option value at t = 0 for a simple base case, where the firm can only choose to activate a learning action (*L*) at T_1, and it can only wait (*W*) at t = 0. The first extension we consider is the optimal timing of learning when early development is also possible. Figure 4 panel (a) shows the possible combinations of decisions.

[Insert figure 4]

In column 3 and 4 of Table 2 we provide numerical results for this scenario of the optimal timing of learning and development. In comparison with the results of the base case we see that optimal values are enhanced and optimal decisions may differ; *L* and *EE* may now be optimal at t = 0. Another extension concerns the availability of other actions to learn or enhance value. For example the firm may have the option to activate two learning actions sequentially at t = 0 and $t = T_1$. Alternatively, the firm may have the option to learn initially and then enhance project value by a control action. We concentrate on the case where the firm can activate both a learning action and a control. The analytic formula (6) provides only a part of this more general problem. Panel b of figure 4 shows all possible combinations of marketing research (learning), improvement actions (controls) and early development that can be made; the analytic formula only provides a subset of these decisions, namely perform *L* and then *C*. Note that the case described in panel b is substantially more complex. For example it allows the firm to perform *L* and then choose at T_1 between *C*, *W*, or *EE*. Columns 5 and 6 of Table 2 provide results for this case where the characteristics of control are $\gamma_c = 0.1$ with $\sigma_c = 0.3$.

[Insert Table 3]

The results show that option values change and more importantly that there is a large region where it pays to proceed with further improvement actions (*C*) immediately. The last two columns of the Table provide numerical results for a similar case as the previous one, the only difference being that the characteristics of control after learning action has been activated are different than if the firm proceeds directly to control (product improvement actions). This reflects the fact that better understanding of relevant attributes of the product is likely to occur after marketing research rather than if the firm goes "blindly" to further developments. Specifically, if the firm performs control directly it gains $\gamma(W,C) = 0.1$, and if the firm performs control after marketing research it gains $\gamma(L,C) = 0.2$ while the volatility of control remains the same in both cases to $\sigma(W,C) = \sigma(L,C) = 0.3$. The results indicate a large change in optimal values and optimal decisions. Under this scenario there is a large region where it is optimal to go for marketing research first so that the firm can later capture a higher effectiveness of control.

Next, we consider a complex scenario with 5 decision points, 2 learning actions and 2 controls with path dependencies. Figure 5 gives a general description of the problem and a base case specification of the parameters of the problem. There are two learning (L_1, L_G) and two control actions (C_1, C_2). In the first phase we can activate either L_1 or C_1, then we can proceed with a pilot project that will give a fraction *m* of the project cash flows *S* and at the same time will create a learning effect (L_G). The first phase actions can be skipped altogether and move directly to the pilot project or even to early development. Furthermore, we allow the firm to also activate a second phase of actions if it has completed the pilot project or if it has completed the first phase of actions. In general we do not allow under this specification to move from control actions to learning but we allow the firm to move from a learning action to a control, specifically from L_1 to C_1, and from L_G to C_2. As shown in figure 5 the volatility of learning directly from the pilot project is set to be double the volatility of the first learning action to show the fact that the pilot project is highly effective in revealing the true demand level. If instead a first phase of learning action has already been activated then the first action resolves half of the total uncertainty and the other half is left for the pilot project to resolve. The volatilities of control actions are all set to 0.30. For the impact of controls we consider the enhanced effect of learning on control impacts thus we assume that the impact of controls doubles if prior action was learning.

[Insert figure 5]

We keep the cost structure of the options simple to $X_{L_1} = X_{C_1} = X_{C_2} = 10, \quad X_G = 20$, the maturity of the option is $T = 5$. Our numerical results provide sensitivities with respect to the growth option parameter *m*, and the importance of learning actions to enhance the impacts of controls that are reflected in parameters $\gamma(L_1, C_1)$ and $\gamma(L_G, C_2)$. Table 3 provides sensitivity with respect to the effectiveness of learning action while keeping the growth option potential to $m = 0.1$.

[Insert Table 3]

The first two columns show the results when neither of the learning actions can improve the impact of the controls that are activated next, columns 3 and 4 provide option values and optimal decisions for the case where only L_1 can effect a better impact for C_1 and the last two columns when both L_1 and L_G can effect better impact, for C_1 and C_2

respectively. The results show that if learning does not provide any additional value-enhancement for the control actions, then it is likely that it will be skipped and the firm will proceed to the controls immediately. If instead, L_1 provides a better impact for the control actions then it is likely that the firm will proceed with learning at $t = 0$. The pilot project L_G will not be preferred over L_1 at $t = 0$ unless it also provides an improved impact for the second phase control C_2 as well. In Table 4 we provide sensitivity with respect to the level of the growth factor.

[Insert Table 5]

As expected, the higher the growth factor the more likely that we will proceed with the pilot project immediately.

V. Conclusions

We analyze investment options with embedded learning (explorative research, marketing research, etc,) and control (attribute or quality improvement, advertisement etc.) actions. The paper extends the analysis of investment options to provide analytic solutions for compound options with embedded optional pilot project built up, learning, and control actions when the project value follows diffusion or jump diffusion process. Analytic formulas allow pricing of compound options with embedded ability to introduce a pilot project, learn, and enhance project value, having in the case of a pure diffusion process Geske (1979) as a special case. The results can easily be extended to multiperiod sequential options, thus providing Carr (1988) as a special case. We show that the availability of options to learn (reduce or eliminate parameter uncertainty) and control can substantially affect project option values and optimal decisions.

We develop an extended numerical model for multistage problems, with multiple interacting controls with path-dependent distributional characteristics and cost structure, and early exercise feature, and apply it to the case of new product development. The most general framework allows optionally the build up of a pilot project before the completion of the final one, and activation of learning and control actions in two phases, one before the pilot project, and one after. Within this framework we demonstrate the importance of learning actions (exploration activities, investigative R&D, marketing research) prior to value-enhancing (attribute enhancing R&D, advertising activities, etc.) actions.

Appendix

In this section we provide the analytic valuation formulas for European compound-growth options, for three cases other than the call on a call option. All formulas are provided for the most general case of jump diffusion, with a control activated at t_1 if the compound-growth option is exercised, and conditional on the activation of another control at t_0. They are as follows:

Compound-growth call on put:

$$C(.|k_0)=\sum_{n(h_1)_1=0}^{\infty}\cdots\sum_{n(h_1)_{N_j}}^{\infty}\sum_{n(h_2)_1=0}^{\infty}\cdots\sum_{n(h_2)_{N_j}}^{\infty}\{p\big(n(h_1)_1,\ldots,n(h_1)_{N_j}\big)p\big(n(h_2)_1,\ldots,n(h_2)_{N_j}\big)$$

$$[Se^{-\delta t_2\sum_{j=1}^{N_j}(\lambda_j\bar{k}_j t_2)+\sum_{j=1}^{N_j}(n(h_1)+n(h_2))\gamma_j+(\gamma_0+\gamma_1)}N(a_{1n(h_1)},-b_{1n(h_2)},-\rho_{n(h_1),n(h_2)})-X_2e^{-rt_2}N(a_{2n(h_1)},-b_{2n(h_2)},-\rho_{n(h_1),n(h_2)})]\}$$

$$+\sum_{n(h_1)_1=0}^{\infty}\cdots\sum_{n(h_1)_{N_j}}^{\infty}\{p\big(n(h_1)_1,\ldots,n(h_1)_{N_j}\big)[mSe^{-\delta t_2-\sum_{j=1}^{N_j}(\lambda_j\bar{k}_j t_1)+\sum_{j=1}^{N_j}n(h_1)\gamma_j+\gamma_0}N(a_{1n(h_1)})+X_1e^{-rt_1}N(a_{2n(h_1)})]\}$$

Compound-growth put on call:

$$C(.|k_0)=\sum_{n(h_1)_1=0}^{\infty}\cdots\sum_{n(h_1)_{N_j}}^{\infty}\sum_{n(h_2)_1=0}^{\infty}\cdots\sum_{n(h_2)_{N_j}}^{\infty}\{p\big(n(h_1)_1,\ldots,n(h_1)_{N_j}\big)p\big(n(h_2)_1,\ldots,n(h_2)_{N_j}\big)$$

$$[X_2e^{-rt_2}N(-a_{2n(h_1)},b_{2n(h_2)},-\rho_{n(h_1),n(h_2)}-Se^{-\delta t_2\sum_{j=1}^{N_j}(\lambda_j\bar{k}_j t_2)+\sum_{j=1}^{N_j}(n(h_1)+n(h_2))\gamma_j+(\gamma_0+\gamma_1)}N(-a_{1n(h_1)},b_{1n(h_2)},-\rho_{n(h_1),n(h_2)})]\}$$

$$+\sum_{n(h_1)_1=0}^{\infty}\cdots\sum_{n(h_1)_{N_j}}^{\infty}\{p\big(n(h_1)_1,\ldots,n(h_1)_{N_j}\big)X_1e^{-rt_1}N(-a_{2n(h_1)})]\}$$

Compound-growth put on put

$$C(.|k_0) = \sum_{n(h_1)_1=0}^{\infty} \dots \sum_{n(h_1)_{N_j}}^{\infty} \sum_{n(h_2)_1=0}^{\infty} \dots \sum_{n(h_2)_{N_j}}^{\infty} \{ p\Big(n(h_1)_1,\dots,n(h_1)_{N_j}\Big) p\Big(n(h_2)_1,\dots,n(h_2)_{N_j}\Big)$$

$$[X_2 e^{-rt_2} N(-a_{2n(h_1)}, -b_{2n(h_2)}, \rho_{n(h_1),n(h_2)}) - Se^{-\delta t_2 \sum_{j=1}^{N_j}(\lambda_j \bar{k}_j t_2) + \sum_{j=1}^{N_j}(n(h_1)+n(h_2))\gamma_j + (\gamma_0+\gamma_1)} N(-a_{1n(h_1)}, -b_{1n(h_2)}, \rho_{n(h_1),n(h_2)})]\}$$

$$- \sum_{n(h_1)_1=0}^{\infty} \dots \sum_{n(h_1)_{N_j}}^{\infty} \{ p\Big(n(h_1)_1,\dots,n(h_1)_{N_j}\Big) X_1 e^{-rt_1} N(-a_{2n(h_1)})]\}$$

The parameters required as input to the bivariate and univariate cumulative normal distribution and the probability of jumps are defined as in the analytic model provided by equation (7) of the main text.

References

Black, F., and M. Scholes, "The Pricing of Options and Corporate Liabilities." *Journal of Political Economy* 81, 637–659, (1973).

Carr, P., "The Valuation of Sequential Exchange Opportunities." *Journal of Finance* 43, 1235-1256, (1988).

Constantinides, G., "Market Risk Adjustment in Project Valuation." *Journal of Finance* 33, 603–616, (1998).

Cox, J., J. Ingersoll, and S. Ross, "An Intertemporal General Equilibrium Model of Asset Prices." *Econometrica* 53, 363–384, (1985).

Dixit, A., and R. Pindyck, *Investments under Uncertainty*. Princeton, New Jersey: Princeton University Press, 1994.

Geske, R., "The Valuation of Compound Options." *Journal of Financial Economics* 7, 63-81, (1979).

Harrison, J. M., and S. Pliska, "Martingales and Stochastic Integrals in the Theory of Continuous Trading." *Stochastic Processes and Their Applications* 11, 313–316, (1981).

Hull, J., and A. White, "Efficient Procedures for Valuing European and American Path-Dependent Options." *Journal of Derivatives* 1, 21-31, (1993).

Korn, R., "Optimal Impulse Control when Control Actions Have Random Consequences." *Mathematics of Operations Research* 22, 639-667, (1997).

Majd, S., and R. Pindyck, "The Learning Curve and Optimal Production under Uncertainty." *RAND Journal of Economics* 20, 331-343, (1989).

Majd, S., and R. Pindyck, "Time to Build, Option Value, and Investment Decisions." *Journal of Financial Economics* 18, 7-27, (1987).

Martzoukos, S., "Real Options with Random Controls and the Value of Learning." *Annals of Operations Research* 99, 305-323, (2000).

McDonald, R. and D. Siegel, "Option Pricing When the Underlying Asset Earns a Below-Equilibrium Rate of Return: A Note." *Journal of Finance* 39, 261–265, (1984).

McDonald, R. and D. Siegel, "The Value of Waiting to Invest." *Quarterly Journal of Economics* 101, 707–727, (1986).

Merton, R., "The Theory of Rational Option Pricing." *Bell Journal of Economics and Management Science* 4, 141–183, (1973a).

Merton, R., “An Intertemporal Capital Asset Pricing Model.” *Econometrica*, 867–887, (1973b).

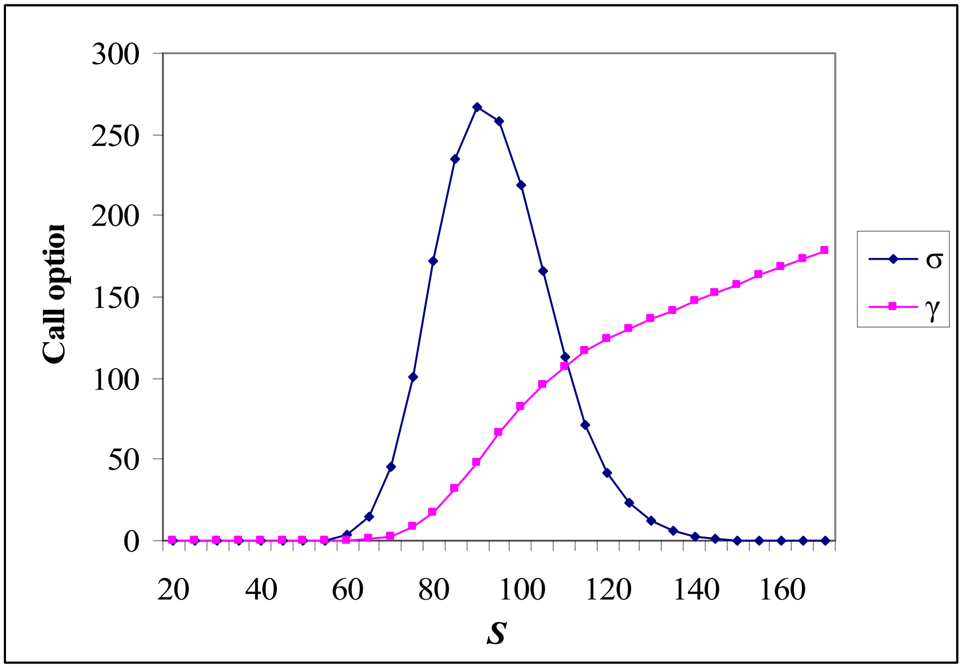
Merton, R., “Option Pricing When the Underlying Stock Returns are Discontinuous.” *Journal of Financial Economics*,3, 125-144, (1976).

Pindyck, R. S., “Investments of Uncertain Cost.” *Journal of Financial Economics* 34, 53-76, (1993).

Thompson, A. C., “Valuation of Path-dependent Contingent Claims with Multiple Exercise Decisions over Time: The Case of Take-or-Pay.” *Journal of Financial and Quantitative Analysis* 30, 271-293, (1995).

Trigeorgis, L., *Real Options: Managerial Flexibility and Strategy in Resource Allocation*. Cambridge: The MIT Press, 1996.

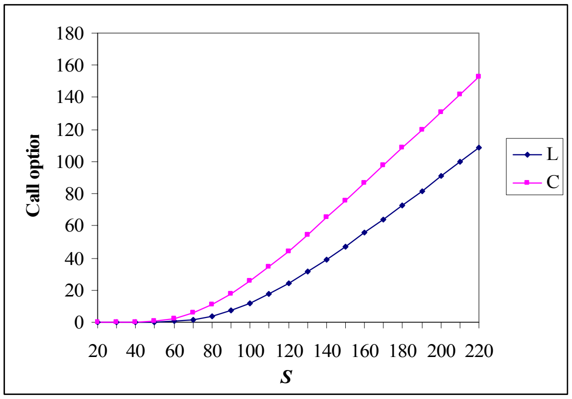
Figure 1: The partial derivative of the call option value with respect to the mean impact (γ), and the volatility (σ) of control



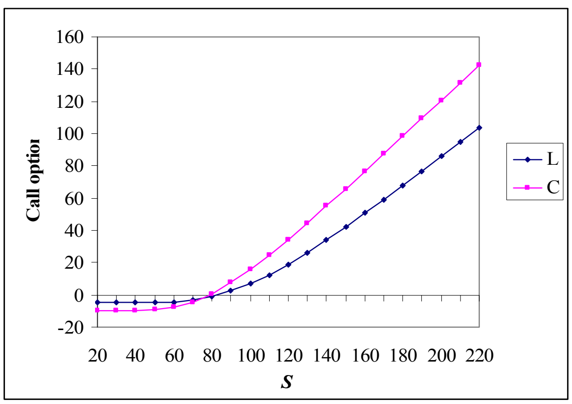
Notes: Numerical results using analytic formula for the control call with S = 100, X = 100, $r = \delta = 0.05$, T = 1 and control parameters $\sigma = 0.1$ and $\gamma = 0.1$.

Figure 2: Values of call option with learning versus control at different costs

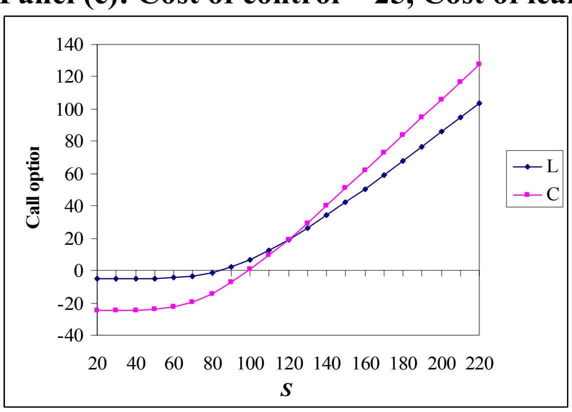
Panel (a): Cost of control and learning is zero



Panel (b): Cost of control = 10, Cost of learning = 5



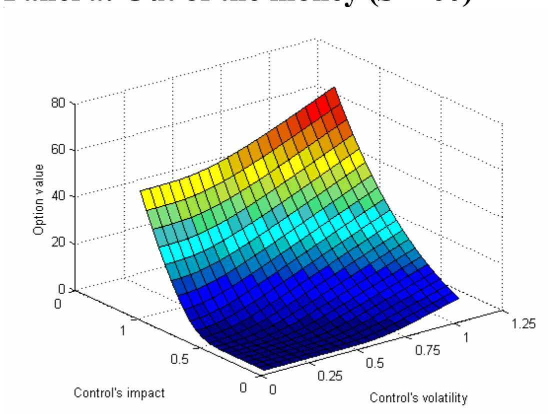
Panel (c): Cost of control = 25, Cost of learning = 5



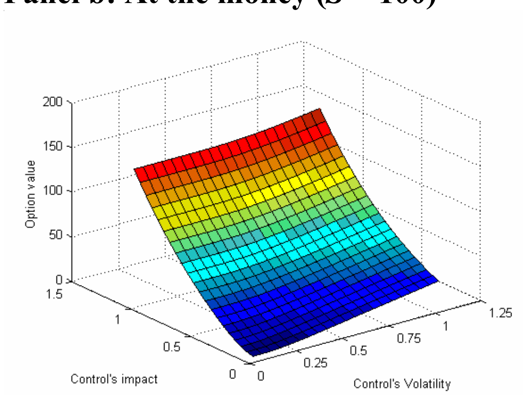
Notes: Numerical results using analytic formula for the control call with X = 100, $r = \delta = 0.05$, $\sigma = 0.1$ $T = 2$. For control $\sigma_i = 0.3$ and $\gamma = 0.2$, and for learning $\sigma_L = 0.3$

Figure 3: Sensitivity of control´s impact and volatility on a compound call option

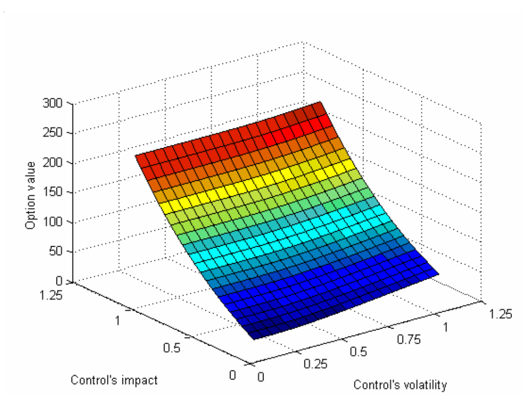
Panel a: Out of the money (S = 60)



Panel b: At the money (S = 100)



Panel c: In the money (S = 140)



Notes: Numerical results using analytic formula for the compound option with controls. Parameters are development cost X = 100, cost of control X_1 = 5, $r = \delta = 0.05$, $\sigma = 0.1$, $T_1 = 1$ and $T_2 = 2$.

Figure 4: Optimal timing of learning (*L*), control (*C*), and early development (*EE*) in a two stage investment problem: The set of possible decisions

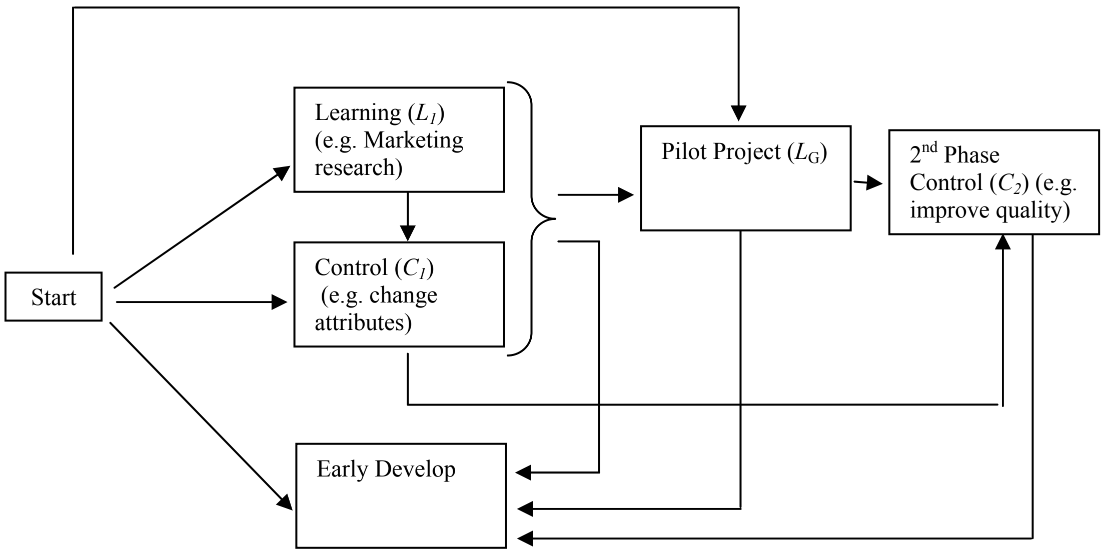
Panel (a): Optimal timing of learning (*L*) and early development (*EE*) with <u>no</u> control action (*C*)

t = 0	*t* = T1
W	*W*
W	*EE*
W	*L*
L	*W*
L	*EE*

Panel (b): Optimal timing of learning (*L*), early development (*EE*), and control action (*C*)

t = 0	*t* = T1
W	*W*
W	*EE*
W	*L*
W	*C*
L	*W*
L	*EE*
L	*C*
C	*W*
C	*EE*

Figure 5: A multi stage investment option with multiple interacting learning and control actions for the new product development problem



Volatility switching matrix of learning and control actions

From \ To	L_1	L_G	C_1	C_2
W	$(0.3)^2$	$2(0.3)^2$	$(0.3)^2$	-
L_1	-	$(0.3)^2$	$(0.3)^2$	-
L_G	-	-	-	$(0.3)^2$
C_1	-	-	-	$(0.3)^2$

Mean impact switching matrix of control actions

From \ To	L_1	L_G	C_1	C_2
W	0	0	0.1	-
L_1	-	0	0.2	-
L_G	-	-	-	0.2
C_1	-	-	-	0.1

Table 1: Compound option with learning: Comparison of numerical and analytic values

		Growth option factor m = 0					
		S = 80		S = 100		S = 120	
Time	**Vol. of Control**	**Analytic**	**Numerical**	**Analytic**	**Numerical**	**Analytic**	**Numerical**
T_2 = 1	0.000	0.000	0.000	1.103	1.103	14.320	14.320
	0.100	0.001	0.001	1.656	1.659	14.839	14.839
	0.200	0.016	0.015	3.773	3.774	16.864	16.865
	0.300	0.282	0.283	7.079	7.083	19.883	19.885
	0.400	1.743	1.747	10.660	10.666	23.341	23.346
	0.500	4.424	4.429	14.266	14.273	26.991	26.997
T_2 = 2	0.000	0.013	0.013	2.123	2.124	14.100	14.100
	0.100	0.027	0.026	2.648	2.651	14.675	14.675
	0.200	0.126	0.125	4.406	4.409	16.547	16.548
	0.300	0.616	0.619	7.203	7.207	19.310	19.313
	0.400	2.038	2.041	10.447	10.451	22.506	22.511
	0.500	4.400	4.404	13.792	13.798	25.906	25.913
T_2 = 5	0.000	0.302	0.301	3.860	3.861	13.635	13.635
	0.100	0.378	0.378	4.244	4.247	14.091	14.091
	0.200	0.668	0.670	5.427	5.431	15.462	15.464
	0.300	1.338	1.340	7.339	7.342	17.560	17.562
	0.400	2.552	2.555	9.747	9.751	20.091	20.094
	0.500	4.319	4.323	12.402	12.407	22.850	22.855

		Growth option factor m = 0.10					
		S = 80		S = 100		S = 120	
Time	**Vol. of Control**	**Analytic**	**Numerical**	**Analytic**	**Numerical**	**Analytic**	**Numerical**
T_2 = 1	0.000	2.964	2.964	8.670	8.667	25.992	25.992
	0.100	3.220	3.219	10.239	10.240	26.537	26.537
	0.200	4.511	4.511	13.344	13.346	28.567	28.568
	0.300	6.707	6.708	16.827	16.831	31.587	31.589
	0.400	9.357	9.360	20.413	20.419	35.045	35.049
	0.500	12.223	12.227	24.019	24.026	38.695	38.701
T_2 = 2	0.000	3.133	3.133	9.857	9.854	25.407	25.406
	0.100	3.506	3.506	11.001	11.002	26.045	26.045
	0.200	4.796	4.796	13.576	13.579	27.957	27.958
	0.300	6.838	6.839	16.674	16.678	30.725	30.727
	0.400	9.299	9.302	19.957	19.962	33.921	33.925
	0.500	11.973	11.978	23.304	23.310	37.321	37.327
T_2 = 5	0.000	3.946	3.943	11.345	11.341	23.977	23.976
	0.100	4.320	4.320	12.004	12.006	24.510	24.510
	0.200	5.400	5.401	13.699	13.701	25.991	25.992
	0.300	7.029	7.030	15.970	15.973	28.137	28.139
	0.400	9.006	9.009	18.527	18.531	30.679	30.682
	0.500	11.187	11.191	21.220	21.225	33.440	33.445

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.10$, $T_1 = \frac{1}{2}T_2$, $\gamma = 0$ and cost of control X_1 = 5. For the numerical lattice we use $N_{sub} = 200$ steps.

Table 2: **Project value for four different scenarios with increasing flexibility and impact**

	Learning only				Learning and control			
	L only at T1		Timing of *L* and *EE*		Timing of L, C and *EE*		Diff. impact of *C* after *L*	
S	**Value**	**Dec. at *t*=0**	**Value**	**Dec. at *t*=0**	**Value**	**Dec. at *t*=0**	**Value**	**Dec. at *t*=0**
240	109.032	*W*	140.000	*EE*	140.000	*EE*	140.000	*EE*
230	101.245	*W*	130.000	*EE*	130.000	*EE*	130.000	*EE*
220	93.457	*W*	120.000	*EE*	120.000	*EE*	120.000	*EE*
210	85.671	*W*	110.000	*EE*	110.000	*EE*	110.000	*EE*
200	77.886	*W*	100.000	*EE*	100.000	*EE*	100.000	*EE*
190	70.106	*W*	90.000	*EE*	90.000	*EE*	90.000	*EE*
180	62.336	*W*	80.000	*EE*	80.000	*EE*	80.690	*L*
170	54.589	*W*	70.000	*EE*	70.000	*EE*	71.520	*L*
160	46.891	*W*	60.000	*EE*	60.000	*EE*	62.472	*L*
150	39.292	*W*	50.000	*EE*	50.000	*EE*	53.567	*L*
140	31.889	*W*	40.000	*EE*	40.480	*C*	44.896	*L*
130	24.806	*W*	30.000	*EE*	31.670	*C*	36.417	*L*
120	18.223	*W*	20.000	*EE*	23.362	*C*	28.458	*L*
110	12.401	*W*	13.272	*L*	15.592	*C*	20.911	*L*
100	7.503	*W*	7.824	*W*	8.656	*W*	13.954	*L*
90	3.833	*W*	3.912	*W*	4.322	*W*	7.934	*L*
80	1.489	*W*	1.499	*W*	1.618	*W*	2.999	*L*
70	0.377	*W*	0.378	*W*	0.395	*W*	0.395	*W*
60	0.052	*W*	0.052	*W*	0.053	*W*	0.053	*W*
50	0.003	*W*	0.003	*W*	0.003	*W*	0.003	*W*
40	0.000	*W*	0.000	*W*	0.000	*W*	0.000	*W*

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.10$, $T_2 = 5$ and $T_1 = \frac{1}{2}T_2$, $\sigma_L = 0.30$ with $X_L = 5$ for all cases. For case I only learning is available at T_1; no early development, no timing of learning. For case II there is optimal timing of learning and development option. For case III there is optimal timing of learning, control and development option with $\gamma(W,C) = \gamma(L,C) = 0.1$ and $\sigma(W,C) = \sigma(L,C) = 0.30$. Case IV is the same as case III but control characteristics are different if prior action is L i.e $\gamma(W,C) = 0.1$, $\sigma(W,C) = 0.30$, $\gamma(L,C) = 0.2$, $\sigma(L,C) = 0.30$. For the numerical lattice we use $N_{sub} = 30$ steps.

Table 3: Multistage investment program with a pilot project option and two phases of learning and controls: Sensitivity with respect to the effectiveness of learning actions

	Growth m = 0.1					
	$\gamma(L_G, C_2) = 0.1$				$\gamma(L_G, C_2) = 0.2$	
	$\gamma(L_1, C_1) = 0.1$		$\gamma(L_1, C_1) = 0.2$		$\gamma(L_1, C_1) = 0.2$	
S	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**
240	155.495	C_1	164.770	L_1	170.096	L_G
230	144.523	C_1	153.281	L_1	158.230	L_G
220	133.564	C_1	141.814	L_1	146.400	L_G
210	122.629	C_1	130.381	L_1	134.615	L_G
200	111.747	C_1	118.995	L_1	122.890	L_G
190	100.925	C_1	107.689	L_1	111.274	L_G
180	90.153	C_1	96.446	L_1	99.761	L_G
170	79.471	C_1	85.296	L_1	88.365	L_G
160	68.965	C_1	74.308	L_1	77.109	L_G
150	58.589	C_1	63.483	L_1	66.080	L_G
140	48.468	C_1	52.891	L_1	55.347	L_G
130	38.700	C_1	42.644	L_1	44.903	L_G
120	29.294	C_1	32.784	L_1	34.831	L_G
110	20.575	C_1	23.535	L_1	25.390	L_G
100	12.514	C_1	14.982	L_1	16.521	L_G
90	6.599	W	7.840	W	9.006	W
80	2.720	W	3.198	W	3.951	W
70	0.755	W	0.849	W	1.180	W
60	0.113	W	0.119	W	0.179	W
50	0.006	W	0.006	W	0.009	W
40	0.000	W	0.000	W	0.000	W

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.1$ and $T = 5$. The problem has as follows: The firm has the option to invest in a first phase of learning or controls (L_1, C_1), develop the project early (*EE*), invest in a pilot project (L_G) and invest in a second phase control action C_2. The sequence of actions and other parameters are given in figure 5 and the cost for each action is $X_{L_1} = X_{C_1} = X_{C_2} = 10$ and $X_G = 20$.

Table 4: New product development (investment program with a pilot project option and two phases of learning and control): Sensitivity with respect to the level *m* of pilot project cash flows

	$\gamma(L_G, C_2) = 0.2$, $\gamma(L_1, C_1) = 0.2$					
	Growth $m = 0$		Growth $m = 0.1$		Growth $m = 0.2$	
S	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**	**Value**	**Dec. at t=0**
240	164,770	L_1	170,096	L_G	194,096	L_G
230	153,281	L_1	158,230	L_G	181,230	L_G
220	141,814	L_1	146,400	L_G	168,400	L_G
210	130,381	L_1	134,615	L_G	155,615	L_G
200	118,995	L_1	122,890	L_G	142,890	L_G
190	107,689	L_1	111,274	L_G	130,274	L_G
180	96,446	L_1	99,761	L_G	117,761	L_G
170	85,296	L_1	88,365	L_G	105,365	L_G
160	74,308	L_1	77,109	L_G	93,109	L_G
150	63,483	L_1	66,080	L_G	81,080	L_G
140	52,891	L_1	55,347	L_G	69,347	L_G
130	42,644	L_1	44,903	L_G	57,903	L_G
120	32,784	L_1	34,831	L_G	46,831	L_G
110	23,535	L_1	25,390	L_G	36,390	L_G
100	14,982	L_1	16,521	L_G	26,521	L_G
90	7,840	*W*	9,006	*W*	17,430	L_G
80	3,198	*W*	3,951	*W*	9,360	*W*
70	0,849	*W*	1,180	*W*	3,920	*W*
60	0,119	*W*	0,179	*W*	1,002	*W*
50	0,006	*W*	0,009	*W*	0,105	*W*
40	0,000	*W*	0,000	*W*	0,002	*W*

Notes: Parameters are $r = \delta = 0.05$, $\sigma = 0.1$ and $T = 5$. The problem has as follows: The firm has the option to invest in a first phase of learning or controls (L_1 , C_1), develop the project early (*EE*), invest in a pilot project (L_G) and invest in a second stage control action C_2 . The sequence of actions and other parameters are given in figure 5 and the cost for each action is $X_{L_1} = X_{C_1} = X_{C_2} = 10$ and $X_G = 20$. $\gamma(L_1, C_1)$ is defined as the mean impact of control C_1 conditional on prior activation of learning action L_1 , and $\gamma(L_G, C_2)$ as the mean impact of C_2 conditional on prior activation of pilot project L_G .