A Stochastic Model of Asset Life

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A Stochastic Model of Asset Life

Abstract

In this paper we propose a stochastic model of asset life under uncertainty when there is a resale option. We consider asset activity life, asset economic life and asset physical life as separate concepts and model asset activity life as the expectation of the first passage time for geometric Brownian motion to a certain resale boundary. We also establish asset economic life as the sum of some first passage time distributions and certain Poisson waiting time distributions. Three convergence results to normal distribution are established. These results throw new lights on the market for used assets.

Key Words: Asset sale, real options, first passage time

JEL Codes: G31, G33

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1. INTRODUCTION

In a seminal model of vintage capital, Malcomson [1975] proposed studied asset life and proves the existence of an upper bound and a lower bound for capital asset. Subsequently, Hilton [1991] proves that the upper and lower bounds in Malcomson converge to a limit. Thus a unique asset life is established under the assumptions of certainty and no resale opportunity.

These are strong assumptions and clearly contradict the existence of markets for considerable number of durable assets, markets that facilitate asset exchanges among interested parties. Quite often, the exchange of used asset is more extensive then the exchange of new assets. In the United States, 88,000 used machine tools were sold while only 40,000 new machine ones were sold in 1960.[1] The ratio of used truck purchases to new truck purchases was 1.2 for panel trucks, 1.3 for multi-top and walk-in trucks, and 1.01 for truck tractors in US in 1977.[2] In a comprehensive paper on asset sale in US, Maskovic and Philips [2001] reveal that over 3 percent of plants and equipment in the US change ownership annually through mergers and acquisitions. No doubt similar data may be found for other assets in other countries. Maskovic and Philips [2001] show that these transactions in used corporate assets have benefited both sellers and buyers and significantly improved resource allocation. This finding is consistent with an early observation of Sen [1962] in the context of international trade in used machines. Sen [1962] shows that exchange in used durable assets may be

[1] Waterson (1964).
[2] Bond (1983).

welfare-improving. Trade in used durable assets has attracted some attention in economics and finance literature. The most celebrated paper of all is perhaps the one on used car market by Akerlof [1970] who was awarded Nobel Price in Economics for his work on such a market. Also in this tradition include the paper by Bond [1983]. The models of Rust [1985a, 1985b] focus on equilibrium characterisation of used asset market.

In this paper we relax both assumptions in Malcomson and Hilton by introducing resale opportunity and uncertainty. Specifically, we allow returns from the employment of a durable asset to fluctuate according to some stochastic processes while allowing an outside resale opportunity. Thus, if the return from using the asset proves disappointing, the asset owner could sell it to another owner who might profitably employ the asset for another use. We demonstrate that once the resale option is allowed, the problem of asset life changes rather drastically. Instead of a unique life, as in Malcomson [1975] and Hilton [1991], a durable asset now acquires a series of lives which we call activity life, each of which pertains to one particular economic activity. The sum of its activity lives makes up the economic life of the asset. As a result, the economic life of a durable asset is a random variable which is in turn a series of random variables.

We are not the first to consider real asset markets from option pricing perspective. Grenadier [1996] and Williams [1995, 1998] have applied the option pricing approach to study real asset market equilibrium and real estate development. They also incorporate informational and agency-theoretical factors. Unlike these authors, we "track down" the change of ownership for individual asset and model its economic life and its components: activity lives. The different perspectives adopted in this paper and that of others all throw lights on different aspects of the market for durable assets. We extend the insights from Brennan and Schwartz (1985), McDonald and Siegel [1986] and Myers and Majd [1990] in that we explicitly model resale opportunity, or abandonment option. In addition, we employ the insights from stochastic processes, as applied to option pricing problems by Black and Cox [1976], Ingersoll [1977], McDonald and Siegel [1986], Dixit [1993], Yaksick [1996] and Grenadier [1996], Bunch and Johnson [2000], Shackleton and Wojakowski [2001], Zhou [2001) and Song [2001].[3] This literature emphasizes certain specific option exercise boundaries and attempt to answer the question of if/when an option is expected to be exercised. By making certain specific assumptions, we obtain the following results. First, we show that asset activity life is the expectation of the first passage time to its resale boundary. Second, we show that an asset's economic life is the sum of first passage time distributions, which, under certain technical conditions, converge to the normal distribution. Thus, instead of a unique deterministic life, an asset's economic life become stochastic when there is a resale option. Our approach has the advantage of enabling us to "track down" the movement of a particular asset. The framework is flexible enough to incorporate realistic features of exchanges in used assets.

Our plan is as follows. In section two, we introduce a stochastic model of asset activity life. We show that an asset's activity life is characterised by the expectation of certain first passage time to a resale boundary. In section three, we demonstrate that the economic life is a random variable which in turn is the sum of the first

[3] Song (2002) contains extensive review of this literature.

passage time distributions. In particular we show that under certain conditions, the economic life distribution converges to a normal distribution whose expectation is the economic life of the asset. In section four we extend the analysis to account for market frictions by introducing Poisson waiting times for transits between different asset activity lives. We show that economic life is now the sum of alternate inverse lognormal and Poisson distributions. Necessary sufficient conditions for finite economic life are provided. In section five, more realism is built into the model by considering interdependence between asset activity lives. Potential applications and extensions of our model are discussed in section six.

2. ASSET ACTIVITY LIFE

Following Myers and Majd [1990], we differentiate three lives for a durable asset: project life, economic life and physical life. Project life corresponds to activity life used in this paper and denotes the period in which the asset is employed for one particular economic activity. Economic life is the time span in which the asset is used for all economic activities. Physical life is a measure denoting the asset in its physical form. Some assets have infinite physical life such as land in that they do not change from one physical form to another. In set theory, we say that project life is a subset of economic life, which in turn is a subset of physical life. More importantly, all three lives are random variables because economic and technical conditions change and new opportunities may arise. An asset may not profitably be used for one activity but it may be employed profitably for another activity. It may not be profitable for one owner but maybe so for another owner. But this requires the change of ownership through the market for used assets. In fact the market for used assets account for a substantial portion of the total market transactions. Sen [1962] and Bond [1983] argue that international exchange of used capital goods account for a substantial volume of international trade. In our model, the market serves to facilitate the transfer of assets from one use to another so that once one activity life terminates another begins and the economic life consists of a series of activity lives.

These exchange opportunities have been studied extensively in the finance literature. Under uncertainty these will exist optimal timing for exercising these options. In this paper we focus on one particular option, the option to sell. Song and Gao [2000] employ the analogy between the decisions to sell and exercising an American put option. It turns out that during an asset's economic life there may be many times that the resale option is exercised by different owners.

Following Song and Gao [2000], let an asset generates a random stream of net present value in accordance with geometric Brownian motion process:

$$\frac{dS}{S} = \mu dt + \sigma dz \qquad (1)$$

S denotes net present value from using the asset, μ is the expected growth rate of the return, σ is its standard deviation and dz is a standard Wiener process. Second, a complete market for the second hand asset exists and the asset can be sold at a constant price denoted as R. This assumption is made for clarity of exposition and may be easily relaxed as in Song and Gao [2001]. Third, let the asset have infinite durability so that wear and tear does not complicate the analysis. Under these and other assumptions, it is possible to obtain an optimal rule for asset sale using option pricing methodology. This optimal exercise rule, first appeared in Samuelson [1965], McKean [1965] and Merton [1973] and applied to asset sale and scrapping by Song and Gao [2000], is of a simple expression:

$$S^* = \frac{2r}{2r + \sigma^2} R \tag{2}$$

This resale rule states that there exists a critical level of S below which the asset should be sold in the second hand market. It is clear that this critical level depends only on three parameters of the model: resale price, riskless rate of interest and the variance rate of the returns from using the asset. Thus this is a constant boundary for the geometric Brownian motion process specified in (1). One insight of this paper is that asset activity life may be approximated by the expectation of the first passage time to this endogenous boundary (2). Specifically, given the evolution of the state variable in (1) and the critical boundary in (2), the problem of determining when S will reach the level derived in (2) boils down to the first passage time for geometric Brownian motion to a fixed level.

Song [2001], Rhys and Song [2001], Shackleton and Wojachowski [2001] shown that the first passage time distribution for geometric Brownian motion to a horizontal boundary is inverse lognormal distribution. This is in contrast to the well known inverse Gaussian distribution for absolute Brownian motion with positive drift first studied by Erwin Schrodinger and Smoluchowski in 1918, Wald and Tweedie in the 1940s, apparently all independent of each other. Johnson and Kotz [1977], Chhikara and Folks [1989] and Seshadri [1998] contain comprehensive treatments of the inverse Gaussian distribution.[4] One cannot apply the inverse Gaussian distribution to most problems of interest in economics and finance. To use these results, care must be taken to avoid confusions. Following Shackleton and Wojachowski [2001], Rhys and Song [2001] and Song (2001), it is straightforward to establish the following theorem.

Theorem 1

If Under uncertainty with resale option, asset activity life is the first passage time distribution for geometric Brownian motion to the optimal resale boundary with the following expectation and variance rate.

$$E(\tau) = \frac{Ln\frac{S^*}{S_0}}{\mu - \frac{1}{2}\sigma^2} \tag{3}$$

$$Var(\tau) = \frac{Ln\frac{S^*}{S_0}}{\left(\mu - \frac{1}{2}\sigma^2\right)^3 \sigma} \tag{4}$$

It is useful to note that the expectation is well defined if certain conditions are met. Specifically, the expectation has positive value if the nominator and the denominator have the same sign. The expectation approaches infinity when the drift equals one half of the variance rate for the original process. Indeed, nothing prevents the denominator from being either sign although we may be certain about the sign for the

[4] It is worth pointing out that Bachelier (1990) derived the first passage time distribution for absolute Brownian motion with zero drift with the celebrated known result of an infinite expectation.

numerator. Note also that if we were to work with absolute Brownian motion, we would have the expected value of first passage time to a fixed level as follows[5]:

$$E(\tau) = \frac{S^* - S_0}{\mu} \tag{5}$$

(5) is very intuitive and expresses the fact that if we know the distance a particle has to travel with certain speed, we obtain the time needed by dividing distance by its speed. For geometric Brownian motion, the correct result is (3) this is because we have transform (1) into log normal form by Ito's lemma as follows:

$$\frac{d[\log S]}{\log S} = (\mu - \frac{1}{2}\sigma^2)dt + \sigma dz \tag{6}$$

Some comparative static results on both moments of the first passage time are collected as follows. They show the effects of various factors on expected activity life and the variability of activity life.

$$\frac{\partial E(\tau)}{\partial \mu} < 0; \frac{\partial E(\tau)}{\partial \sigma^2} < 0 \forall \mu > 0$$

$$\frac{\partial E(\tau)}{\partial S^*} > 0; \frac{\partial E(\tau)}{\partial S_0} < 0$$

$$\frac{\partial Var(\tau)}{\partial S^*} > 0; \frac{\partial Var(\tau)}{\partial S_0} < 0$$

$$\frac{\partial Var(\tau)}{\partial \mu} < 0, \frac{\partial Var(\tau)}{\partial \sigma} < 0$$

[5] This result was also obtained by Yaksick (1996) using martingale method.

These results may be used to test the model predictions. In particular, although a higher variance rate of the driving process will lower the resale boundary, through Ito's lemma, it may increase or decrease the drift rate for the lognormal process. Thus the net effects of variance rate on expected activity life are ambiguous. However, if specific assumption is made on the sign and magnitude of the drift rate for (1), then we may make precise prediction on the sign of uncertainty on activity life.

We should also point out that the drift rate, unlike in option pricing, plays a crucial rule in first passage time problems because it is the drift rate that determine the time it takes a random variable to travel certain distance. One will by now also note the different measures adopted in this paper and that of Sarkar [2000] who considers the probability of reaching an investment boundary *within* a given time period. Through analytical and numerical simulations, Sarkar [2000] concludes that although uncertainty raises optimal investment threshold, there is no clear relation between uncertainty and the probability to invest within a given period. For further clarifications of these important issues, we refer the readers to Rhys and Song [2001].

3. ECONOMIC LIFE AS SUM OF FIRST PASSAGE TIME DISTRIBUTIONS

That economic life of a durable asset is not unique, rather, consists of a series of activity lives is implicit in Myers and Majd [1990] who recognise that although an asset's activity life may be terminated for one activity, the economic life of the asset may continue in another activity. As we have argued in section one, the existence of the second hand market in consumable goods such as cars is to facilitate the continuation of an asset's economic life. What we shall demonstrate is that the sum of these random activity lives constitutes an asset's economic life.

An example from particle radiation may be useful here. Suppose a certain counter is used to register emitting particles. Assume that the incoming particles constitute a Poisson process. A particle reaching the counter when it is free is registered but locks the counter for a fixed duration. Particles reaching the counter during a locked period have no effect whatsoever. If we start the process at an instant when a new particle reaches a free, counter, we have then two renewal processes. The primary process-the incoming traffic-is a Poisson process, its inter-arrival times have an exponential distribution, and the successive registrations form a secondary renewal process in which the inter-arrival times represent the sum of the primary process and an exponential random variable. Thus the number of registrations within a fixed time interval is approximately normal.[6]

In the remainder of this section, we establish a convergence result for the sum of independently distributed random variables. We defer the treatment of variable distributions in section four in which waiting times are introduced. Denote the series of activity lives as $\tau_i \left(i = 1,...,\infty \right)$ and the economic life of the asset as T. Then T may be expressed as the sum of activity lives:

$$T = \sum_{i=1}^{n} \tau_i \qquad (7)$$

[6] Feller (1966).

In this section we assume that once an activity life is terminated, another immediately starts. Asset transfer is instantaneous without any frictions. Incorporating transactions costs is straightforward, however if we restrict to the case of fixed or proportional transactions costs. The problem is now reduced to an evaluation of the sum of random variables all following inverse lognormal distributions. This result has been summarised in section two and is contained in Ryhs and Song [2001]. Using this result, it is easy to establish the following.

Theorem 2

The economic life of capital asset T is sum of independent but not necessarily identical Inverse Lognormal Distributions.

To prove this theorem, we require the following assumptions to hold. First, asset activity lives are independent but not necessarily identically distributed. Second, they have individual means and variances. And the following additional notations are used. Let activity lives $\tau_i\,(i-1,...,n)$ have means $m_i\,(i=1,...,n)$ and variances $v_i\,(i=1,...,n)$ so that the mean and variance of the sum $T=\sum_1^n \tau_i$ are given by

$$m = m_1 + ...,+m_n$$
$$\sigma^2 = \sigma_1^2 + ...,+\sigma_n^2$$

Now the Central Limit Theorem asserts, that, under very general conditions with respect to the distributions of the activity lives, the density function of the standardised sum

$$\frac{T-m}{\sigma}$$

will for large n be approximately equal to normal distribution. This may be expressed as follows:

$$\underset{n\to\infty}{Lim} P\left(\frac{T-m}{\sigma} \le x\right) = \phi(x) \qquad (8)$$

Note that a sufficient condition for the validity of the Theorem is that all activity lives have the same distribution, as we have assumed. This common distribution may be chosen quite arbitrarily, provided only that their first and second moments exist. In particular, if these activity lives have identical distribution with common mean and variance, then their sum will have mean nm_o and variance rate $\sigma_o \sqrt{n}$. Furthermore, the distribution of the economic life of the asset will be asymptotically normal and the arithmetic mean of its economic life

$$\bar{T} = \frac{\tau_1 + ...,+\tau_n}{n}$$

is asymptotically normal

$$\left(m_o, \frac{\sigma_o}{\sqrt{n}}\right) \qquad (9)$$

In general, asset activity lives will not be identical and independent due to the shocks to different owners and their initial conditions. Then the condition required establishing the theorem is that each activity life makes an insignificant contribution to the asset's economic life.

4. THE CASE OF POISSON WAIITNG TIMES

In section three, we have considered asset economic life with instantaneous exchange so that once one activity life is terminated, another one commences immediately.

Although this is a neat and plausible result, it abstracts from reality. Many asset sales take time: buyers need to be found, price need to be negotiated so that the asset may stay in transit for sometime before another activity life could start. In this section, introduce some realism and consider time spent between activity lives. We then establish another convergence result.

For simplicity, suppose transit times follow Poisson process. We do this for two reasons. First, Poisson distribution has been found useful for modelling waiting time in stochastic processes. Without empirical evidence suggesting otherwise, it may be a first approximation. Second, the tractability of Poisson distribution simplifies presentation without loss of generality. All transit times between successive activity lives follow independent Poisson distributions:

$$p_i = \frac{\lambda_i^k e^{\lambda_i}}{k!} \qquad (10)$$

with expectations λ_i .

It is obvious that the economic life of the asset is simply the sum of these alternate inverse lognormal distributions and Poisson distributions. Write T as the asset's economic life. Clearly T is the sum of the first passage time distributions and Poisson distributions. Suppose the means and variances of asset economic lives exist such that

$$m_n = \sum_1^{\frac{n}{2}} \mu_i + \sum_1^{\frac{n}{2}} \lambda_i$$

$$s_n^2 = \sum_1^{\frac{n}{2}} \sigma_i^2 + \sum \lambda_i \tag{11}$$

Following Feller [Vol.1: 253-256), it is straightforward to establish the following theorem.

Theorem 3

A sufficient and necessary condition of the Central Limit Theorem for variable distributions is the Lindeberg Condition which states that The Central Limit Theorem holds whenever for every ε > 0 the truncated variables U_k defined by

$$U_k = X_k - \mu_k \forall |X_k - \mu_k| \leq \varepsilon s_n$$

$$U_k = 0 \forall |X_k - \mu_k| > \varepsilon s_n$$

satisfy conditions $s_n \to \infty$ and $\frac{1}{s_n^2}\sum_1^n E(U_k^2) \to 1$. This theorem implies that every uniformly bound sequence of mutually independent random variables obeys the Central Limit Theorem.[7]

5. THE CASE OF DEPENDENT ACTIVITY LIVES

Thus far we have assumed that asset activity lives are independently distributed. This may be a reasonable assumption for many durable assets. However, it may not correspond to all durable assets. There are in reality many intertemporal externalities associated with asset use. Land is probably a good example. It is well known that different uses of a piece of land may affect its fertility and other attributes differently and sometime irreversibly. Growing one crop may exhaust its fertility while another enhances its fertility. Thus one use lengthens and another shortens activity life. The exact patterns of these correlations vary according to specific circumstances. To handle the complex correlations among activity lives requires further refinement of the model, demanding a version of the central limit theorem for dependent random variables. The result is stated in the following theorem.

Theorem 4

A finite economic life exists and is the sum of series of dependent random variables which converge to a normal distribution.

The proof of this theorem follows that of Hoeffding and Robbins [1948], George Marsalia [1954], Billingsley [1961], Loeve (1969: 377) and Hansen [1982], Hansen and Singleton [1983], Hansen and Scheinkman [1996].

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[7] Feller (1966).

6. DISCUSSIONS

In this paper we introduce the distinctions of physical life, economic life and activity life for durable asset and model asset activity lives and economic life as random variables when there is a resale opportunity. We show that activity lives are the expectations of series of first passage times for geometric Brownian motion to certain horizontal boundaries. We also demonstrate asset economic life as sum of first passage time distributions. The basic model has been extended to incorporate market frictions in terms of transit times. We also incorporate dependency of activity lives. Since uncertainty plays a prominent role in our model, we are able to show that observed asset lives are random variables. Our results are in sharp contrast to the models of Malcomson [1976] and Hill [1991] in which only one deterministic asset life exists.

Although motivated by asset sale and scrapping problem, we claim that the structure of our model is quite general and may have applications in economics and finance. Here we mention two applications. First, the model may be a basis for testing the rationale of voluntary liquidations in corporate finance. Preliminary results by Elisasson, Gao and Song [2001] show that some model predictions are supported by empirical evidence from voluntary liquidations. In any case, our model provides an alternative framework for investigating asset sale and liquidations to that of Shleifer and Vishny [1992], Pulvino [1998], Mehran, *et al* [1998] and Maksimovic and Philips [2001]. One advantage of our model is to regard asset sale as rational and value maximising behaviour. From this we derive the critical action boundary and we then calculate the first passage time. The first moment gives average time to liquidation, bankruptcy and investment and abandonment rules. Furthermore, the comparative static results are readily testable. Second, the first passage time approach used in this paper has applications in pricing financial options. Bunch and Johnson [2000] has used it to price American put option. It may also be used to study returns from holding Mertonian options as in Shackleton and Wojakowski [2001]. Thirdly, the first passage time approach advocated in this paper may also be used to characterise some portfolio rules, optimal trading rules when tax options are present. For these applications, see Cox and Huang [1989] and Merton [1990], Constantinides and Ingersoll [1984]. Song [2002] contains other applications.

Extending the basic model in this paper may be a worthwhile topic for future work. The asset in this paper has infinite durability. Although there are some assets with this property such as land, gold and platinum, most assets have finite durability. For finitely durable assets, the issue of depreciation and maintenance will undoubtedly complicate things. However, Rhys [2000] shows that a similar approach may be used to tackle it. Other options such as repair and replacement may be incorporated, an issue raised and unresolved in a classic paper of Miller and Modigliani.[8] The replacement issue has been taken up by Mauer and Ott [1995].

[8] Miller and Modigliani, 1963: 434, 442.

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