Managerial Flexibility, Agency Costs and Optimal Capital Structure

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Abstract

This paper investigates the effect of managerial flexibility on the choice of capital structure for a firm and the corresponding valuation of its long term debt. We consider a general model in continuous time where the manager (who is not a shareholder) of a firm with long term debt in place may dynamically switch strategies at random times so as to maximize his expected discounted compensation. The manager may bear personal costs due to bankruptcy of the firm and the firm enjoys a tax shield on its interest payments to creditors. Under general assumptions on the nature of the strategies available to the manager, we show the existence of and derive explicit analytical characterizations for the optimal policies for the manager. We then derive the optimal policies for the firm that can hypothetically contract for managerial behavior ex ante, i.e. before debt is in place. We investigate the implications of these results for the optimal capital structure for the firm in the presence of managerial flexibility and the valuation of its long term debt. We also obtain precise quantitative characterizations of the agency costs of debt due to managerial flexibility in a very general context and show that they are very significant when compared with the tax advantages of debt thereby implying that managerial flexibility is a very important determinant of the choice of optimal capital structure for a firm. We carry out several numerical simulations with different choices of underlying parameter values to calculate the optimal leverage, agency costs , corporate debt values and bond yield spreads and study the comparative statics of these quantities with respect to the parameters characterizing the strategies available to the manager. The optimal leverage levels predicted by our model correspond very well with average leverage levels observed in the marketplace.

Key Words: Managerial Flexibility, Agency Theory, Capital Structure, Stochastic Control, Asset Substitution

JEL Classification Codes: C60, C73, D80, D92, G30, C73

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Introduction

Since the pioneering work of Modigliani and Miller (M-M) [1958, 1963], the investigation of the capital structure decision for a firm has been one of the cornerstones of research in financial economics in general and corporate finance in particular. The M-M assumptions of perfect markets and the independence of the investment and financing decisions for a firm lead to the conclusion that the value of a firm is independent of its capital structure.

The seminal paper of Jensen and Meckling (J-M) [1976] critically analyzed the M-M assumption of independence of the investment and financing decisions for a firm. The paper argued persuasively that equity holders of a leveraged firm, for example, could make use of asset substitution to extract value from bondholders after debt is in place. This phenomenon creates agency costs that must be controlled thereby forging an inextricable link between the capital structure of the firm and its investment decisions. However, increased risk-taking may limit the ability of the firm to shield itself from taxes through leverage so these conflicting forces would seem to indicate that there is, in fact, an optimal amount of debt that a firm should issue, and hence, an optimal capital structure for a firm.

The research of Jensen and Meckling spawned a large amount of theoretical and empirical research investigating how the choice of capital structure is influenced by these considerations. However, until recently, there were no attempts to completely integrate and synthesize the different approaches within one realistic unified framework.

This problem has been addressed by several recent papers.[1] Using models differing in some important underlying assumptions, Leland [1998] and Ericsson [2001] succeeded in precisely quantifying how the capital structure of a firm is influenced by ex-post flexibility of shareholders in choosing risk and how the presence of debt distorts a firm's ex-post choice of risk. In particular, the papers succeeded in valuing debt and leveraged equity in the presence of firm risk-taking thereby obtaining a precise characterization of the agency costs of debt associated with asset substitution by equity holders. However, in spite of the considerable progress that has been made, there are several questions that remain unanswered.

Leland [1998] and Ericsson [2001] both implicitly assume that the manager of the firm always behaves in the interests of shareholders. They consider the situation where the firm may alter its risk through hedging, but do not explicitly introduce the real-world *expected return* of the firm's overall assets (including the instruments used for hedging or risk-taking) that will be affected, in general, by increased or decreased firm risk-taking. This is, of course, not an issue when the cash flows of the unlevered assets of the firm are generated by marketed securities (as is usually assumed in the extant literature) and the firm's goal is to maximize the *market value* of the firm and the shareholders' goal is to

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[1] See, for example, Kim, Ramaswamy and Sundaresan [1993], Leland [1994a, 1994b], Leland and Toft [1996], Mella-Barral and Perraudin [1997], Anderson and Sundaresan [1997], Mello and Parsons [1992] and Brennan and Schwarz [1985], Green and Talmor [1986], Fischer, Heinkel and Zechner [1989].

maximize the market value of their equity since the entire analysis can then be carried out under the risk-neutral measure in which case the expected return of all traded assets is the risk-free rate.

However, in the presence of *managerial flexibility*, i.e. when the manager's incentives need not correspond with those of shareholders, the *expected returns* of the available strategies are crucial since the manager's goal is to maximize the *real world* expected utility of the cash flows that make up his compensation. More precisely, in the situation where the manager is not a shareholder or his compensation is derived from actual real world cash flows generated by the firm's operations his optimal policies would be, in general, dependent on the expected returns as well as the volatilities of the available strategies. Therefore, increased or decreased risk taking would, in general, affect not only the risk of the firm's cash flows, but also the expected growth. In the situation where the manager is compensated with the traditional structure of a base salary and a bonus that is proportional to firm profits (net of debt payments) or with options on firm stock, it is easy to see the nature of the conflict between the manager's and the firm's (or shareholders') interests.

In this paper, we study the optimal asset substitution problem for the manager of a firm in continuous time. The manager may dynamically switch between different strategies over an infinite time horizon so as to maximize his expected utility[2]. We also assume that the manager bears nonzero personal costs if the firm goes bankrupt. We derive explicit analytical expressions for the optimal policies for the manager under very general assumptions on the nature of the available strategies. We use these results to derive the optimal capital structure of the firm in the presence of managerial flexibility and the corresponding valuation of the firm's debt under additional assumptions on the servicing of debt during financial distress.

We then study the optimal asset substitution problem for a firm that can hypothetically contract for managerial behavior ex ante, i.e. before debt is in place. We explicitly derive the optimal policies for the firm in this situation and elucidate the nature of the conflict between the manager's interests and those of the firm.

These investigations and the explicit analytical results obtained allow us to obtain precise quantitative characterizations of the agency costs associated with managerial flexibility by employing the measure proposed by Leland [1998], i.e. we compare the hypothetical ex ante situation where the firm can contract for the policies to be followed by the manager and the real situation where managerial policy cannot be contracted. Through several numerical simulations, we demonstrate that, in general, the agency costs of managerial flexibility are very significant when compared with the tax advantages of debt thereby significantly affecting the leverage of the firm.

These results should be contrasted with the results of Leland's numerical simulations that indicate that the agency costs of debt associated with shareholder asset substitution are very insignificant when compared with the tax advantages of debt and that, therefore, shareholder asset substitution is not a significant determinant of the optimal capital structure of the firm. We thereby conclusively demonstrate that managerial flexibility is a far more significant determinant of capital structure of the firm than asset substitution driven by shareholders' interests.

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[2] This is in sharp contrast with the model of Ericsson [2001] where risk shifting can only occur once and is irreversible, but is in conformity with the model of Leland [1998].

The existence of analytical expressions for the manager's optimal policies also allows us to easily investigate the dependence of capital structure, agency costs, and firm value on the menu of strategies available to the manager. Moreover, our framework is broad and general enough to also obtain an accurate quantitative characterization of the costs of *asymmetric information* between the manager and the firm's investors where the investors only know the risks (or volatilities) of the strategies available to the firm, but not their expected returns.

In summary, this paper proposes and investigates a broad and general framework that provides answers to the following questions:

1. Given that the manager of a leveraged firm has the flexibility to choose between different strategies with different risks and expected returns, what is his optimal dynamic ex-post policy (i.e. after debt is in place)?
2. What are the implications of these considerations for the valuation of the firm and therefore, its choice of capital structure?
3. How do these issues affect the agency costs of debt and how significant are the agency costs of debt associated with managerial flexibility when compared with the tax advantages of debt?
4. What are the costs of asymmetric information between the manager and the firm's investors in the situation where the firm's investors only know the volatilities (or risks) of the available strategies but not their expected returns?

The principal results of the paper can be summarized as follows.

- It is EITHER optimal for the manager to always choose the high risk (and high expected return) strategy, OR to choose the low risk (and low expected return) strategy when the value of the firm's assets is below an endogenously derived threshold and the high risk (and high expected return) strategy when the value of the firm's assets is above the threshold. We provide analytical necessary and sufficient conditions for each of the above scenarios.
- In contrast, it is EITHER optimal for the firm to always choose the low risk strategy, OR to choose the high risk strategy when the value of the firm's assets is below an endogenously derived threshold and the low risk strategy above the threshold. Again, we provide analytical necessary and sufficient conditions for each of the above scenarios.
- For reasonable choices of parameter values, the agency costs of debt due to managerial flexibility may be as high as 75% of the tax advantages of debt and the optimal leverage of the firm may be as much as 25% lower than it would be in the absence of managerial flexibility. The optimal leverage levels predicted by the model lie between 20% and 30% which correspond very well with average leverage levels observed in the market.
- Managerial flexibility is therefore a very significant determinant of optimal capital structure.

From a mathematical standpoint, the manager's optimal policy problem is a stochastic control problem. In contrast with the papers of Leland [1998] and Ericsson

[2001], we rigorously prove the existence of optimal policies for the manager and derive analytical characterizations for them without resorting to numerical calculations or simulations. As we shall see later, within our framework, it is far from obvious at the outset that the optimal policies would always be of the "switching" type and that the usual technique of "smooth pasting" would give rise to optimal policies. This is, in fact, rigorously demonstrated by using well-known verification results for optimal policies and the value function for the stochastic control problem under consideration. In particular, we are able to show global uniqueness of the optimal policies derived under general assumptions.

Our detailed analysis of the optimal asset substitution problem for the firm that can contract for managerial behavior ex ante reveals that the prevailing economic intuition that the firm will always prefer higher risk close to bankruptcy is not always true. We provide a *necessary* and *sufficient* condition for the firm to increase risk close to bankruptcy. In contrast, if it is optimal for the manager to switch strategies, he will always prefer to decrease risk close to bankruptcy and we provide necessary and sufficient conditions for this to occur. These results dramatically illustrate the implications of the conflict between the manager's and the firm's interests.

Obtaining precise characterizations of the manager's optimal policies and the firm's optimal policies in the presence of long term debt sheds light on the important problem of the design of optimal compensation schemes for the manager that can align managerial incentives with those of the firm and thus eliminate or minimize agency costs.

Apart from the specific problem that is investigated in this paper, the results of the paper are, we believe, of independent interest since they have implications in a much more general context. Since we propose and solve the problem of optimal asset substitution in continuous time for a manager with a compensation structure of the *bonus type*, our results can be directly applied to the investigation of the problem of deriving the optimal policies for the portfolio manager of a mutual fund (or hedge fund) in a general incomplete market with an *option-like* compensation structure.

The plan for the paper is as follows. In **Section 1**, we present the model under consideration. In **Section 2**, we derive the optimal policies for the manager. In **Section 3**, we derive the optimal policies for the firm that can hypothetically contract for managerial behavior. In **Sections 4** and **5,** we investigate the implications of the results of **Sections 2** and **3** for the optimal capital structure for the firm in the presence of managerial flexibility, the associated agency costs and present numerical results that study their dependence on the nature of the available asset substitution strategies. **Section 6** concludes the paper. All detailed proofs appear in the Appendix.

1. The Model

Throughout the paper, we consider a filtered probability space (Ω, F, P, F_t) with the filtration F_t (completed and augmented) generated by two independent Brownian motions B_1, B_2.

A firm has a certain amount of long-term debt in place that is completely amortized, i.e. the firm is liable for an interest (coupon) payment of q per unit time over an infinite time horizon. The manager of the firm is assumed to be risk-neutral and need not behave in the interests of shareholders or the firm, i.e. there are potential agency problems arising from the manager's actions not conforming to the interests of shareholders or the firm. The manager is compensated with the traditional structure of a fixed base salary and a bonus that is proportional to the profits of the firm (net of interest payments) and his discount factor or opportunity cost parameter is b so that the discount factor for cash flows at time t is $\exp(-bt)$[3]. We assume that the manager's compensation contract is not renegotiated and that he is retained as long as the firm is in operation. The manager's goal is to choose firm strategies so as to maximize his (discounted) expected utility of cash flows comprising his compensation. In contrast with Leland [1998], we assume that the firm does not retire its debt or restructure it at intermediate times.

$P(.)$ represents a state variable that determines the cash flows from the firm's operations. We assume (as is usually done in the extant literature) that the cash flows associated with the firm's operations are spanned by marketed securities and that the cash flows (per unit time) arising from the firm's operations (before coupon payments to creditors and corporate taxes) are proportional to the value of the state variable $P(.)$ and equal to $lP(t)$ at time t. $P(.)$ is the price process of a traded asset that has a cash payout ratio of d per unit time.

As long as the firm's cash flows exceed the required interest payments, they are used to service debt. If they are lower than the required interest payments, we assume that the cash flows from the firm's operations go to bondholders. The remaining portion of the interest payments due may be serviced either entirely or in part by shareholders until an exogenous level p_b when bankruptcy occurs and the firm is liquidated. Liquidation occurs at a proportional cost a to the firm. Since we assume that bankruptcy and liquidation occur simultaneously, we make distinction between bankruptcy and liquidation and use the terms interchangeably throughout the paper.

Our assumption of exogenous bankruptcy differs from that of Leland [1998] who assumes that shareholders inject additional capital to service debt until their value of equity is zero so that bankruptcy occurs endogenously. Typically, however, shareholders of a publicly traded firm are a diffuse group so that the only feasible way to service debt is to issue new equity that can be difficult, if not impossible, when the firm is in financial distress. Moreover, a significant percentage of the outstanding debt of several firms is unsecured that significantly reduces the bargaining power of the firm's bondholders. In this situation, it is quite likely that interest payments due to bondholders will not be serviced entirely when the firm is in financial distress (Anderson and Sundaresan 1996, Mella-Barral and Perraudin 1997). The assumption of exogenous bankruptcy where debt need not be serviced entirely in financial distress is appropriate in a large number of situations in the real world where it has been observed empirically that bondholders are persuaded by shareholders to accept concessions prior to formal bankruptcy and liquidation proceedings.

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[3] All the results of the paper hold if the manager is periodically compensated with executive options with a fixed strike price. We only assume that the manager is compensated with a base salary and a bonus proportional to firm profits (net of debt payments) for expositional convenience.

There are of course several possible choices for the liquidation trigger level p_b. It could be the level at which net earnings after interest fall below zero (Kim, Ramaswamy and Sundaresan 1993) or it could be the level at which the value of unlevered equity falls below the outstanding debt principal. We, however, keep our treatment as general as possible by assuming that bankruptcy occurs at some level p_b that may be a function of the coupon rate q. This general framework has the additional advantage of allowing us to investigate the comparative statics of the agency costs of debt and the optimal capital structure of the firm with the exogenous bankruptcy level.

Since we are primarily concerned with the effect of managerial flexibility on the firm's choice of capital structure, i.e. the magnitude of the agency costs of debt due to managerial flexibility, at this point we do not make any specific assumptions about how debt is serviced when cash flows from the firm's operations are not sufficient to meet interest payments. Since we have assumed that bondholders get at least the firm's cash flows in this situation, bankruptcy is exogenous and losses are not carried over, debt service represents a redistribution of wealth between bondholders and shareholders that does not affect the overall cash flows of the firm and hence, its value. Debt service during financial distress may either be determined by a covenant in the bond indenture or may be negotiated by shareholders and bondholders afterwards (Anderson and Sundaresan 1996, Mella-Barral and Perraudin 1997).

The actual form of debt service during financial distress would of course affect the value of debt and the optimal leverage of the firm. In a later section, we make specific assumptions about the form of debt service and derive the value of debt and the optimal capital structure of the firm. In this paper, we assume that the manager of the firm is not a shareholder and is replaced when bankruptcy occurs so that the manager bears costs due to bankruptcy. In reality, the management is usually assigned a large portion of the responsibility for financial distress and, it is therefore, quite likely that the manager would bear substantial personal costs due to bankruptcy. The results of the paper hold if the manager bears any strictly nonzero cost due to bankruptcy. For notational convenience, we however assume throughout that the manager is replaced with no severance compensation. We also assume throughout that the Absolute Priority Rule (APR) is enforced on liquidation, i.e. creditors obtain all the proceeds (after costs) from liquidation.

Available Strategies for the Manager

At any instant of time, the manager of the firm can switch between two strategies *without cost*. The state variable $P(.)$ evolves in the *real world* as follows under the two strategies :

(1a)
$$dP(t) = P(t)[(m_1' - d)dt + s_{11}dB_1(t) + s_{12}dB_2(t)] \quad \text{Strategy 1}$$
$$dP(t) = P(t)[(m_2' - d)dt + s_{21}dB_1(t) + s_{22}dB_2(t)] \quad \text{Strategy 2}$$

Note : The state variable processes under the two strategies need not be perfectly correlated with each other.

Therefore, we may rewrite equations (1a) as follows :

(1b)
$$dP(t) = P(t)[m_1 dt + s_1 dB_1^*(t)] \quad \text{Strategy 1}$$
$$dP(t) = P(t)[m_2 dt + s_2 dB_2^*(t)] \quad \text{Strategy 2}$$

where B_1^*, B_2^* are (not necessarily perfectly correlated) F_t – Brownian motions with

$$s_1 = \sqrt{s_{11}^{\;2} + s_{12}^{\;2}}, m_1 = m_1' - d$$
$$s_2 = \sqrt{s_{21}^{\;2} + s_{22}^{\;2}}, m_2 = m_2' - d$$

Thus, if the manager has initially chosen strategy 1 and switches to strategy 2 at time t^*, then the evolution of the state variable $P(.)$ for times $t > t^*$ is described by the drift and volatility parameters (m_2, s_2) until it switches back to strategy 1 in which case the evolution is governed by the drift and volatility parameters (m_1, s_1). Thus, the state variable process $P(.)$ is always continuous. We assume that m_1, m_2, s_1, s_2 are constants, $b > m_1 > m_2 > 0$ [4] and $s_1 > s_2$, but don't make any further assumptions on their values. Thus, strategy 1 has a higher drift (or higher expected return) and higher volatility (or higher risk) than strategy 2 that is consistent with the usual tradeoff between expected return and variance. Therefore, the manager's policies Γ may be described as follows:

(2)
$$\Gamma \equiv \{t_1, t_2, t_3, ...\}$$

where t_i are increasing F_t – stopping times (reflecting the fact that the manager's decisions cannot anticipate the future) representing the instants where the manager switches strategies. The goal of the manager is to choose his policy to maximize his expected discounted compensation that is given by

(3)
$$U_\Gamma(p) = E[\int_0^{t_b} \exp(-bt)\, f dt + g \int_0^{t_b} \exp(-bt)[\, l P_\Gamma(t) - q]^+ dt\,],$$

where f is the fixed salary per unit time and the second term is the discounted expected utility of consumption of the variable salary or bonus. g is the fixed proportion of firm profits (net of debt payments) representing the manager's bonus. $P_\Gamma(.)$ represents the state variable process when the manager follows strategy Γ described in (2) and t_b is the time (stopping time) at which bankruptcy (or liquidation) occurs.

[4] It is very easy to see that if $b < m_1$, the value function for the manager's optimization problem (3) is infinite.

If $u(.)$ is the value function of the dynamic optimization problem (3), then we can use traditional dynamic programming arguments (see e.g. Oksendal 1998) to write down the following formal Hamilton-Jacobi-Bellman equation for u:

$$(4)\qquad \begin{aligned} &-bu+\sup_{i=1,2}[m_i p u_p+\frac{1}{2}s_i^{\,2}p^2u_{pp}]+f+g(lp-q)^+=0, p\in(p_b,\infty) \\ &u(p_b)=0 \end{aligned}$$

where $u(p_b)=0$ above represents the boundary condition for the manager's value function u at the bankruptcy level p_b. In the dynamic programming framework, the variable p above represents the value of the state variable $P(.)$ so that the term $f+g(lp-q)^+$ is the instantaneous rate of compensation of the manager. Therefore, in regions where it is optimal for the manager to choose strategy 1, we would expect to have

$$(5)\qquad \begin{aligned} &L^1(u)+f+g(lp-q)^+=0 \\ &L^2(u)+f+g(lp-q)^+\le 0 \end{aligned} \quad \text{where } L^1(u)=-bu+m_1pu_p+\frac{1}{2}s_1^{\,2}p^2u_{pp},$$

and in regions where it is optimal for the manager to choose strategy 2, we would have

$$(6)\qquad \begin{aligned} &L^2(u)+f+g(lp-q)^+=0 \\ &L^1(u)+f+g(lp-q)^+\le 0 \end{aligned} \quad \text{where } L^2(u)=-bu+m_2pu_p+\frac{1}{2}s_2^{\,2}p^2u_{pp}.$$

We shall now state without proof the following standard verification result.

Proposition 1
If $u:[p_b,\infty)\to R_+$ is a continuous function that is twice differentiable on (p_b,∞) satisfying the HJB equation (4), then u is the value function of the manager's optimization problem (3).

Proof. See e.g. Oksendal [1998].

This completes the formulation of the model and the mathematical preliminaries.

2.Optimal Policies for the Manager

In this section, we shall show the existence of and explicitly derive the optimal switching policies for the manager for all possible pairs of strategies 1 and 2 characterized by the drift-volatility parameters (m_1,s_1) and (m_2,s_2) satisfying $b>m_1>m_2>0$ and $s_1>s_2$. It is important to emphasize here that it is far from obvious at the outset that optimal policies of the switching type even exist for the manager. We will show that

- It is EITHER optimal for the manager to choose strategy 1, i.e. the high risk strategy always OR optimal for the manager to choose strategy 2 close to bankruptcy and switch to strategy 1 when the firm moves away from bankruptcy.
- We provide analytical necessary and sufficient conditions for each of the two cases above.

We shall begin by introducing two quadratic equations that are intimately related to the derivation of the value function for the manager.

$$
\begin{aligned}
&\frac{1}{2}s_1{}^2x^2+(m_1-\frac{1}{2}s_1{}^2)x-b=0\\
&\frac{1}{2}s_2{}^2x^2+(m_2-\frac{1}{2}s_2{}^2)x-b=0
\end{aligned}
\qquad (7)
$$

Each of the equations above has two real roots, one of which is strictly positive and the other strictly negative. Let us denote the positive and negative roots of the equations above by h_1^+, h_1^- and h_2^+, h_2^- respectively. Throughout the paper, we shall assume that $h_1^+ \neq h_2^+, h_1^- \neq h_2^-$, i.e. the available strategies are such that the roots of equations (10) are all distinct[5]. We can now prove the following lemma that collects properties of the roots of equations (7) that we will use frequently.

Lemma 1

(1) $1 < h_i^+ < \frac{b}{m_i}$ *for* $i = 1,2$

(2) $h_1^+ < h_2^+$

Proof. In the Appendix.

We shall now proceed to the explicit description of the manager's optimal policies and value function for different choices of the pair of available strategies. By the result of part (2) of **Lemma 1**, since we must have $h_1^+ < h_2^+$, there are only two different scenarios described by the ordering of the *negative roots* h_1^-, h_2^- of equations (7) that depend on the drift-volatility parameters characterizing the two strategies.

[5] This assumption avoids unnecessarily complicating the statements of several propositions.

Case 1 : $h_1^- < h_2^- < h_1^+ < h_2^+$

We can show that the roots are distributed as above when (roughly) the difference between the drifts of the available strategies $m_1 - m_2$ is large compared with the difference in the volatilities $s_1 - s_2$. The following proposition completely characterizes the optimal policy for the manager in this case.

Proposition 2
The optimal policy for the manager is to choose strategy 1 throughout, i.e. asset substitution will never occur. Moreover, this is the unique optimal policy for the manager.

Proof. In the Appendix.

The intuition for the above result is that even though the risk of strategy 1 is higher than the risk of strategy 2, its drift or expected return is much higher so that, due to the convexity of his compensation when the firm is solvent, it is optimal for the manager to choose strategy 1 always and never shift to strategy 2 even if the firm is close to bankruptcy and he risks losing his job.

Case 2 : $h_2^- < h_1^- < h_1^+ < h_2^+$

This corresponds (roughly) to the situation where the difference in the drifts of the strategies $m_1 - m_2$ is small compared with the difference in the volatilities $s_1 - s_2$.

We shall show that the manager's unique optimal policies are to choose strategy 2 whenever the value of the state variable $p \leq p_*$ and strategy 1 whenever $p > p_*$ for some p_* with $p_b \leq p_* < \infty$. In other words, the manager's (stationary) optimal policies are to switch strategies whenever the value of the state variable crosses p_*. In the degenerate case where $p_* = p_b$, the optimal policy is to choose strategy 1 throughout.

The intuition for this result is that, in this case, the difference in the risks of the two strategies outweighs the difference in their expected returns so that when the firm's performance is mediocre, the manager would prefer to lower risk in order to preserve his job. However, when the firm is performing extremely well, the manager would prefer the strategy with higher expected return even though it has higher risk. These results are driven intrinsically by the fact that the manager's compensation is *convex* in the firm's operational cash flows when it is solvent. We will provide precise analytical characterizations of the optimal policies. Since the arguments are rather involved we shall present the results in the form of a series of propositions. Before proceeding, we need to introduce some notation.

For each r with $p_b \le r < \infty$, let u_r be the value function of the policy (of the manager's expected discounted compensation given by (3)) where the manager chooses strategy 2 for $p \le r$ and strategy 1 for $p > r$. We can clearly distinguish two scenarios:

A: $p_b \le r \le \frac{q}{l}$

In this case, we see from the results we have obtained thus far (see the proof of **Proposition 2)**, that

$$
(8)\qquad
\begin{aligned}
u_r &= A_r p^{h_2^+} + B_r p^{h_2^-} + \frac{f}{b};\ p_b \le p < r \\
&= C_r p^{h_1^+} + D_r p^{h_1^-} + \frac{f}{b};\ r \le p < \frac{q}{l} \\
&= E_r p^{h_1^-} + \frac{f-gq}{b} + \frac{glp}{b-m_1};\ p \ge \frac{q}{l} \\
u_r(p_b) &= 0
\end{aligned}
$$

where the subscripts indicate the explicit dependence on the switching point r.

B: $r \ge \frac{q}{l}$

In this case, we see that

$$
(9)\qquad
\begin{aligned}
u_r &= A_r p^{h_2^+} + B_r p^{h_2^-} + \frac{f}{b};\ p_b \le p < \frac{q}{l} \\
&= C_r p^{h_2^+} + D_r p^{h_2^-} + \frac{f-gq}{b} + \frac{glp}{b-m_2};\ \frac{q}{l} \le p < r \\
&= E_r p^{h_1^-} + \frac{f-gq}{b} + \frac{glp}{b-m_1};\ p \ge r \\
u_r(p_b) &= 0
\end{aligned}
$$

where the subscripts indicate the explicit dependence on the switching level r. We have retained the same variables for the coefficients defining the value functions in the two cases for the sake of notational brevity. The coefficients are uniquely determined by the condition that the value function is continuously differentiable for $p > p_b$ and continuous at p_b.

We can now state the following proposition that provides a *necessary and sufficient* condition for the unique optimal policy for the manager to choose strategy 1 throughout.

Proposition 3
A necessary and sufficient condition for the choice of strategy 1 throughout to be the unique optimal policy for the manager is

(10) $L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=p_b+} \leq 0$

Note : *u_{p_b} denotes the value function of the policy of switching strategies at $p = p_b$, i.e. of choosing strategy 1 throughout so that the expression $L^2(u_{p_b}) + f + g(lp-q)^+$ is a function of p. This function is evaluated at the point $p = p_b+$, i.e. the limit as $p \to p_b$ in (10).*

Proof. In the Appendix.

When condition (10) does not hold, we will show that there exists a switching point $p_* > p_b$ such that the policy of choosing strategy 2 for $p \leq p_*$ and strategy 1 for $p > p_*$ is optimal with u_{p_*} defined as in (8) or (9) being the corresponding optimal value function.

We note from (8) and (9) that the value function u_r is twice differentiable on (p_b, ∞) except possibly at $p = r$. The basic idea in the proofs of the results that follow is to show the existence of a value p_* such that the value function u_{p_*} is twice differentiable[6] at $p = p_*$ and hence on (p_b, ∞). We will then show that the hypotheses of **Proposition 1** are satisfied to conclude that the policy is therefore optimal.

Before proceeding with the statements of the results, we prove the following lemma that we will use frequently in our proofs.

Lemma 2
If u_r is defined as in (8) or (9) with $p_b < r < \infty$ then a necessary and sufficient condition for u_r to be twice differentiable at $p = r$ (and therefore everywhere) is

(11) $L^2(u_r) + f + g(lp-q)^+ |_{p=r+} = 0$

Moreover, this is also equivalent to the condition

(12) $L^1(u_r) + f + g(lp-q)^+ |_{p=r-} = 0$

which is in turn equivalent to

(13) $L^1(u_r) + f + g(lp-q)^+ |_{p=r} = L^2(u_r) + f + g(lp-q)^+ |_{p=r} = 0$

[6] This is the "super contact" condition discussed by Dumas [1991] and Mella-Barral and Perraudin [1997]

Proof. In the Appendix

By the result of the above lemma, it therefore suffices to show either (11) or (12) in order to show that the value function is twice differentiable. We are now ready to state our results

Proposition 4

If

(14) $L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=p_b+} > 0$

and

(15) $L^2(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l+} \leq 0$

then there exists p_* *with* $p_b < p_* \leq q/l$ *such the policy of choosing strategy 2 for* $p \leq p_*$ *and strategy 1 for* $p > p_*$ *is the unique optimal policy for the manager and* u_{p_*} *defined in (8) is the corresponding optimal value function.*

Proof. In the Appendix.

The result of the above proposition tells us that when conditions (14) and (15) hold it is optimal for the manager to switch strategies at $p_* \leq q/l$, i.e. when the firm's cash flows cannot meet required debt payments. The following proposition shows that conditions (14) and (15) are both necessary and sufficient for asset substitution to occur at $p_* \leq q/l$.

Proposition 5

If condition (14) holds and

(16) $L^2(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l+} > 0$

then there exists p_* *with* $\dfrac{q}{l} < p_* < \infty$ *such the policy of choosing strategy 2 when* $p_b < p \leq p_*$ *and strategy 1 for* $p > p_*$ *is the unique optimal policy for the manager.*

Proof. In the Appendix

Since the volatility of strategy 1 is greater than that of strategy 2 by definition, the results of **Propositions 3, 4** and **5** clearly imply that if it is ever optimal for the manager to switch strategies, he will always (roughly) choose the *low risk* strategy, i.e. strategy 2 close to bankruptcy and the *high risk* strategy, i.e. strategy 1, away from bankruptcy. As we shall see in the next section, the optimal policies for the firm are exactly the reverse, i.e. the firm will always choose the *high risk* strategy close to bankruptcy and the *low risk* strategy away from bankruptcy. This result is driven by the fact that the firm's payoff structure is *concave* while the manager's payoff structure is *convex*. This is exactly the source of managerial flexibility that is the primary focus of this paper. We shall now condense the results of **Propositions 3, 4 and 5** into the following corollary.

Corollary 1

a) *A necessary and sufficient condition for the manager to choose strategy 1 throughout, i.e. for asset substitution to never occur is*

$$L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=p_b+} \leq 0$$

b) *Necessary and sufficient conditions for the manager to switch from strategy 2 to strategy 1 at* $p_* \leq \dfrac{q}{l}$ *are*

$$L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=p_b+} > 0$$
$$L^2(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l+} \leq 0$$

c) *Necessary and sufficient conditions for the manager to switch from strategy 2 to strategy 1 at* $p_* > \dfrac{q}{l}$ *are*

$$L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=p_b+} > 0$$
$$L^2(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l+} > 0$$

d) *It is never optimal for the manager to choose strategy 2 , i.e. the low risk strategy, throughout.*

5.Optimal Policies for the Firm

In this section, we shall explicitly derive the optimal policies for the firm, i.e. the policies that maximize the *market value* of the firm when the firm can hypothetically contract for managerial behavior ex ante. We will show that

- the optimal policies are EITHER to choose strategy 2, i.e. the low risk strategy throughout OR to switch from the high risk strategy, i.e. strategy 1 at low asset values to the low risk strategy, i.e. strategy 2 at high asset values.
- We provide analytical necessary and sufficient conditions for each of the cases above.

We consider the general situation where debt has tax benefits and there are costs associated with bankruptcy. Bankruptcy is assumed to occur *exogenously* as discussed in **Section 1**. When the firm's cash flows are not enough to meet interest payments completely, we assume that all the cash flows go to the firm's creditors. Any additional payments are negotiated *without cost* between shareholders and creditors and represent a redistribution of wealth between them that does not affect overall firm cash flows. The tax rate is denoted by t and the exogenous bankruptcy level by p_b which is a function of the coupon rate q. If the firm is financed entirely through equity, i.e. the firm is not leveraged and $q=0$, there is no possibility of bankruptcy or liquidation and $p_b=0$ in this case. The cash flows per unit time associated with the firm after debt is in place and when the firm is solvent are therefore given by

$$\begin{aligned} C(P(t)) &= lP(t)+tq;\ P(t)\ge q/l \\ &= lP(t)+t\,lP(t);\ p_b \le P(t) \le q/l \end{aligned} \tag{17}$$

Since our goal in this section is the maximization of the *market value* of the firm, we work under the *risk neutral measure* under which the drifts of both strategies are equal to $r-d$ where r is the risk free rate and d is the cash flow rate for the state variable $P(.)$ that we have assumed to be the price process of some traded asset in the market. Therefore, the state variable $P(.)$ evolves as follows under the *risk neutral measure*:

$$\begin{aligned} dP(t) &= (r-d)P(t)dt + s_1 P(t)dB_1^*(t);\ \text{Strategy 1} \\ dP(t) &= (r-d)P(t)dt + s_2 P(t)dB_2^*(t);\ \text{Strategy 2} \end{aligned} \tag{18}$$

For the unlevered firm, we easily see from (17) and (18) with $p_b=0$ and $t=0$ (there are no tax advantages) that the value of the firm when the value of the state variable $P(t)=p$ is given by $V(p)=\dfrac{lp}{d}$. Therefore, in the presence of debt, the value of the firm at the exogenous bankruptcy or liquidation level p_b is given by $V(p_b)=(1-a)\dfrac{lp_b}{d}$, since liquidation occurs at a proportional cost a and $\dfrac{lp_b}{d}$ is the value of unlevered equity at p_b. The goal is to maximize the *market value* of the firm that is given by

$$V_\Gamma(p) = E_\Gamma\Big[\int_0^{t_b} \exp(-rt)C(P(t))dt + (1-a)\frac{lp_b}{d}\Big] \quad \text{under the switching policy } \Gamma. \tag{19}$$

As in (4), (5), (6) we can introduce the formal Hamilton-Jacobi-Bellman equation associated with the firm's optimization problem

$$\begin{aligned} &-ru + \sup_{i=1,2}\Big[(r-d)pu_p + \frac{1}{2}s_i^{\,2}p^2u_{pp}\Big] + C(p) = 0, p\in(p_b,\infty) \\ &u(p_b) = \frac{(1-a)lp_b}{d} \end{aligned} \tag{20}$$

and the generators of the strategies available to the firm

$$(21)\qquad \begin{aligned} L^1(u) &= -ru + (r-d)pu_p + \frac{1}{2}s_1^{\,2}p^2u_{pp}, \\ L^2(u) &= -ru + (r-d)pu_p + \frac{1}{2}s_2^{\,2}p^2u_{pp} \end{aligned}$$

Analogous to (7), we obtain two quadratic equations intimately related to the value functions of the firm :

$$(22)\qquad \begin{aligned} &\frac{1}{2}s_1^{\,2}x^2 + (r-d-\frac{1}{2}s_1^{\,2})x - r = 0 \quad \text{with roots } r_1^+, r_1^- \\ &\frac{1}{2}s_2^{\,2}x^2 + (r-d-\frac{1}{2}s_2^{\,2})x - r = 0 \quad \text{with roots } r_2^+, r_2^- \end{aligned}$$

By the results of **Lemma 1**, we see that

$$(23)\qquad r_2^- < r_1^- < r_1^+ < r_2^+ \quad \text{if} \quad s_1 > s_2$$

Before actually deriving the optimal policies for the firm, we can prove the following general result that allows us to gain important intuition and insight into the nature of the optimal policies for the firm.

Proposition 6
Suppose the firm chooses a policy that involves switching strategies whenever the state variable crosses a value in the finite set $\{p_1, p_2, ..., p_n\}$ *where* $p_1 < p_2 < ... < p_n$. *Then a necessary (but not sufficient) condition for the policy to be optimal is that* $\frac{q}{l} > p_n$, *i.e. any switching of strategies by the firm when it is profitable (net of debt payments) is never optimal.*

Proof. In the Appendix.

Proposition 7

A necessary condition for a stationary switching policy to be optimal is that the firm chooses the less risky (less volatile) strategy, i.e. strategy 2, for $p \geq q/l$.

Proof. In the Appendix.

We can now state the following corollary of the previous two propositions.

Corollary 2
Suppose a firm chooses a policy that involves risk shifting at a single value p^* of the state variable. Then necessary conditions for optimality of the policy are that $p^* < q/l$ and that the firm switches from the high risk strategy to the low risk strategy at p^*.

Remark : One of the results in the corollary above, i.e. a stationary optimal policy for the firm that involves risk shifting at a single level p^* must choose the high risk strategy close to the exogenous bankruptcy level and switch to the low risk strategy at p^* is strongly supported by economic intuition and is , in fact, quite well known in the finance and economics literature. Several authors have assumed the result without actually proving it in a general continuous time setting. But the result that the optimal switching level p^* should be less than q/l , i.e. the firm will shift its risk when it is still unprofitable is far from obvious intuitively[7].

We can now state the following proposition that completely specifies the optimal policies for the firm.

Proposition 8
With exogenous bankruptcy at the level $p_b \le q/l$, there exists p^*, with $p_b \le p^* < q/l$ such that the unique optimal policy for the firm is to choose the high volatility (high risk) strategy, i.e. strategy 1 for $p \le p^*$ and the low volatility (low risk) strategy, i.e. strategy 2 for $p \ge p^*$.

Proof. In the Appendix.

Remark
By the result of the above proposition, we see that contrary to what is generally assumed in the literature, it is not always optimal for a firm to increase risk close to bankruptcy. The proof of the proposition provides a very precise *necessary* and *sufficient* condition under which it is optimal for the firm to increase risk close to bankruptcy.

Intuitively, the reasons for the conflict of interest between manager and the firm is that the firm's goal is to maximize its *market value*, i.e. its expected discounted cash flows in a *risk neutral* world and that the firm's cash flows described by (17) are *concave* in the value of the state variable $P(.)$. This is in contrast with the fact that the manager's

---

[7] We can, in fact, use arguments similar to those used in the proofs of the previous propositions (we shall omit the analysis here for the sake of brevity[7]) to show that shareholders maximizing the market value of their equity will also never shift risk at values greater than q/l , i.e. risk shifting will always occur when the firm is still unprofitable. This result probably partially explains why the agency costs associated with asset substitution by shareholders is very small (as reported by Leland [1998]) and insignificant compared with the tax advantages of debt.

goal is to maximize his expected discounted cash flows in the *real* world and that his compensation is *convex* in the value of the state variable $P(.)$.

4.Optimal Capital Structure and Agency Costs

We can use the results of the previous sections to explicitly derive the agency costs of debt due to managerial flexibility. By comparing the hypothetical situation where the manager's policies can be contracted for *ex ante*, (i.e. before debt is in place) and the actual situation where the manager's interests may conflict with those of the firm, i.e. he chooses his policies *ex post*, (i.e. after debt is in place), we can obtain a measure of the agency costs of debt due to managerial flexibility that is inspired by Leland [1998].

Although the primary focus of this paper is the derivation of optimal managerial and firm policies and the quantification of agency costs, we can also derive the optimal capital structure of the firm and the valuation of the firm's debt with additional assumptions about the servicing of debt when the firm is in financial distress, i.e. its cash flows are unable to meet interest payments entirely. In the formulation of our model, recall that we did not make any specific assumptions about the servicing of debt in financial distress except that creditors obtain at least all cash flows from the firm's operations. The actual form of debt servicing does not affect either the manager's optimal policies or the value of the firm in the presence of exogenous bankruptcy as long as losses are not carried over. However, the servicing of debt clearly affects the valuation of the firm's debt and its optimal leverage.

In this section, we assume that when the firm is in financial distress, equity holders inject capital to service debt entirely as long as the value of equity is positive. If the (endogenously determined) level p_e at which the value of equity falls to zero is greater than the exogenous liquidation or bankruptcy level p_b, then all cash flows from the firm's operations when $p \leq p_e$ go to bondholders including the proceeds from liquidation if it occurs. It is very easy to see that if p_e exists, it must be less than q/l which is the point at which the firm's cash flows are not sufficient to meet interest payments. We emphasize that this just represents a redistribution of wealth between creditors and shareholders that does not alter overall firm-related cash flows.

Our assumption that equity holders inject capital to service debt as long as the value of equity is positive is similar to that of Leland [1998], Mella-Barral and Perraudin [1997] and others. But, in contrast, we do not in general assume that control of the firm transfers to bondholders as soon as the value of equity becomes zero. This allows for more generality in the modeling of financial distress and also corresponds with what is observed in reality in the case of several financially distressed firms where they continue to operate in the face of protracted bankruptcy and liquidation proceedings even though their equity is practically worthless. It is important to emphasize here that it may be possible for equity holders to service debt entirely till the liquidation level p_b in which case the value of equity is greater than zero for $p > p_b$.[8]

The Ex-Post Value Functions and Optimal Capital Structure

In **Section 2**, we have shown that the optimal policies for the manager are always to choose strategy 2 for $p \le p_*$ and strategy 1 for $p > p_*$ for some p_* with $p_b \le p_* < \infty$. In particular, when $p_b = p_*$, the optimal policy is to choose strategy 1 throughout. We can now use the results of the previous section to obtain the value function of the firm corresponding to this policy. We clearly have three different possibilities:

Case 1 : $p_* = p_b$

In this case, the value function of the firm is given by

$$
\begin{aligned}
v(p) &= Ep^{r_1^-} + \frac{lp}{d} + \frac{tq}{r};\ p \ge \frac{q}{l} \\
&= Cp^{r_1^+} + Dp^{r_1^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_b < p \le \frac{q}{l} \qquad (24) \\
v(p_b) &= (1-a)\frac{lp_b}{d}
\end{aligned}
$$

The determination of the value of the firm's debt is complicated by our assumption of the form of debt service during financial distress. If p_e is the (endogenously determined) level at which the value of equity falls to zero, then the value of debt is given by

$$
\begin{aligned}
d(p) &= Jp^{r_1^-} + \frac{q}{r};\ p \ge p_e \\
&= v(p);\ p < p_e \qquad (25)
\end{aligned}
$$

Case 2: $p_b < p_* \le \dfrac{q}{l}$

In this case, the value function of the firm is given by

---

[8] Our modeling of financial distress therefore combines features of the endogenous bankruptcy models (Leland [1998], Leland and Toft [1994]) and the exogenous bankruptcy models (Ericsson 2001, Kim, Ramaswamy and Sundaresan 1993).

$$
(26)\quad
\begin{aligned}
v(p) &= Ep^{r_1^-} + \frac{lp}{d} + \frac{tq}{r};\ p \ge \frac{q}{l} \\
&= Cp^{r_1^+} + Dp^{r_1^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_* < p \le \frac{q}{l} \\
&= Ap^{r_2^+} + Bp^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_b < p \le p_* \\
v(p_b) &= (1-a)\frac{lp_b}{d}
\end{aligned}
$$

If $p_e < p_*$, the value of the firm's debt is given by

$$
(27a)\quad
\begin{aligned}
d(p) &= Jp^{r_1^-} + \frac{q}{r};\ p > p_* \\
&= Fp^{r_2^+} + Gp^{r_2^-} + \frac{q}{r};\ p_e < p \le p_* \\
d(p) &= v(p);\ p \le p_e
\end{aligned}
$$

and if $p_e \ge p_*$, it is given by

$$
(27b)\quad
\begin{aligned}
d(p) &= Jp^{r_1^-} + \frac{q}{r};\ p > p_e \\
&= Fp^{r_1^+} + Gp^{r_1^-} + \frac{q}{r};\ p_* < p \le p_e \\
d(p) &= v(p);\ p \le p_*
\end{aligned}
$$

Case 3: $\frac{q}{l} < p_* < \infty$

In this case, the firm's value function is given by

$$
(28)\quad
\begin{aligned}
v(p) &= Ep^{r_1^-} + \frac{lp}{d} + \frac{tq}{r};\ p \ge p_* \\
&= Cp^{r_2^+} + Dp^{r_2^-} + \frac{lp}{d} + \frac{tq}{r};\ \frac{q}{l} < p < p_* \\
&= Ap^{r_2^+} + Bp^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_b < p \le \frac{q}{l}
\end{aligned}
$$

with $v(p_b) = (1-a)\frac{lp_b}{d}$

and the value of its debt (since $p_e \le \frac{q}{l} < p_*$) is given by

$$d(p) = Jp^{r_1^-} + \frac{q}{r};\ p \geq p_*$$

(29) $$= Hp^{r_2^+} + Ip^{r_2^-} + \frac{q}{r};\ p_e < p < p_*$$

$$d(p) = v(p);\ p \leq p_e$$

We have obtained precise necessary and sufficient conditions for each of the 3 cases above in **Section 2**. Just as in Leland [1998], the optimal ex-post leverage is obtained by maximizing the firm's initial value $v(p_0)$ (where p_0 is the initial value of the unlevered assets of the firm) as a function of the coupon rate q.

The coefficients in the expressions above are determined by the conditions that the value function and value of debt are continuous for $p \geq p_b$ and differentiable for $p > p_b$. This implies that the level p_e is endogenously determined by the condition that the value of equity, i.e. the difference between the value of the firm and the value of debt, is zero at e and its first derivative is also zero (the smooth pasting condition). If no such point exists, i.e. the value of equity is positive for all $p > p_b$, then we set $p_e = p_b$ in the above expressions.

The Ex-Ante Value Functions and Optimal Capital Structure

In order to obtain the agency costs of debt due to managerial flexibility, we use the procedure suggested by Leland [1998] to investigate the hypothetical situation where managerial policy can be contracted for ex ante, i.e. the firm can choose policies so as to maximize firm value. In this case, we can directly apply the results of **Section 3** to write down the value functions of the firm. The optimal policies of the firm are to choose strategy 1, i.e. the high risk strategy for $p \leq p^*$ and the low risk strategy, i.e. strategy 2 for $p > p^*$ where $p_b \leq p^* < \frac{q}{l}$. As we have discussed in **Section 3**, there are only two possibilities :

Case 1: $p^* = p_b$

The firm's value function is given by

$$v(p) = Ep^{r_2^-} + \frac{lp}{d} + \frac{tq}{r};\ p \geq \frac{q}{l}$$

(30) $$= Cp^{r_2^+} + Dp^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_b < p \leq \frac{q}{l}$$

$$v(p_b) \quad = (1-a)\frac{lp_b}{d}$$

and the value of its debt in terms of the (endogenously determined) level p_e at which the value of equity falls to zero is given by

$$
(31)\qquad
\begin{aligned}
d(p) &= Jp^{r_2^-} + \frac{q}{r};\ p \ge p_e \\
&= v(p);\ p < p_e
\end{aligned}
$$

Case 2: $p_b < p^* < \frac{q}{l}$

The firm's value function is given by

$$
(32)\qquad
\begin{aligned}
v(p) &= Ep^{r_2^-} + \frac{lp}{d} + \frac{tq}{r};\ p \ge \frac{q}{l} \\
&= Cp^{r_2^+} + Dp^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p^* < p \le \frac{q}{l} \\
&= Ap^{r_1^+} + Bp^{r_1^-} + \frac{lp}{d} + \frac{t\,lp}{d};\ p_b < p \le p^* \\
v(p_b) &= (1-a)\frac{lp_b}{d}
\end{aligned}
$$

and the value of its debt if $p_e < p^*$ is given by

$$
(33a)\qquad
\begin{aligned}
d(p) &= Jp^{r_2^-} + \frac{q}{r};\ p \ge p^* \\
&= Hp^{r_1^+} + Ip^{r_1^-} + \frac{q}{r};\ p_e < p \le p^* \\
&= v(p);\ p \le p_e
\end{aligned}
$$

and if $p_e > p^*$ is given by

$$
(33b)\qquad
\begin{aligned}
d(p) &= Jp^{r_2^-} + \frac{q}{r};\ p \ge p_e \\
&= Hp^{r_2^+} + Ip^{r_2^-} + \frac{q}{r};\ p_e < p \le p^* \\
&= v(p);\ p \le p_e
\end{aligned}
$$

We obtain the optimal ex-ante leverage by maximizing the firm's value function as a function of the coupon rate q. The coefficients and the endogenous level p_e in all cases above are determined by the conditions that the value function and the value of debt be continuous for $p \geq p_b$ and differentiable for $p > p_b$. If p_e does not exist, i.e. the value of equity is positive for all $p \geq p_b$, then we set $p_e = p_b$ in the above expressions. If $v_{post}(p_0), v_{ante}(p_0)$ denote the optimal value functions (i.e. value functions at the optimal leverage) ex post and ex ante respectively where p_0 is the initial value of the state variable, then as in Leland [1998], we have

(34) Agency Costs $= v_{ante}(p_0) - v_{post}(p_0)$

In the next section, we present the results of several numerical simulations we have carried out that allow us to evaluate the significance of managerial flexibility as a determinant of capital structure and the valuation of corporate debt.

5.Numerical Simulations

In all the numerical simulations whose results we present, we have assumed that the exogenous liquidation level p_b is proportional to the coupon rate q. More precisely, we assume that

(35) $p_b = e\dfrac{q}{l}$ where $0 < e \leq 1$

Recall that q / l is the "illiquidity threshold", i.e. it is the value of the state variable below which the cash flows from the firm's operations are not sufficient to meet interest payments.

A. Agency Costs due to Managerial Flexibility

We have numerically implemented the results of the previous section to evaluate the optimal ex ante and ex post value functions of the firm in order to derive the agency costs from (34). **Tables 1** and **2** present our results for different choices of the parameter values of our model. As is clear from the tables, the agency costs of debt due to managerial flexibility are, in general, very significant in comparison with the tax advantages of debt. The difference in the leverage the firm may take on ex ante and ex post is also very significant for reasonable choices of parameter values. When contrasted with the results of Leland [1998], these results demonstrate that managerial flexibility is a far more significant determinant of the optimal capital structure of the firm than asset substitution driven by shareholders' interests.

We have also displayed the optimal risk-shifting points for the manager and the optimal risk-shifting points for the firm at the optimal leverage, the exogenous liquidation levels and the levels at which equity values fall to zero. Since the value of the unlevered assets of the firm and the value of the state variable are in one-one correspondence with each other, we describe these points in terms of the value of the unlevered assets at these points. The initial value of the unlevered assets is always assumed to be 100. We display the *credit spread* at the optimal ex post leverage that is the difference between the interest rate the firm pays and the interest rate it would pay if its strategy were risk free.

B. Variation of Agency Costs with the Drifts of the Strategies

It is also interesting to investigate how agency costs vary with the drifts of the available strategies. **Figure 1** displays the variation of agency costs with the drift of strategy 2 assuming that the drift of strategy 1 and the volatilities of both strategies are kept fixed. The values of the other parameters are the same as in the case of **Table 1.** This investigation also allows us to easily quantify the agency costs of debt due to *asymmetric information* between the manager and investors regarding the expected returns of the available strategies. If the investors only know the risks of the available strategies, but not their expected returns, then both equity and debt would be valued assuming the *worst-case* scenario. **Figure 1** shows that the agency costs when investors do not know the drift of strategy 2 would be just under 3.5%.

C. Variation of Optimal Ex Ante and Ex Post Leverage with Drifts

Figure 2 displays the variation of the optimal ex ante and ex post leverage of the firm with the drift of strategy 2 with the values of the other parameters being the same as in the case of **Figure 1**. The ex ante leverage does not vary with the drift of strategy 2 since the ex ante optimal policies of the firm do not depend on the drifts of the strategies as explained in **Section 2** since firm value maximization is under the risk neutral measure.

D. Variation of Agency Costs with the Volatilities of the Strategies

Figure 3 displays the variation of agency costs with the volatility of strategy 2 assuming that the volatility of strategy 1 and the drifts of both strategies are kept fixed. We notice that the agency costs go to zero as the volatilities of the strategies converge since the strategies are then indistinguishable from the standpoint of the firm so that the optimal ex ante and ex post value functions would converge.

E. Variation of Optimal Ex Ante and Ex Post Leverage with Volatility of Strategy 2

Figure 4 displays the variation of the optimal ex ante and ex post leverage of the firm with the volatility of strategy 2.

From **Figures 2** and **4**, we see that the optimal ex post leverage, i.e. the optimal leverage of the firm in the presence of managerial flexibility varies between 10% and 30% that corresponds quite well with average leverage levels observed in the market.

6. Conclusions

In this paper, we have studied the problem of optimal asset substitution in continuous time for the manager of a firm whose incentives need not correspond with those of shareholders. The manager, who is assumed to be risk-neutral, may dynamically switch between two strategies with different risks and expected returns and also bears significant personal costs due to the bankruptcy or liquidation of the firm.

We demonstrated that the manager's unique optimal policies are to choose the low risk (and low expected return strategy) whenever the unlevered asset value of the firm is below an endogenously derived threshold and the high risk (and high expected return strategy) whenever the unlevered asset value of the firm is above the threshold and presented precise necessary and sufficient conditions for the location of the risk-shifting threshold.

We then investigated the optimal policies for the firm that can hypothetically contract for managerial behavior and demonstrated that the optimal policies for the firm are to choose the high risk strategy whenever the unlevered asset value is below an endogenously derived threshold and the low risk strategy whenever the unlevered asset value is above the threshold. We demonstrated that the threshold is always below the illiquidity threshold, i.e. the firm will switch strategies when it is unprofitable net of contractual debt payments.

The fundamental dichotomy between the optimal behavior of the manager and that of the firm is the principle contributor to the agency costs of debt due to managerial flexibility. We demonstrated the significance of managerial flexibility as a determinant of optimal capital structure through several numerical simulations.

Our analytical characterizations of optimal managerial and firm behavior provide insights into the problem of designing optimal compensation contracts for the manager that could analyze managerial incentives with those of the firm and thus hopefully eliminate managerial flexibility.

Limitations and Extensions

In this paper, we have examined the situation where the firm issues perpetual debt. This is an idealization of reality that has been imposed for analytical tractability. Although this is a good approximation for a firm issuing long term debt, it is clearly not valid in the modeling of medium of short term debt. It would be interesting to examine the influence of managerial asset substitution on the capital structure of a firm issuing medium or short term debt. It is quite likely that the model would not be amenable to analytical results, but one could, in principle, adopt a numerical approach similar to that of Anderson and Sundaresan [1996] to investigate this problem. Such an analysis would contribute to the important goal of studying the *optimal debt structure* of a firm in the presence of managerial flexibility.

APPENDIX

Proof of Lemma 1

(1) We notice that

$$0 > m_i - b = \frac{1}{2}s_i^{\,2}(1)^2 + (m_i - \frac{1}{2}s_i^{\,2})(1) - b = \frac{1}{2}s_i^{\,2}(1-h_i^+)(1-h_i^-) \text{ for } i = 1,2$$

where the first inequality above follows from our hypothesis that $b > m_i$ and the last equality follows from the definitions of the roots of equations (10). It follows from the above that we must have $h_1^- < 1 < h_1^+$. Next, we note that

$$\frac{1}{2}s_i^{\,2}(\frac{b}{m_i} - h_i^+)(\frac{b}{m_i} - h_i^-) = \frac{1}{2}s_i^{\,2}(\frac{b}{m_i})^2 + (m_i - \frac{1}{2}s_i^{\,2})\frac{b}{m_i} - b = \frac{1}{2}s_i^{\,2}(\frac{b^2}{m_i^{\,2}} - \frac{b}{m_i}) > 0$$

since $\frac{b}{m_i} > 1$. It follows from the above that $h_i^+ < \frac{b}{m_i}$.

(2) We have

$$\frac{1}{2}s_1^{\,2}h_2^{+2} + (m_1 - \frac{1}{2}s_1^{\,2})h_2^+ - b = \frac{1}{2}s_1^{\,2}(h_2^{+2} - h_2^+) + m_1 h_2^+ - b > \frac{1}{2}s_2^{\,2}(h_2^{+2} - h_2^+) + m_2 h_2^+ - b = 0$$

since $m_1 > m_2, s_1 > s_2, h_2^+ > 1$. Therefore,

$$\frac{1}{2}s_1^{\,2}(h_2^+ - h_1^+)(h_2^+ - h_1^-) = \frac{1}{2}s_1^{\,2}h_2^{+2} + (m_1 - \frac{1}{2}s_1^{\,2})h_2^+ - b > 0$$

It follows that h_2^+ must be greater than h_1^+, h_1^-, i.e. $h_1^+ < h_2^+$. This completes the proof of the lemma. ♦

Proof of Proposition 2

The proof proceeds by the explicit construction of a function u satisfying the hypotheses of **Proposition 1** thereby implying that it is the value function. In the process, we shall also show that the optimal policy for the manager is to choose strategy 1 throughout. We distinguish two regions :

Region I : $p_b \le p < q/l$

Since our hypothesized optimal policy is to choose strategy 1, (5) implies that the value function u must satisfy

(A1) $\quad L^1(u) = -bu + m_1 p u_p + \frac{1}{2}s_1^{\,2}p^2 u_{pp} + f = 0$.

It is well known that any solution to (A1) has the general form

$$u = Ap^{h_1^+} + Bp^{h_1^-} + \frac{f}{b}$$

By the boundary condition on u at $p = p_b$, we have

(A2) $$Ap_b^{h_1^+} + Bp_b^{h_1^-} + \frac{f}{b} = 0$$

Therefore,

(A3) $$u_I = Ap^{h_1^+} + Bp^{h_1^-} + \frac{f}{b},\ u_I(p_b) = 0$$

where the subscript on u denotes that this is the value function in region I.

Region II : $p > q/l$
In this case , the value function must satisfy

(A4) $$L^1(u) + f + g(lp - q) = 0$$

The general solution to the above equation must be of the form

$$u = Cp^{h_1^+} + Dp^{h_1^-} + \frac{f}{b} + \frac{glp}{b - m_1} - \frac{gq}{b}$$

For very large values of p, the manager is almost certain to obtain cash flows at the rate $f + glp - gq$ so that we must have $C = 0$ in the equation above. Therefore,

(A5) $$u_{II} = Dp^{h_1^-} + \frac{f}{b} + \frac{glp}{b - m_1} - \frac{gq}{b}$$

We define our hypothesized value function u to be equal to u_I in region I and equal to u_{II} in region II. For the function u to be C^1, we match its value and its first derivative at the boundary $p = q/l$ thereby obtaining

(A6) $$A(q/l)^{h_1^+} = \frac{(1 - h_1^-)gq}{(b - m_1)(h_1^+ - h_1^-)} + \frac{h_1^- gq}{b(h_1^+ - h_1^-)} = \frac{(b - m_1 h_1^-)gq}{b(b - m_1)(h_1^+ - h_1^-)} > 0$$

since $h_1^- < 0$. Hence, we see that

(A7) $$A > 0$$

Since $u_I(p_b)=0$, we must have

$$A(p_b)^{h_1^+}+B(p_b)^{h_1^-}=-\frac{f}{b}$$

Since $A>0$ from (A7), we must have

(A8) $B<0$

By the result of **Proposition 1**, u is the value function if and only if

(A9)
$$\begin{aligned}&L^2(u_I)+f+g(lp-q)^+\leq 0\\ &L^2(u_{II})+f+g(lp-q)^+\leq 0\end{aligned}$$

We see that

(A10)

$$\begin{aligned}&L^2(u_I)+f+g(lp-q)^+=L^2(Ap^{h_1^+}+Bp^{h_1^-}+\frac{f}{b})+f+0\\ &=Ap^{h_1^+}(\frac{1}{2}s_2{}^2(h_1^+)^2+(m_2-\frac{1}{2}s_2{}^2)h_1^+-b)+Bp^{h_1^-}(\frac{1}{2}s_2{}^2(h_1^-)^2+(m_2-\frac{1}{2}s_2{}^2)h_1^--b)\\ &=Ap^{h_1^+}\frac{1}{2}s_2{}^2(h_1^+-h_2^+)(h_1^+-h_2^-)+Bp^{h_1^-}\frac{1}{2}s_2{}^2(h_1^--h_2^+)(h_1^--h_2^-)\end{aligned}$$

where the last equality follows from the fact that h_2^+,h_2^- are the roots of the second equation in (7). Since $h_1^-<h_2^-<h_1^+<h_2^+$ by hypothesis and $A>0,B<0$ from (A7) and (A8), we easily see that

(A11) $L^2(u_I)+f+g(lp-q)^+<0$

By construction, u is twice differentiable at $p=\frac{q}{l}$ since the value and first derivatives of u_I,u_{II} are equal at $p=\frac{q}{l}$ and

$$L^1(u_I)+f+g(lp-q)^+|_{p=q/l}=L^1(u_{II})+f+g(lp-q)^+|_{p=q/l}=0$$

by construction. Therefore, (A11) clearly implies that

(A12) $L^2(u_{II})+f+g(lp-q)^+|_{p=q/l}<0$

We now note from (A5) that

$$(A13)\quad \begin{aligned} &L^2(u_{II}) + f + g(lp-q)^+ = L^2(Dp^{h_1^-} + \frac{f-gq}{b} + \frac{glp}{(b-m_1)}) + f + g(lp-q)^+ \\ &= Dp^{h_1^-}\frac{1}{2}s_2^{\,2}(h_1^- - h_2^+)(h_1^- - h_2^-) + \frac{glp(m_2-m_1)}{(b-m_1)} \end{aligned}$$

where the second equality above is obtained as in (A10). Since $h_1^- < h_2^- < h_1^+ < h_2^+$ and $m_2 < m_1$, we see from the above that if $D \le 0$,

$$L^2(u_{II}) + f + g(lp-q)^+ < 0$$

On the other hand, if $D > 0$, the first term in the last expression in (A13) is positive. However, we note that in this case, $Dp^{h_1^-}\frac{1}{2}s_2^{\,2}(h_1^- - h_2^+)(h_1^- - h_2^-) + \frac{glp(m_2-m_1)}{(b-m_1)}$ is a *decreasing* function of p. Since $L^2(u_{II}) + f + g(lp-q)^+ |_{p=q/l} < 0$ from (A12), we conclude that

$$L^2(u_{II}) + f + g(lp-q)^+ < 0 \text{ for } p \ge q/l \text{ and therefore in region II.}$$

It follows that

$$(A14)\quad L^2(u_{II}) + f + g(lp-q)^+ < 0$$

From (A11) and (A14) we see that the function u satisfies the hypotheses of **Proposition 1** and is therefore the value function of the manager's optimization problem. Moreover, the fact that the inequalities (A11), (A14) are strict implies (from standard programming arguments[9]) that the policy of choosing strategy 1 throughout is the *unique* optimal policy for the manager. This completes the proof of the proposition. ♦

Proof of Proposition 3

We shall only prove the sufficiency of condition (10). The necessity follows directly from the results of **Propositions 4** and **5**. By (8) we have

$$(A15)\quad \begin{aligned} u_{p_b} &= C_{p_b} p^{h_1^+} + D_{p_b} p^{h_1^-} + \frac{f}{b};\ p_b \le p < \frac{q}{l} \\ &= E_{p_b} p^{h_1^-} + \frac{f-gq}{b} + \frac{glp}{b-m_1};\ p \ge \frac{q}{l} \\ u_{p_b}&(p_b) = 0 \end{aligned}$$

[9] These are available from the author upon request.

By the matching of the value and first derivative at $p = q/l$, we obtain as in (A6) in the proof of **Proposition 2** that

(A16) $$C_{p_b}(q/l)^{h_1^+} = \frac{(1-h_1^-)gq}{(b-m_1)(h_1^+-h_1^-)} + \frac{h_1^- gq}{b(h_1^+-h_1^-)} = \frac{(b-m_1h_1^-)gq}{b(b-m_1)(h_1^+-h_1^-)} > 0$$

By the condition $u_{p_b}(p_b) = 0$, we see that $D_{p_b} < 0$. Therefore,

(A17) $$C_{p_b} > 0, D_{p_b} < 0$$

We now note that for $p_b \le p < q/l$,

(A18)
$$\begin{aligned} &L^2(u_{p_b}) + f + g(lp-q)^+ = L^2(u_{p_b}) + f \\ &= C_{p_b} p^{h_1^+} \frac{1}{2} s_2^{\,2}(h_1^+-h_2^-)(h_1^+-h_2^+) + D_{p_b} p^{h_1^-} \frac{1}{2} s_2^{\,2}(h_1^- - h_2^-)(h_1^- - h_2^+) \end{aligned}$$

Since $h_2^- < h_1^- < h_1^+ < h_2^+$ by hypothesis, we see from (A17) that the first term in the last expression above is negative and the second term is positive. Therefore, the expression is a *decreasing function* of p for $p_b \le p \le q/l$. It therefore follows from hypothesis (10) of the proposition that

(A19) $$L^2(u_{p_b}) + f + g(lp-q)^+ < 0 \text{ for } p_b \le p \le q/l.$$

For $p \ge q/l$,

(A20)
$$\begin{aligned} &L^2(u_{p_b}) + f + g(lp-q)^+ = \\ &E_{p_b} p^{h_1^-} \frac{1}{2} s_2^{\,2}(h_1^- - h_2^-)(h_1^- - h_2^+) + \frac{glp(m_2-m_1)}{b-m_1} \end{aligned}$$

If $E_{p_b} \ge 0$, we easily see from the fact that $m_2 < m_1$ and $h_2^- < h_1^- < h_1^+ < h_2^+$ that both terms in the second expression above are negative. Therefore,

$$L^2(u_{p_b}) + f + g(lp-q)^+ \text{ for } p \ge q/l.$$

On the other hand, if $E_{p_b} < 0$, the first term in the last expression in (A20) is positive and the second term is negative. But we note that

$$E_{p_b} p^{h_1^-} \frac{1}{2} s_2^{\,2}(h_1^- - h_2^-)(h_1^- - h_2^+) + \frac{glp(m_2-m_1)}{b-m_1}$$

is a *decreasing function* of p. From (A19) and the fact that u_{p_b} is twice differentiable by construction, it follows that $L^2(u_{p_b}) + f + g(lp-q)^+ |_{p=q/l} < 0$. Therefore,

$$\text{(A21)} \quad L^2(u_{p_b}) + f + g(lp-q)^+ < 0 \text{ for } p \geq q/l\,.$$

Therefore, from (A19) and (A21),

$$L^2(u_{p_b}) + f + g(lp-q)^+ < 0 \text{ for } p > p_b\,.$$

By the result of **Proposition 1** and the fact that the inequality above is strict, it follows that choosing strategy 1 throughout is the unique optimal policy for the manager and that u_{p_b} is the optimal value function. This completes the proof. ♦

Proof of Lemma 2

Recall that u_r is the value of the policy of choosing strategy 2 for $p \leq r$ and strategy 1 for $p > r$. Therefore, by construction,

$$L^2(u_r) + f + g(lp-q)^+ = 0 \text{ for } p < r\,.$$

Hence,

$$\text{(A22)} \quad L^2(u_r) + f + g(lp-q)^+ |_{p=r-} = 0$$

If (11) holds , then we easily see by subtracting (A22) from (11) that

$$\text{(A23)} \quad \frac{1}{2} s_2^{\,2} [\frac{d^2}{dp^2} u_r |_{p=r+} - \frac{d^2}{dp^2} u_r |_{p=r-}] = 0$$

from which it follows that u_r is twice differentiable at r and therefore everywhere. Conversely, if u_r is twice differentiable at r, we easily see from (A22) that (11) must hold. Conditions (12) and (13) can be shown to be equivalent to (11) by exactly analogous arguments. This completes the proof. ♦

Proof of Proposition 4

We begin by noting that the function $L^2(u_r) + f + g(lp-q)^+ |_{p=r+}$ is a continuous function of r. It therefore follows from (14) and (15) that there exists p_* with $p_b < p_* \leq q/l$ such that

(A24) $L^2(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*+} = 0$

We shall show that p_* is the required *optimal switching point.* By the result of **Proposition 1**, we need to show that

(A25)
$$\begin{aligned} &L^1(u_{p_*}) + f + g(lp-q)^+ < 0 \ \text{ for } p_b < p < p_* \\ &L^2(u_{p_*}) + f + g(lp-q)^+ < 0 \ \text{ for } p > p_* \end{aligned}$$

By the result of **Lemma 2**, (A24) implies that u_{p_*} is twice differentiable at $p = p_*$ and that

(A26) $L^1(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*} = L^2(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*} = 0$

From the definition (8) of u_{p_*}, (A24), (A26) imply that

(A27)
$$\begin{aligned} &A_{p_*}\frac{1}{2}s_1{}^2(h_2^+ - h_1^+)(h_2^+ - h_1^-)p_*{}^{h_2^+} + B_{p_*}\frac{1}{2}s_1{}^2(h_2^- - h_1^+)(h_2^- - h_1^-)p_*{}^{h_2^-} = 0 \\ &C_{p_*}\frac{1}{2}s_2{}^2(h_1^+ - h_2^+)(h_1^+ - h_2^-)p_*{}^{h_1^+} + D_{p_*}\frac{1}{2}s_2{}^2(h_1^- - h_2^+)(h_1^- - h_2^-)p_*{}^{h_1^-} = 0 \end{aligned}$$

Since $h_2^- < h_1^- < 0 < h_1^+ < h_2^+$ by hypothesis and u_{p_*} must be an increasing function of p, the conditions (A27) imply that

(A28) $A_{p_*} > 0, B_{p_*} < 0, C_{p_*} > 0, D_{p_*} < 0$

We now see that

$$A_{p_*}\frac{1}{2}s_1{}^2(h_2^+ - h_1^+)(h_2^+ - h_1^-)p^{h_2^+} + B_{p_*}\frac{1}{2}s_1{}^2(h_2^- - h_1^+)(h_2^- - h_1^-)p^{h_2^-}$$

must be an *increasing function* of p and

$$C_{p_*}\frac{1}{2}s_2{}^2(h_1^+ - h_2^+)(h_1^+ - h_2^-)p^{h_1^+} + D_{p_*}\frac{1}{2}s_2{}^2(h_1^- - h_2^+)(h_1^- - h_2^-)p^{h_1^-}$$

must be a *decreasing function* of p. Since both the expressions above are equal to zero at $p = p_*$, we easily conclude that

(A29)
$$\begin{aligned} &L^1(u_{p_*}) + f + g(lp-q)^+ < 0 \ \text{ for } p_b < p < p_* \\ &L^2(u_{p_*}) + f + g(lp-q)^+ < 0 \ \text{ for } p_* < p < q/l \end{aligned}$$

We can use arguments identical to those used in the proof of **Proposition 3** to show that

$$\text{(A30)}\quad \begin{aligned} &L^2(u_{p_*}) + f + g(lp-q)^+ = \\ &E_{p_*} p^{h_1^-} \frac{1}{2} s_2^{\,2}(h_1^- - h_2^-)(h_1^- - h_2^+) + \frac{glp(m_2 - m_1)}{b - m_1} < 0 \text{ for } p \geq \frac{q}{l} \end{aligned}$$

(A29) and (A30) together imply that (A25) holds. Therefore, u_{p_*} is the optimal value function by the result of **Proposition 1** and the policy of switching from strategy 2 to strategy 1 at p_* is optimal. This completes the proof. ♦

<u>Proof of Proposition 5</u>

We begin by noting that by the definitions (8) and (9) of $u_{q/l}$,

$$\text{(A31)}\quad L^2(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l-} = 0$$

and

$$\text{(A32)}\quad L^1(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l+} = 0$$

Subtracting (A31) from (16), we see that

$$\text{(A33)}\quad \frac{1}{2} s_2^{\,2} \left(\frac{q}{l}\right)^2 \left[\frac{d^2}{dp^2}(u_{q/l})|_{p=q/l+} - \frac{d^2}{dp^2}(u_{q/l})|_{p=q/l-}\right] > 0$$

This implies that

$$\text{(A34)}\quad \frac{1}{2} s_1^{\,2} \left(\frac{q}{l}\right)^2 \left[\frac{d^2}{dp^2}(u_{q/l})|_{p=q/l+} - \frac{d^2}{dp^2}(u_{q/l})|_{p=q/l-}\right] > 0$$

(A32) and (A34) clearly imply that

$$\text{(A35)}\quad L^1(u_{q/l}) + f + g(lp-q)^+ |_{p=q/l-} < 0$$

We need to show the existence of $p_* > q/l$ such that

$$\text{(A36)}\quad L^1(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*-} = 0$$

which would imply by the result of **Lemma 2**, that u_{p_*} is twice differentiable at p_* and that

(A37) $L^1(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*} = L^2(u_{p_*}) + f + g(lp-q)^+ |_{p=p_*} = 0$

We prove this by first showing that

(A38) $\lim_{r\to\infty} L^1(u_r) + f + g(lp-q)^+ |_{p=r-} = \infty$

As $r \to \infty$, the value function u_r clearly approaches the value function u_∞ of the policy of choosing strategy 2 throughout. It is easy to see that the functional form of u_∞ is

$$
\begin{aligned}
& u_\infty(p) = A_\infty p^{h_2^+} + B_\infty p^{h_2^-} + \frac{f}{b}; p_b < p < \frac{q}{l} \\
\text{(A39)} \qquad & \qquad = C_\infty p^{h_2^-} + \frac{f-gq}{b} + \frac{glp}{b-m_2}; p > \frac{q}{l} \\
& u_\infty(p_b) = 0
\end{aligned}
$$

We now note that

$$
\begin{aligned}
& \lim_{p\to\infty} L^1(u_\infty) + f + g(lp-q)^+ = \\
\text{(A40)} \quad & \lim_{p\to\infty} [C_\infty p^{h_2^-}(\frac{1}{2}s_1{}^2(h_2^-)^2 + (m_2 - \frac{1}{2}s_1{}^2)h_2^- - b) + \frac{glp(m_1-m_2)}{b-m_2}] = \infty
\end{aligned}
$$

as $h_2^- < 0$. The result (A40) implies that (A38) holds[10]. It now easily follows by continuity that (A36) holds and therefore (A37) holds by the result of **Lemma 2**. We shall now show that p_* is the required "optimal switching point" where u_{p_*} is defined by (9). By the result of **Proposition 1**, we clearly need to show that

$$
\begin{aligned}
& L^1(u_{p_*}) + f + g(lp-q)^+ < 0 \text{ for } p_b < p < p_* \\
\text{(A41)} \quad & L^2(u_{p_*}) + f + g(lp-q)^+ < 0 \text{ for } p > p_*
\end{aligned}
$$

For $p > p_*$,

(A42) $L^2(u_{p_*}) + f + g(lp-q)^+ = E_{p_*}\frac{1}{2}s_2{}^2 p^{h_1^-}(h_1^- - h_2^-)(h_1^- - h_2^+) + \dfrac{glp(m_2-m_1)}{b-m_1}$

Since $m_1 > m_2$ and $h_2^- < h_1^- < 0 < h_1^+ < h_2^+$ by hypothesis, (A37) holds only if

(A43) $E_{p_*} < 0$

[10] Strictly this needs to be shown rigorously, but the arguments are quite straightforward and are available from the author upon request.

It now follows that the expression $E_{p_*}\frac{1}{2}s_2^{\ 2}p^{h_1^-}(h_1^- - h_2^-)(h_1^- - h_2^+) + \frac{g\,lp(m_2 - m_1)}{b - m_1}$ is a *decreasing function* of p . Therefore, (A37) implies that

(A44) $L^2(u_{p_*}) + f + g(\,lp - q)^+ < 0$ for $p > p_*$

Using the fact that u_{p_*} is twice differentiable at $p = p_*$, we can show that (A43) implies that the coefficients C_{p_*}, D_{p_*} in the definition (9) of u_{p_*} satisfy

(A45) $C_{p_*} > 0, D_{p_*} < 0$

Now, using the fact that u_{p_*} is twice differentiable at $p = q/\,l$, we can show that the coefficients A_{p_*}, B_{p_*} in the definition (9) of u_{p_*} satisfy

(A45) $A_{p_*} > 0, B_{p_*} < 0$

We can use arguments identical to those we have been using so far to show that $L^1(u_{p_*}) + f + g(\,lp - q)^+$ is an *increasing function* of p for $p < p_*$. (A36) and (A37) now clearly imply that

(A46) $L^1(u_{p_*}) + f + g(\,lp - q)^+ < 0$ for $p < p_*$

We have therefore shown that (A41) holds and hence, the hypotheses of **Proposition 1** are satisfied. Therefore, u_{p_*} is the optimal value function and the policy of switching from strategy 2 to strategy 1 at p_* is optimal. This completes the proof. ♦

Proof of Proposition 6

Suppose $p_n \geq q/\,l$ and the firm chooses strategy 1 for $p \geq p_n$ and switches from strategy 2 to strategy 1 at $p = p_n$. If u is the value function of the policy, we must have

$$L^1(u) + C(p) = 0 \text{ for } p \geq p_n$$
$$L^2(u) + C(p) = 0 \text{ for } p_{n-1} \leq p \leq p_n$$

where $L^1(.), L^2(.)$ are defined in (21) and $C(.)$ is defined in (17). Since $p_n \geq q/\,l$ by assumption, we have from (17)

$$L^1(u) + \,lp + tq = 0 \text{ for } p \geq p_n$$

The above equation has the general solution

(A47) $u = Ap^{r_1^-} + Bp^{r_1^+} + \frac{lp}{d} + \frac{tq}{r}$

As $p \to \infty$, clearly $u(p) \to \frac{lp}{d} + \frac{tq}{r}$ since the firm is profitable with probability very close to one. Therefore, $B = 0$ in (A47). Further, $\frac{lp}{d} + \frac{tq}{r}$ is the value of cash flows associated with the firm in the hypothetical situation when it enjoys tax advantages of debt whether it is profitable or not and faces no bankruptcy costs. In reality, it faces the possibility of bankruptcy and its tax shield is only $t\,lp \le tq$ for $p \le q/l$. Therefore, in (A47), we must have $A < 0$. In other words, the value function of any policy is always strictly less than $\frac{lp}{d} + \frac{tq}{r}$.

By the results of **Proposition 1** and **Lemma 2**, for the policy to be optimal, u must be twice differentiable at p_n. Since

$$L^2(u) + lp + tq\,|_{p=p_n-} = 0; L^1(u) + lp + tq\,|_{p=p_n+} = 0$$

we must have

$$L^1(u) + lp + tq = L^2(u) + lp + tq = 0 \text{ at } p = p_n$$

But

$$L^2(u) + lp + tq\,|_{p=p_n} = A\frac{1}{2}s_2^{\,2}(r_1^- - r_2^-)(r_1^- - r_2^+)p_n^{\,r_1^-} \neq 0$$

since $A < 0$ and $r_1^- \neq r_2^-, r_1^- \neq r_2^+$.

This contradiction shows that we must have $p_n < q/l$ for the hypothesized policy to be optimal. We can use exactly similar arguments to arrive at a contradiction if the firm switches from policy 1 to policy 2 at $p = p_n$. Therefore, a necessary condition for the hypothesized policy to be optimal is that $p_n < q/l$. This completes the proof. ♦

Proof of Proposition 7

Suppose to the contrary that the firm chooses strategy 1 for $p \ge q/l$. Then, by arguments identical to those in the proof of the previous proposition, we must have

(A48) $\quad Ap^{r_1^-} + \dfrac{lp}{d} + \dfrac{tq}{r}$ for $p \geq q/l$ where $A < 0$

By the result of **Proposition 1**, a necessary condition for optimality of the policy is that

$L^2(u) + lp + tq \leq 0$ for $p \geq q/l$

From (A48),

$$L^2(u) + lp + tq = A\frac{1}{2}s_2{}^2(r_1^- - r_2^-)(r_1^- - r_2^+)p^{r_1^-} \text{ for } p \geq q/l$$

By (23), we have $r_2^- < r_1^- < r_2^+$. Since $A < 0$, we see that

$L^2(u) + lp + tq > 0$ for $p \geq q/l$

Hence, the policy cannot be optimal and this completes the proof.

<u>Proof of Proposition 8</u>

We shall use arguments similar to those used in the proofs of **Propositions 3, 4** and **5** in the previous section. We consider the set of policies defined by the parameter r with $p_b \leq r \leq q/l$ where the firm chooses the high volatility strategy for $p \leq r$ and the low volatility strategy for $p \geq r$. We denote the corresponding value function by u_r. We shall then show that the required "optimal switching level" p^* is given by

$$\text{(A49)} \quad \begin{aligned} p^* &= r^* \text{ if } L^1(u_{r^*}) + lp + t\,lp\,|_{p=r^*} = L^2(u_{r^*}) + lp + t\,lp\,|_{p=r^*} = 0 \text{ and } r^* > p_b \\ &= p_b \text{ if such an } r^* \text{ does not exist} \end{aligned}$$

Therefore, the optimal policy for the firm is to choose the low volatility strategy throughout if the first condition in (A49) does not hold.

Step 1 : Derivation of u_r for fixed r such that $p_b \leq r \leq q/l$.
As in (8), we can express the functional form of the value function u_r as follows:

$$\begin{aligned} u_r(p) &= A_r p^{r_1^+} + B_r p^{r_1^-} + \frac{lp}{d} + \frac{t\,lp}{d}; p_b < p \leq r \\ &= C_r p^{r_2^+} + D_r p^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d}; r < p \leq \frac{q}{l} \end{aligned}$$

(A50)
$$= E_r p^{r_2^-} + \frac{lp}{d} + \frac{tq}{r}; p > \frac{q}{l}$$
$$u_r(p_b) = (1-a)\frac{lp_b}{d}$$

By the same arguments as the ones following (A47) in the proof of **Proposition 6**, we can conclude that $E_r < 0$ above. Therefore, for $p > \frac{q}{l}$,

(A51) $L^1(u_r) + lp + tq = \frac{1}{2} s_1{}^2 E_r (r_2^- - r_1^-)(r_2^- - r_1^+) p^{r_2^-} < 0$

In particular,

(A52) $L^1(u_r) + lp + tq|_{p=q/l+} < 0$ for *all* values of r such that $p_b \leq r \leq q/l$.

By the smooth matching of the value function at $p = q/l$, it therefore follows that

(A53) $L^1(u_r) + lp + t\,lp|_{p=q/l-} < 0$

We now have two possible scenarios.

Case 1 :

Suppose

(A54) $L^1(u_{p_b}) + lp + t\,lp|_{p=p_b+} \leq 0$ [11]

We shall show that choosing strategy 2 throughout is optimal for the firm. By the result of **Proposition 1**, we only need to show that

(A55) $L^1(u_{p_b}) + C(p) \leq 0$ for all $p \geq p_b$.

For $p \geq q/l$,the above follows from (A51).

For $p_b \leq p \leq q/l$, we have from (A50)

[11] **Note :** u_{p_b} is the value function of the policy that chooses strategy 2 for all values above bankruptcy.

(A56)
$$L^1(u_{p_b}) + C(p) = L^1(C_{p_b} p^{r_2^+} + D_{p_b} p^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d}) + dp + t\,lp$$
$$= \frac{1}{2} s_1^{\,2} [C_{p_b}(r_2^+ - r_1^+)(r_2^+ - r_1^-) p^{r_2^+} + D_{p_b}(r_2^- - r_1^+)(r_2^- - r_1^-) p^{r_2^-}]$$

By the smooth matching of value functions at $p = q/l$ and some algebra, we can show that

(A57) $$C_{p_b} = \frac{tq}{(r_2^+ - r_2^-)} [\frac{(r-d)\,r_2^- - r}{rd}] (\frac{q}{l})^{-r_2^+} < 0$$

since $r > d,\ r_2^- < 0$.

If $D_{p_b} \le 0$, then from (A56) we easily see that $L^1(u_{p_b}) + C(p) < 0$ as required since $r_2^- < r_1^- < r_1^+ < r_2^+$ from (23). Therefore (A55) is true and the policy of choosing strategy 2 throughout is optimal.

On the other hand, if $D_{p_b} > 0$, then we see that the last term in (A56) is a *decreasing function* of p. Therefore, (A54) clearly implies that (A55) is true and the policy of choosing strategy 2 throughout is optimal. Therefore, it is optimal for the firm to choose strategy 2 throughout if (A54) holds.

Case 2:

Suppose

$$L^1(u_{p_b}) + lp + t\,lp|_{p=p_b+} > 0$$

From (A52) and an application of the intermediate value theorem and continuity, we see that there exists a value r^* with $p_b \le r^* < q/l$ such that

(A58) $$L^1(u_{r^*}) + lp + t\,lp|_{p=r^*+} = 0$$ [12]

We shall show that r^* is the required "optimal switching level" p^*.By the result of **Lemma 2**, (A58) implies that u_{r^*} is twice continuously differentiable at $p = r^*$ and that

(A59) $$L^1(u_{r^*}) + lp + t\,lp|_{p=r^*} = L^2(u_{r^*}) + lp + t\,lp|_{p=r^*} = 0$$

---

[12] **Note :** In the above, u_{r^*} is the value of the policy of choosing strategy 1 for $p \le r^*$ and strategy 2 for $p > r^*$.

In order to show the optimality of this policy, and since (A51) holds, it remains to show by the result of **Proposition 1** that

(A60)
$$L^1(u_{r^*}) + lp + t\,lp < 0 \text{ for } r^* < p \le q/l$$
$$L^2(u_{r^*}) + lp + t\,lp < 0 \text{ for } p_b \le p < r^*$$

We see that for $r^* \le p \le q/l$

(A61)
$$L^1(u_{r^*}) + C(p) = L^1(C_{r^*} p^{r_2^+} + D_{r^*} p^{r_2^-} + \frac{lp}{d} + \frac{t\,lp}{d}) + lp + t\,lp$$
$$= \frac{1}{2} s_1{}^2 [C_{r^*} p^{r_2^+} (r_2^+ - r_1^+)(r_2^+ - r_1^-) + D_{r^*} p^{r_2^-} (r_2^- - r_1^+)(r_2^- - r_1^-)$$

By the smooth matching of the value functions at $p = q/l$, we have

(A62) $$C_{r^*} = \frac{tq}{(r_2^+ - r_2^-)}[\frac{(r-d)r_2^- - r}{rd}](\frac{q}{l})^{-r_2^+} < 0$$

From (A62) and the fact that $r_2^- < r_1^- < r_1^+ < r_2^+$ (23), we see that the first term in the last expression in (A61) is negative for all p. Therefore, (A59) can only hold if

(A63) $$D_{r^*} > 0.$$

In this case, we see from (A61) and the fact that $r_2^- < r_1^- < r_1^+ < r_2^+$ (23) that $L^1(u_{r^*}) + C(p)$ is a *decreasing function* of p for $r^* \le p \le q/l$. Therefore, (A59) implies that

$$L^1(u_{r^*}) + C(p) = L^1(u_{r^*}) + lp + t\,lp < 0 \text{ for } r^* < p \le q/l$$

It only remains to show that

$$L^2(u_{r^*}) + lp + t\,lp < 0 \text{ for } p_b \le p < r^*$$

By (A60), we have $L^2(u_{r^*}) + lp + t\,lp \,|_{p=r^*} = 0$.

By (A50), for $p_b \le p < r^*$,

(A64)
$$L^2(u_{r^*}) + lp + t\,lp =$$
$$\frac{1}{2} s_2{}^2 [A_{r^*}(r_1^+ - r_2^+)(r_1^+ - r_2^-) p^{r_1^+} + B_{r^*}(r_1^- - r_2^+)(r_1^- - r_2^-) p^{r_1^-}]$$

From the smooth matching of value functions at $p = r^*$ and some tedious algebra, we obtain

(A65) $$A_{r^*} = [\frac{r_2^+ - r_1^-}{r_1^+ - r_1^-}]C_{r^*}(r^*)^{r_2^+ - r_1^+} + [\frac{r_2^- - r_1^-}{r_1^+ - r_1^-}]D_{r^*}(r^*)^{r_2^- - r_1^+}$$

Since $r_2^- < r_1^- < r_1^+ < r_2^+$ from (23) and $C_{r^*} < 0, D_{r^*} > 0$ from (A62) and (A63), we easily see that $A_{r^*} < 0$ from the above expression.

From (A64) and the fact that $r_2^- < r_1^- < r_1^+ < r_2^+$, we now see that since $L^2(u_{r^*}) + lp + t\,lp < 0|_{p=r^*} = 0$, we must have $B_{r^*} > 0$. We now see from (A64) that $L^2(u_{r^*}) + lp + t\,lp$ is a *decreasing* function of p for $p_b \le p \le r^*$. Therefore, since $L^2(u_{r^*}) + lp + t\,lp < 0|_{p=r^*} = 0$, we see that $L^2(u_{r^*}) + lp + t\,lp < 0$ for $p_b \le p < r^*$.

Therefore, by the result of **Proposition 2**, the policy of choosing strategy 1 for $p \le r^*$ and strategy 2 for $p > r^*$ is optimal. This completes the proof. ♦

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TABLE 1 : AGENCY COSTS AND OPTIMAL LEVERAGE

Drift of Strategy 1	0.4	**Lambda**	0.12
Volatility of Strategy 1	0.6	**Delta**	0.04
Drift of Strategy 2	0.3	**Tax Rate**	0.2
Volatility of Strategy 2	0.3	**Bankruptcy Cost**	0.1
Risk-Free Rate	0.05	**Bankruptcy Parameter**	0.5
Manager Opportunity Cost	0.5	**Initial Unlevered Asset Value**	100
EX POST		**EX ANTE**	
Value Function	100.9076	**Value Function**	104.2154
Value of Debt	21.95922	**Value of Debt**	32.714435
Leverage	21.7617%	**Leverage**	31.3912%
Coupon Rate	1.2	**Coupon Rate**	2.5
Optimal Switching Point	22.2	**Optimal Switching Point**	31.25
Zero Equity Value Point	15	**Zero Equity Value Point**	31.25
Exogenous Bankruptcy Point	15	**Exogenous Bankruptcy Point**	31.25
Agency Costs	3.308%		
Tax Advantages of Debt	4.215%		

TABLE 2 : AGENCY COSTS AND OPTIMAL LEVERAGE

Drift of Strategy 1	0.4	**Lambda**	0.12
Volatility of Strategy 1	0.4	**Delta**	0.04
Drift of Strategy 2	0.3	**Tax Rate**	0.2
Volatility of Strategy 2	0.2	**Bankruptcy Cost**	0.1
Risk-Free Rate	0.05	**Bankruptcy Parameter**	0.25
Manager Opportunity Cost	0.5	**Initial Unlevered Asset Value**	100
EX POST		**EX ANTE**	
Value Function	105.1631	**Value Function**	111.7093
Value of Debt	51.77586	**Value of Debt**	87.26455
Leverage	49.2339%	**Leverage**	78.1175%
Coupon Rate	4.1	**Coupon Rate**	5.5
Optimal Switching Point	32.8	**Optimal Switching Point**	34.37499
Zero Equity Value Point	28.7	**Zero Equity Value Point**	53.625
Exogenous Bankruptcy Point	25.625	**Exogenous Bankruptcy Point**	34.37499
Agency Costs	6.546%		
Tax Advantages of Debt	11.709%		
Credit Spread	1.51		

Fig. 1: Variation of Agency Cost wtih Drift of Strategy 2

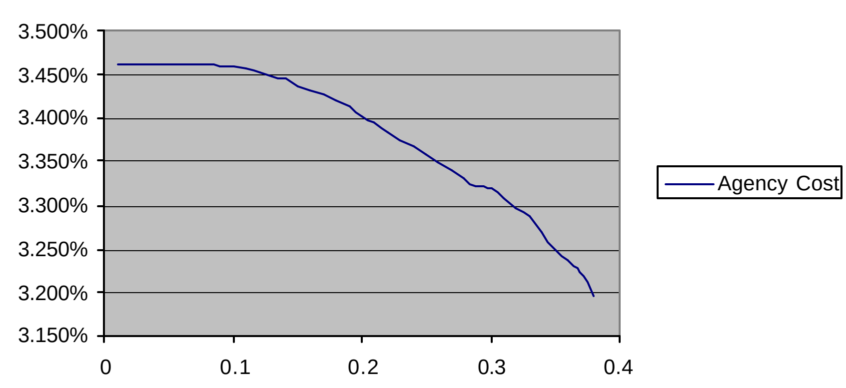
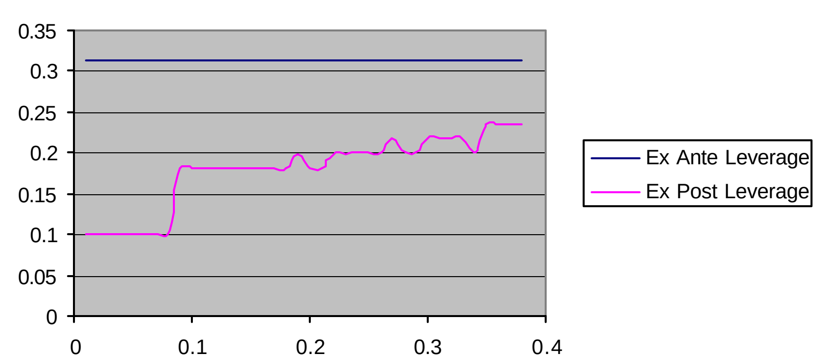


Fig. 2: Variation of Optimal Ex Ante and Ex Post Leverages with Drift of Strategy 2



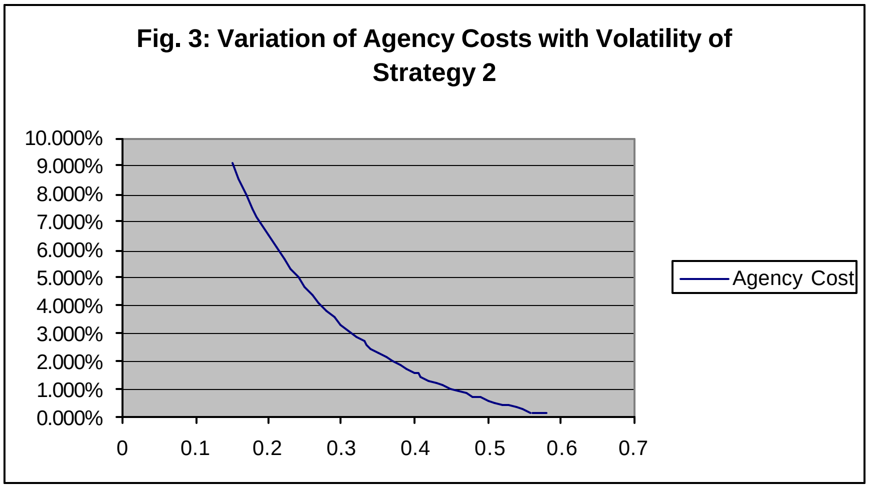


Fig. 3: Variation of Agency Costs with Volatility of Strategy 2
10.000%
9.000%
8.000%
7.000%
6.000%
5.000%
4.000%
3.000%
2.000%
1.000%
0.000%
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
Agency Cost

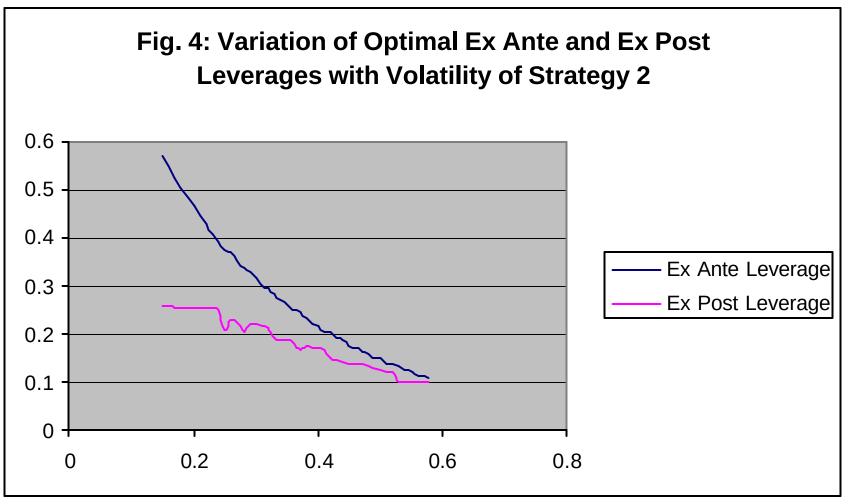


Fig. 4: Variation of Optimal Ex Ante and Ex Post Leverages with Volatility of Strategy 2
0.6
0.5
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
Ex Ante Leverage
Ex Post Leverage