TIMING ADVANTAGE: LEADER/ FOLLOWER VALUE FUNCTIONS IF THE MARKET SHARE FOLLOWS A BIRTH AND DEATH PROCESS

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Abstract

For a duopoly environment, we model the leader and follower real value functions assuming that the leader's "market share" evolves according to an immigration (birth) and death process. We derive analytical solutions for the follower and leader options to invest, and numerical solutions for the leader's optimal investment timing. Then we calculate the partial derivatives of the leader and follower value functions to market share, birth/death parameters, and market profitability. This model is possibly more realistic than that proposed by some other authors studying the advantages of being first (and also being a follower).

We show that over certain ranges of the parameter values, the leader and follower real options to wait to invest, and not to wait to invest, are sometimes surprising, but possibly on refection plausible. The follower's value function is usually less sensitive (and of opposite sign) than the leader's value function to market share or the rate of customer arrivals/departures until the expected revenue exceeds the follower's trigger investment level, but the sensitivity is dependent on the relative parameters, particularly the revenue/trigger. The follower's trigger increases with market share, the immigration-death ratio and revenue volatility. The leader's value function "deltas" are highly sensitive and unstable as revenues approach the follower's trigger, confirming the adage, if you're ahead, "watch the competition".

I INTRODUCTION

There is a developing real option literature that considers that firms are not in a monopoly setting and focuses on the option of not waiting. Fearing the move of competitors, many times firms act in order to achieve the advantages of being first, balanced against the advantages of the option to wait. In a duopoly setting where one of the firms is the first one to enter, from now on defined as the leader, and the other one the follower, there are some advantages and disadvantages for assuming either of the roles. The leader has normally advantages in distribution, product line breadth, product line and especially market share (Tellis and Golder, 1996). The follower can have lower adoption costs and a reduction in uncertainty (Hope, 1997), through "learning from the leader's mistakes". An adequate model to determine investment/entry timing should consider the strategic policies of each firm and consequently include the advantages and disadvantages of each role.

The advantages of leaders establishing a dominant market share have been documented using the PIMS[1] database. More than seventy per cent of current market leaders are market pioneers. Tellis and Golder (1996) argue although being first does not necessarily induce an advantage, it certainly creates an opportunity. When the pioneer is alone in the market, the leader enjoys the revenues of a monopolist; when other firms enter, the pioneer can continue to be the leader or not and that will depend on his ability to satisfy costumers and innovate.

Spatt and Sterbenz (1985) consider learning and pre-emption. Smets (1991) considers a strategic setting where firms can act under the fear of pre-emption. Grenadier (1996) applies the model to real estate market. The effect of incomplete information is analysed by Lambrecht and Perraudin (1997); strategic competition in Kulatilaka and Perotti (1998); the advantage of being first with the network advantage of adopting with others in Mason and Weeds (2000); R&D competition in Weeds

[1] Profit Impact of Market Strategies.

(2000); and Tsekrekos (2002) studies the sensitivity of the leader and follower value function to market share, assumed to be constant after the follower enters.

We relax the constant market share assumption to reflect a possibly more realistic environment, where the market and the market share reflects new customers arriving =(birth or immigration process) and old customers departing =(death process) Section two develops this model. Section three derives the partial derivatives of the follower and leader value functions to changes in market share, birth/death parameters, volatility and market profitability. Section four concludes.

II MARKET SHARE and BIRTH/DEATH PROCESSES MODEL

In common with Smets (1991), Weeds (2000) and Tsekrekos (2002), we develop a model where two competing firms have the option to enter the market; the leader will invest earlier and will benefit from securing a higher market share than its competitors. Operating the market will yield a net revenue flow x_t that evolves according to a geometric Brownian motion given by:

$$dx_t = \mu x_t dt + \sigma x_t dw_t \tag{1}$$

where μ is the drift parameter, σ is the standard deviation, and dw_t is the increment of a standard Wiener process. We assume lognormality of revenues and do not consider operating costs explicitly, so there is no option to abandon. The underlying game is a Stackelberg leader-follower: the leader receives a monopolistic revenue flow $x_t dt$ when alone in the market. When the follower enters, that revenue will be shared with the leader having a higher market share, "a". The market share is presented in a deterministic setting; the leader will have a revenue flow of a x_t and the follower (1-a) x_t.

II.1- THE FOLLOWER'S VALUE FUNCTION

Let $V_0^F(x)$ represent the value of the follower in the region where it is not yet optimal to invest. This option gives the follower a capital gain or loss according to the evolution of the market. In the continuation region the value of the follower is given by:

$$rV_0^F(x)dt = E[dV_F^0(x)] \tag{2}$$

Using Ito's lemma we obtain the differential equation:

$$\frac{1}{2}\sigma^2 x^2 V''^F_0(x) + \mu x V'^F_0(x) - rV_0^F(x) = 0 \tag{3}$$

This has the general solution:

$$V_0^F(x) = Ax^{\beta_1} + Bx^{\beta_2} \tag{4}$$

where A and B are constants, and β_1 and β_2 are:

$$\beta_1 = \frac{1}{2} - \frac{\mu}{\sigma^2} + \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} > 1 \tag{5}$$

$$\beta_2 = \frac{1}{2} - \frac{\mu}{\sigma^2} - \sqrt{\left(\frac{\mu}{\sigma^2} - \frac{1}{2}\right)^2 + \frac{2r}{\sigma^2}} < 0 \tag{6}$$

We assume that if our state variable reaches zero it will stay there forever, meaning that zero is an absorbing barrier, and so obtain the following boundary condition:

$$V_0^F(0) = 0 \tag{7}$$

Since as the state variable goes to zero the function has to decrease, B in (4) has to be equal to zero, so our solution becomes:

$$V_0^F(x) = Ax^{\beta_1} \tag{8}$$

Let $V_1^F(x)$ denote the value of the follower in the stopping region, the region where it is optimal to invest. In this region the follower invests a fixed cost K in order to receive perpetually certain revenue that will be multiplied by the market share.

Consider now that the initial market share "a" evolves according to a random process, more specifically an immigration-death model where new clients arrive according to a Poisson process and once there, they can leave at any time. The lifetime of each individual as a client has an exponential distribution. This model gives a realistic representation of many practical situations like molecules of gas in a given space. New particles can enter at any time and the Poisson process represents their arrival pattern. Once inside the molecules can leave at any time, so the exponential distribution also provides a good model for the time spent inside the space[2]. This process can also be used to model the number of people in a shop, or use of a telephone system, adoption of 3G mobile facility, or net new internet banking customers. In the U.K., football pay-TV viewing developed first in one media (B-SkyB) and some customers were expected to migrate to another (ITV Digital). (In this case, the follower's expectations on customer migration rates from the leader turned out to be irrationally exuberant, as ITV Digital failed.)

The immigration-death process has an equilibrium distribution (the distribution of the population size at time t approaches a limiting distribution as t increases). This equilibrium distribution gives the expected proportion of time spent in each state in the long run. Consider an immigration-death model where the individuals join the population according to a Poisson process at rate λ and the lifetime distribution of each individual is exponentially distributed, M(ν). Let X(t) equal the population size at time t, X(t) is asymptotically Poisson distributed with parameter $\rho = \lambda / \nu$, (see the Appendix) where birth (or immigration) has the parameter λ, and death ν:

$$P(X(t) = n) \to \frac{\rho^n}{n!} e^{-\rho} \tag{9}$$

[2] See Karlin and Taylor (1975), Chapter 4.

Now assume that the "market share" is not constant, so some new clients will arrive and others will leave, and that the immigration-death model is appropriate for this phenomenon. We define "market share" broadly as the multiplier for a standard revenue x. The multiplier is itself adjusted over time by a parameter ρ, which is immigration (λ) divided by death (ν) (new customers adjusted for old and new customers leaving). In the stopping region, the follower receives perpetually the expected value of the active project with no option value, so the expected value will be given by:

$$V_1^F(x_t)dt = \int_0^{\infty} (1 - a\rho) x e^{-\gamma t} dt \tag{10}$$

where $\gamma = r - \mu + \rho$.

Solving (10) we obtain the function for the follower in the stopping region:

$$V_1^F(x) = \frac{x(1 - a\rho)}{\gamma} \tag{11}$$

As usual, the optimal investment rule is found by solving for the boundary between the continuation and the stopping regions. The boundary is the trigger point x_F. If the value of the state variable is smaller than the trigger, the optimal decision for the investor is not to invest, i.e. to continue in the continuation region; if it exceeds the trigger, then the follower should invest. At the boundary two conditions must be satisfied; the value matching requires that when the state variable reaches the trigger the investor will invest so that:

$$V_0^F(\bar{x}) = V_1^F(\bar{x}) - K \tag{12}$$

and the smooth-pasting condition requires that the derivatives of the functions match at the boundary:

$$V_0^F{}'(\bar{x}) = V_1^F{}'(\bar{x}) \tag{13}$$

Conditions (12) and (13) imply:

$$x_F = \frac{K\beta_1\gamma}{(1-a\rho)(\beta_1-1)} \tag{14}$$

and:

$$A = \frac{K\left(\dfrac{K\beta_1\gamma}{(1-a\rho)(\beta_1-1)}\right)^{-\beta_1}}{\beta_1-1} \tag{15}$$

Putting equations (8), (11), (14) and (15) together we obtain the value function of the follower:

$$V_F(x) = \begin{cases} \dfrac{K}{\beta_1-1}\left(\dfrac{x}{x_F}\right)^{\beta_1} & if \quad x < x_F \\ \dfrac{x(1-a\rho)}{\gamma} - K & if \quad x \geq x_F \end{cases} \tag{16}$$

Figure 1 shows the sensitivity of the follower's option to enter to volatility, and as expected, that the option increases with volatility. Figure 2 shows the sensitivity of the follower's option to the initial market share, and that as the market share of the follower diminishes so does the value of the option to enter the market. Note that the sensitivity rate declines as the leader's market share increases. Figure 3 shows the sensitivity of the follower's option to the parameter ρ. Since ρ explains the evolution of the market share of the leader, increases in ρ implies an increase in the leader's future market share. Thus as a ratio greater than one increases, due to an increase in net immigration, the option value of the follower will decrease, because the probability of obtaining those clients is decreasing.

According to our model the optimal strategy for the follower is stated in proposition 1.

Proposition 1. *In a duopoly setting where the market share evolves according to a immigration-death model, the optimal entry time for the follower conditional on a previous entrance of the leader is given by:*

$$T_F = \inf\left\{ t \geq T_L : x_t \geq \frac{K}{(\beta_1 - 1)}\left(\frac{x}{x_F}\right)^{\beta_1} \right\} \tag{17}$$

where T_L is the trigger time for the leader.

II.2 THE LEADER'S VALUE FUNCTION

Until the follower enters the market, the leader's decision either to enter the market or to wait may seem identical to the single setting framework. So the basic idea, following Dixit and Pindyck (1994) would be that there exists an optimal time to enter that will maximise the firm's value. Until that moment the firm should wait to invest and its value is explained by the option to wait. When that moment is reached, the firm should invest and its value function is given by the present value of the revenues in perpetuity. The possible problem with the option to wait is that it excludes the case where companies do not have the possibility of waiting, and also that not waiting can itself be an important option.

First mover advantage should make pre-emption attractive, and pre-emption should lead to early adoption by the leader. Examples where the value of being the first can become very important are: the location of a building because this can determine how profitable it will be and once it is built you can not change its location; and the decision of companies to have a website. A first mover company might buy cheaper domain names and obtain lower staff costs and better access to resources. However, the value of the clients obtained by being first is offset by not learning by others' mistakes.

Often if a company does not make an investment immediately, it loses either the investment opportunity, or the chance of success is diminished. Our model is not concerned with what happens to the leader prior to investment. We are assuming that the fear of pre-emption leads to a possible early entrance in the market, or in other words that the option to wait is nullified by the fear of not achieving the advantage of being first.

Once entering the market, the leader has no further action to take. It will enjoy monopolistic revenues until the moment that the follower enters the market and will share them with the follower afterwards. The value function of the leader, before the follower enters the market, can be explained by the following equation:

$$V^L(x) = E\left[\int_0^{T_F} e^{-r\tau} x d\tau\right] + E_t\left[e^{-rT_F}\right]\frac{a\rho x_F}{\gamma} - K \tag{18}$$

The first function of equation (18) represents the monopolistic revenues received by the leader until the follower enters the market.

Let the expectations terms of equation (18) be respectively $f(x) = E[e^{-rT_F}]$ and $g(x) = E\left[\int_0^{T_F} e^{-r\tau} x d\tau\right]$, where x follows a geometric Brownian motion as described in (1). Over the time interval dt we can write the first expectation as:

$$f(x) = e^{-rdt} E[f(x + dx)] \tag{19}$$

Using Ito's lemma we obtain the following partial differential equation:

$$\frac{1}{2}\sigma^2 x^2 f''(x) + \mu x f'(x) - rf(x) = 0 \tag{20}$$

with a general solution:

$$f(x) = Cx^{\beta_1} + Dx^{\beta_2} \tag{21}$$

where β_1 and β_2 are as defined previously in equations (5) and (6) respectively, and f(x) represents the expectation of the discounted term at the risk free rate, during the time T_F. So, we can submit equation (21) to two boundaries: as our state variable x tends to the trigger price of the follower x_F, the optimal time to invest T_F will be very small; obviously the moment that it reaches zero our function f(x) will be one so $\lim_{x \to x_F} f(x) = 1$. The other boundary is defined as x goes to zero. If the revenues go to zero, the follower will not enter in the short run implying that its optimal time to enter will be very large, so that f(x) will tend to zero $\lim_{x \to 0} f(x) = 0$. This last boundary implies that D in equation (21) has to equal zero, and the first one implies that $C = \frac{1}{x_F^{\beta_1}}$.

So, our equation (21) becomes:

$$f(x) = \left(\frac{x}{x_F} \right)^{\beta_1} \tag{22}$$

In the same way, g(x) satisfies the following partial differential equation:

$$\frac{1}{2}\sigma^2 x^2 g''(x) + \mu x g'(x) - r g(x) + x = 0 \tag{23}$$

with a general solution:

$$g(x) = E x^{\beta_1} + F x^{\beta_2} + \frac{x}{r - \mu} \tag{24}$$

and subject to two boundary conditions: as x goes to zero, g(x) will tend to zero implying that F in equation (24) has to be zero; on the other hand as x tends to x_F the monopoly revenues will also tend to zero because the follower will enter the market. This last boundary condition implies that: $E = -\frac{x_F^{(1-\beta_1)}}{r - \mu}$. So equation (24) becomes:

$$g(x) = -\frac{x_F^{1-\beta_1}}{r-\mu} x^{\beta_1} \tag{25}$$

Substituting f(x) and g(x) back into equation (18) we obtain:

$$V^L(x) = \begin{cases} \dfrac{x}{r-\mu} + Ex^{\beta_1} + \dfrac{a\rho x_F}{\gamma}\left(\dfrac{x}{x_F}\right)^{\beta_1} - K & if \quad x < x_F \\ \dfrac{a\rho x}{\gamma} & if \quad x \geq x_F \end{cases} \tag{26}$$

where Ex^{β_1} is an option like term that captures the negative effect that the entry of the follower will have on the leader's value function.

The value functions of the leader and of the follower are shown in Figure 4. The value function of the leader is almost always higher than that of the follower. It is possible, as can be seen in the figure, for the follower to have a higher value function when the revenues are very low. In this case the follower has not yet entered the market while the leader has already invested. We can also observe that when the follower enters the two functions almost meet tangentially[3]. Dixit and Pindyck (1994) describe this as a smooth-pasting-like property of present values.

The value function of the leader is more complicated than that of the follower. It is concave until the trigger time of the follower is reached and at that precise moment its slope is discontinuous. This happens because the follower's decision changes discontinuously at x_F (Dixit and Pindyck, 1994). The two curves meet at a point that we will designate as x_L; this point should be the trigger point of the leader since until that point its value function is negative, following the equalisation principle of Fudenberg and Tirole (1985).[4]

[3] The leader always has a higher value function because we are assuming a first mover advantage.
[4] At the leader's investment point the expected payoff of the two firms must be equal. If this were not the case, one firm would have an incentive to deviate and the proposed outcome would not be an equilibrium.

Although we cannot obtain an explicit general expression for x_L, we can prove that this expression has a root strictly below x_F [5]. If we evaluate V(x) at x=0 using our value functions for the leader and the follower, we obtain:

$$V(0) = -K \quad < 0$$

and evaluating V(x) at x_F:

$$V(x_F) = \frac{(2a\rho - 1)K\beta_1}{(1 - a\rho)(\beta_1 - 1)} \qquad > 0$$

Since V(x) is continuous on the interval $(0, x_F)$, it has at least one root in that interval. Uniqueness of the root x_F can be proved while demonstrating strict concavity of V(x) over the same interval. The second derivative of V(x) is:

$$V''(x) = -\frac{\beta(\beta - 1)\left(\frac{K}{\beta - 1} + \frac{x_F}{r - \mu} - \frac{a\rho x_F}{\gamma}\right)\left(\frac{x}{x_F}\right)}{x_F^{\ 2}} < 0$$

So, the root is unique, with V(x)<0 for $V(x) \in (0, x_L)$ and V(x)>0 for $V(x) \in (x_L, x_F)$. Thus we have shown that there exists a single point belonging to the interval $(0, x_F)$ at which the leader and the follower have the same value. At any point below that interval the follower has a higher value, meaning that the only motive that can explain a rational leader entering the market is fear of pre-emption. After passing the trigger point of the leader, the leading firm benefits from the advantage of being the leader, in this special case from a higher market share that evolves according to a immigration-death process. In Figure 4 we can see that until the leader trigger point is achieved[6] the leader incurs losses while the follower has a positive value function.
The stopping time for the leader is:

[5] This implies that the trigger point exists and it is unique.
[6] The trigger point for the leader for r=.09, μ=0.02, K=5, σ=0.1, a=0.55 and ρ=1.01 is 0.35 (calculated numerically).

Proposition 2. *The optimal leader strategy is to invest as soon as the revenues reach* x_L. *In other words the optimal time for the leader to invest is:*

$$T_L = \inf\{t \geq 0 : x \in [x_L, x_F]\} \tag{27}$$

III VALUE FUNCTION PARTIAL DERIVATIVES

In studying the behaviour of our value functions, we derive in this section some partial derivatives, namely we study the sensitivity of our value functions to changes in the market share, in the immigration-death ratio, the revenues and also the sensitivity of the trigger function of the follower to volatility.

The partial derivatives of the value functions to market share ("MS Δ") are:

$$\frac{\partial V_F}{\partial a} = \begin{cases} -\dfrac{\rho x \left(\dfrac{x}{x_F}\right)^{\beta_1 - 1}}{\gamma} < 0 & if \quad x < x_F \\ -\dfrac{\rho x}{\gamma} < 0 & if \quad x \geq x_F \end{cases} \tag{28}$$

$$\frac{\partial V_L}{\partial a} = \begin{cases} \dfrac{\beta_1 K \rho\left(\beta_1 (r-\mu) + \rho\left(-1 + \beta_1 (1 - ar + a\mu)\right)\right)}{(\beta_1 - 1)(a\rho - 1)^2 (r-\mu)} \left(\dfrac{x}{x_F}\right)^{\beta_1} > 0 & if \quad x < x_F \\ \dfrac{\rho x}{\gamma} > 0 & if \quad x \geq x_F \end{cases} \tag{29}$$

Figures 5 and 6 show that the MS Δ of the follower and leader have contrasting reactions to different revenue levels. Note that in our model we are assuming that the initial market share is shared by two parties. Since "a" represents the leader's initial share, the sign of the partial derivatives are consistent with the leader's value increasing with a, prior to the follower's entry. Consequently, an increase in the initial market share of the leader will imply a decrease in the market share of the follower, so the value function of the follower has to decrease with the market share of the leader. The slope of Figure 2 is negative, and the curve concave; the slope of Figure 5 is negative, and convex (at least before the trigger). Another interesting, though expected conclusion, is that since an increase in the market share implies an increase in the value function of the leader, and a decrease in that of the follower, and since market share appears in the model as an advantage of the leader, pre-emption is obvious and seems to justify what is described in the literature as the fear of pre-emption.

The market share as a pre-emption advantage is further pronounced prior to the entrance of the follower. After the follower enters, the leader continues to benefit from increases in its market share, and the follower continues to have a decrease in its value function but the magnitude of changes in the market share is exactly the same for both, obviously of different sign[7]. But prior to the follower's entrance the difference between the two functions is not only in sign but also in magnitude. Although the market share as a pre-emptive factor will always constitute an advantage over the follower, the higher advantage relative to the follower, will occur during the time interval that the follower is inactive. The optimal time for the follower to invest increases with a and consequently the leader will enjoy monopolistic revenues for longer[8].

The partial derivatives of the value functions to the immigration-death ratio ("Ratio Δ") are:

$$\frac{\partial V_F}{\partial \rho} = \begin{cases} \dfrac{\beta_1 K \left(\dfrac{x}{x_F}\right)^{1-\beta_1} \left(-\dfrac{x}{x_F \gamma} - \dfrac{a(\beta_1 - 1)x}{\beta K \gamma}\right)}{\beta_1 - 1} < 0 & if \quad x < x_F \\ -\dfrac{(1-a\rho)x}{\gamma^2} - \dfrac{ax}{\gamma} < 0 & if \quad x \geq x_F \end{cases} \tag{30}$$

[7] Note that after the follower enters, the partial derivatives of the value functions are exactly the same, except for the sign.
[8] The relative higher advantage of the value function of the leader compared to the follower prior to the entrance of the latter can also be seen in Figure 4.

(31)

$$\frac{\partial V_L}{\partial \rho} = \begin{cases} \dfrac{\left(\dfrac{x}{x_F}\right)^{\beta_1}\left(\beta_1 K\left(\rho\left(-1-\beta_1\left(-1+a^2(r-\mu)^2\right)\right)+(r-\mu)\left(-1+\beta_1+a\beta_1 r-a\beta_1\mu\right)\right)\right)}{(\beta_1-1)(a\rho-1)^2(r-\mu)\gamma} > 0 & \\ \qquad if \quad x < x_F & \\ \dfrac{ax}{\gamma} - \dfrac{a\rho x}{\gamma^2} > 0 \qquad if \quad x \geq x_F \end{cases}$$

Figures 7 and 8 show that the Ratio Δ's are similar to the MS Δ's, after the follower's trigger. An increase in the ratio, that is the number of new clients divided by the ones that leave, signifies an increase in the evolving market share of the leader and consequently leads to the same conclusions that as for parameter a. When $x < x_F$, the sensitivity of the follower's and the leader's value functions to changes in ρ will depend on the parameter values, particularly (x/x_F) times some variables divided by ρ for the follower, and the same ratio times some variables multiplied and divided by ρ for the leader. For illustrative parameters herein, K=5, r=.09, μ=.02, σ=.10, a=.55, over a range of ρ=.97 to 1.07, the follower's value function is less sensitive than the leader's value function to changes in ρ when x is slightly less than x_F and more sensitive to changes in ρ when x is slightly greater than x_F but always of opposite sign.

The partial derivatives of the value functions to the revenues (delta) are:

$$\frac{\partial V_F}{\partial x} = \begin{cases} \dfrac{(1-a\rho)\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{\gamma} > 0 & if \quad x < x_F \\ \dfrac{1-a\rho}{\gamma} > 0 & if \quad x \geq x_F \end{cases} \tag{32}$$

$$\frac{\partial V_L}{\partial x} = \begin{cases} \dfrac{1}{r-\mu} - \dfrac{\beta_1\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{r-\mu} + \dfrac{a\beta_1\rho\left(\dfrac{x}{x_F}\right)^{\beta_1-1}}{\gamma} >< 0 & if \quad x < x_F \\ \dfrac{a\rho}{\gamma} > 0 & if \quad x \geq x_F \end{cases} \tag{33}$$

The value function of the follower behaves as expected; delta is always positive. As the revenues increase, so does the follower's value function. Curiously the value function of the leader decreases as the state variable increases, as shown in Figure 9, while $x < x_F$. There is a trade-off between monopoly revenue enjoyed by the leader and the likelihood that the monopoly will end with the follower's entry.

Thus the leader's value function MS Δ, Ratio Δ, and delta are highly sensitive (and change signs) to expected revenues slightly below the follower's trigger revenues. In a broad sense, "delta" hedging of the leader's value function would be very complex, and probably confounded by transaction costs.

Finally, volatility is one of the most important parameters in option pricing. From the literature, we expect an increase in real option value with volatility. Since the value functions and trigger points are complex functions of volatility we computed only the "vega" of the trigger function of the follower, which is:

$$\frac{\partial x_F}{\partial \sigma} = \frac{4K\gamma\left(\mu\sigma^2\left(1+\sqrt{1+\frac{4\mu^2}{\sigma^4}+\frac{8r}{\sigma^2}-\frac{4\mu}{\sigma^2}}\right)-2\mu^2-2r\sigma^2\right)}{(a\rho-1)\sigma\left(2\mu+\sigma^2-2\sigma^2\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}\right)^2\sqrt{\left(\frac{\mu}{\sigma^2}-\frac{1}{2}\right)^2+\frac{2r}{\sigma^2}}} > 0 \qquad (34)$$

The vega behaves as expected, that is increases in volatility lead to increases in the trigger value function of the follower, because the option value of waiting until some of the uncertainty will be resolved increases. This result is confirmed in Figures 1 and 10. Figure 10 shows that the follower's trigger increases (almost) linearly with volatility.

Figure 11 shows the behaviour of the trigger function of the follower divided by that of the leader as volatility and the immigration-death parameters increase. Volatility increases induce an increase of higher magnitude in the trigger function of the follower compared to the leader. This result leads to the conclusion that the follower's decision to invest is more affected by volatility than that of the leader, probably because the follower is the one being pre-empted. Notice that as the advantage of being the first increases, so does the ratio of the follower/leader triggers. When the advantage of being first is very high, the follower will desire even more that the uncertainty is resolved before investing, meaning that the follower will attribute higher value to the option to wait. In contrast, although there is a higher value for the leaders' option to wait while facing higher volatility, first mover advantage gives a lower value relative to the follower.

IV. CONCLUSION

For a duopoly environment, we model the leader and follower value functions assuming that the leader's "market share" evolves according to an immigration (birth) and death process. We define "market share" broadly as the multiplier for a standard revenue x. The multiplier is itself adjusted over time by a parameter ρ, which is immigration (λ) divided by death (ν) (new customers adjusted for old and new customers leaving). We derive analytical solutions for the options to invest, and numerical solutions for the leader's optimal investment trigger. Then we calculate the partial derivatives of the leader and follower value functions to market share, birth/death parameters, and market profitability. This model is possibly more realistic than that proposed by some other authors studying the advantages of being first (and also being a follower).

We show that over certain ranges of the parameter values, the leader and follower real options to wait to invest, and not to wait to invest, are sometimes surprising and not immediately intuitive. The follower's value function is less sensitive than the leader's value function to market share or the rate of customer arrivals/departures, until the expected revenue exceeds the follower's trigger investment level. The follower's trigger increases with market share, the immigration-death ratio and revenue volatility.

The leader's value function "deltas" are highly sensitive and unstable as revenues approach the follower's trigger, confirming the adage, if you're ahead, "watch the competition".

APPENDIX The Immigration-death Model

Immigration: individuals join the population according to a Poisson process at rate λ.
Death: the lifetime distribution of each individual is exponential, $M(\nu)$.
The overall birth and death rates of this process are:

$\beta_x = \lambda$, $\nu_x = \nu x$

The Kolmogorov forward equation is equation (A1)

$$\frac{d}{dt} p_x(t) = \beta_{x-1} p_{x-1}(t) + \nu_{x+1} p_{x+1}(t) - (\beta_x + \nu_x) p_x(t) \tag{A1}$$

So the differential-difference equations for the immigration-death model are:

$$\frac{d}{dt} p_x(t) = \lambda p_{x-1}(t) + \nu(x+1) p_{x+1}(t) - (\lambda + \nu x) p_x(t) \quad x=1,2\ldots \tag{A2}$$

$$\frac{d}{dt} p_x(t) = \nu p_1(t) - \lambda p_0(t) \quad x=0 \tag{A3}$$

The equilibrium distribution

The equilibrium distribution, if it exists, is found by putting $\frac{d}{dt} p_x(t) = 0$ in the forward equations, and solving them:

For x=0, $\nu p_1 - \lambda p_0 = 0$, so $p_1 = \frac{\lambda}{\nu} p_0$

For x=1, $\lambda p_0 + 2\nu p_2 - (\lambda + \nu) p_1 = 0$, so $p_2 = \frac{\lambda}{2\nu} p_1$, $p_1 = \frac{\lambda^2}{2\nu^2} p_0$

For x=2, $\lambda p_1 + 3\nu p_3 - (\lambda + 2\nu) p_2 = 0$, so $p_3 = \frac{1}{3!}\left(\frac{\lambda}{\nu}\right)^3 p_0$

For general x, $p_x = \frac{1}{x!}\left(\frac{\lambda}{\nu}\right)^x p_0$

For a proper probability distribution, $\sum_{x=0}^{\infty} p_x = 1$

$$\sum_{x=0}^{\infty} p_x = p_0 \sum_{x=0}^{\infty} \frac{1}{x!}\left(\frac{\lambda}{\nu}\right)^x = p_0 e^{\frac{\lambda}{\nu}} \qquad , \ p_0 = e^{-\frac{\lambda}{\nu}}$$

The equilibrium is

$$p_x = \frac{e^{-\frac{\lambda}{\nu}}\left(\frac{\lambda}{\nu}\right)^x}{x!}, \qquad x=0,1\ldots$$

This is a Poisson distribution with parameter $\rho = \frac{\lambda}{\nu}$.
Assuming that the conditional probability of the market share having a certain value in a short interval of length dt is ρdt, and the density function of the time that the company takes to achieve a certain market share a is given by $\rho e^{-\rho t}$, then equation (9) follows.

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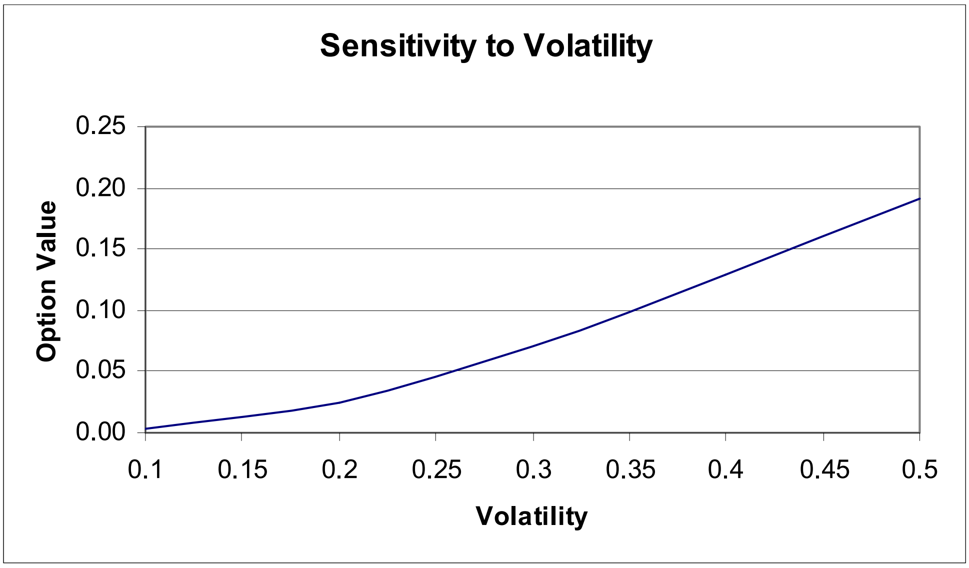
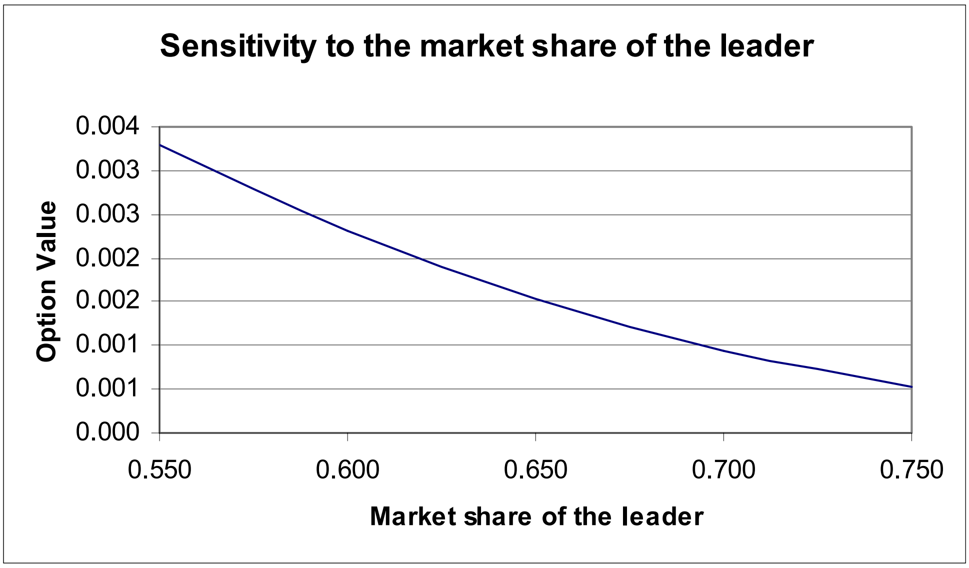
Figure 1- Sensitivity of the follower's option to wait to volatility



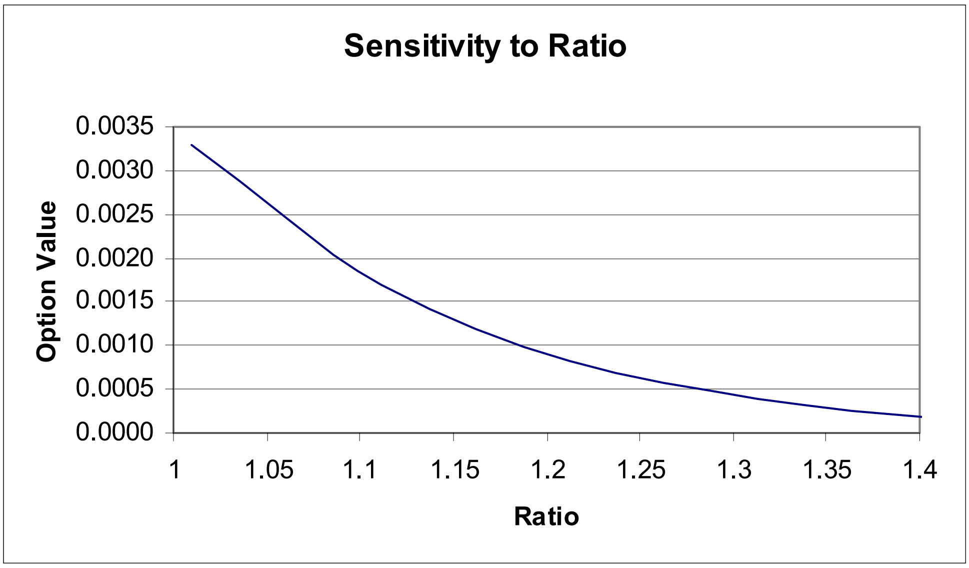
The parameters are: μ=0.02, K=5, r=0.09, a=0.55, ρ=1.01 and x=2.

Figure 2- Sensitivity of the follower's option to wait to the market share



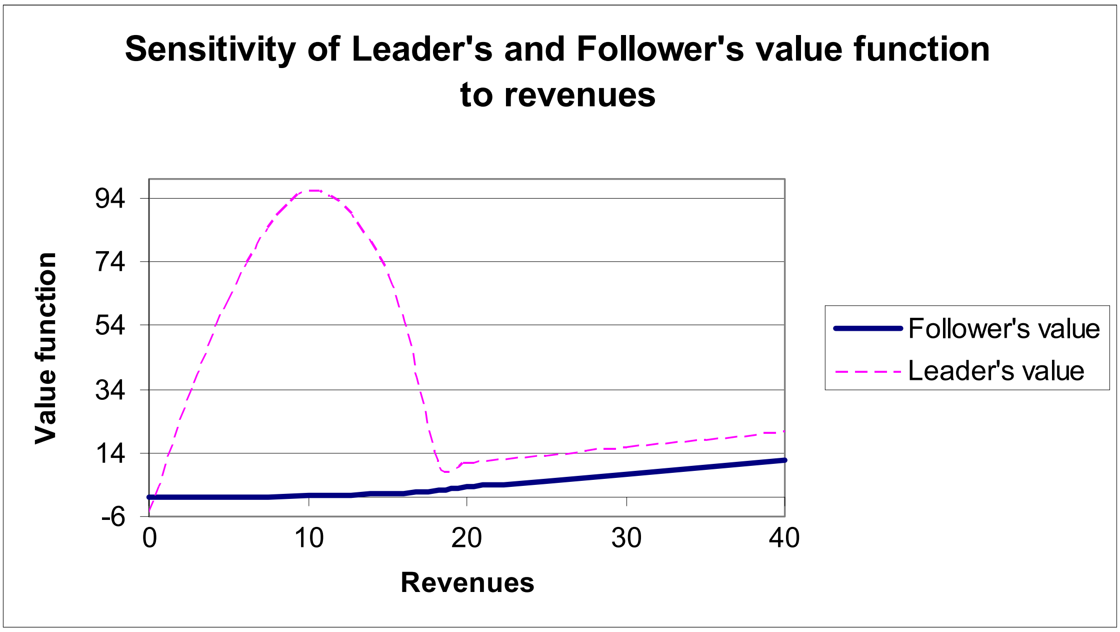
The parameters are: μ=0.02, K=5, r=0.09, σ=0.1, ρ=1.01 and x=2.

Figure 3-Sensitivity of the follower's option to wait to the immigration/death parameter



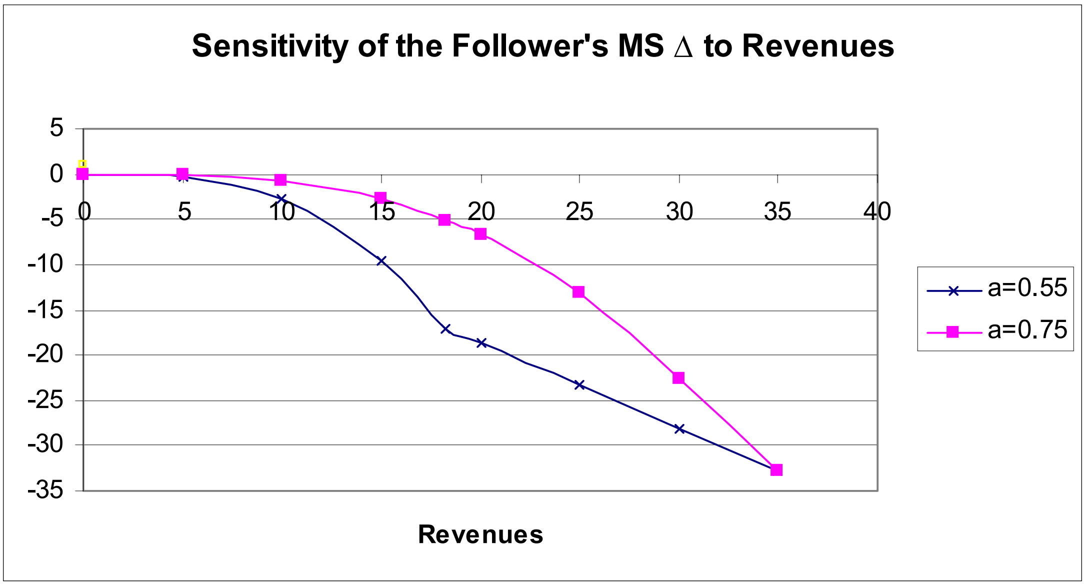
The parameters are: μ=0.02, K=5, r=0.09, σ=0.1, a=0.55 and x=2. Ratio=immigration/death rates.

Figure 4-The follower and leader's value functions



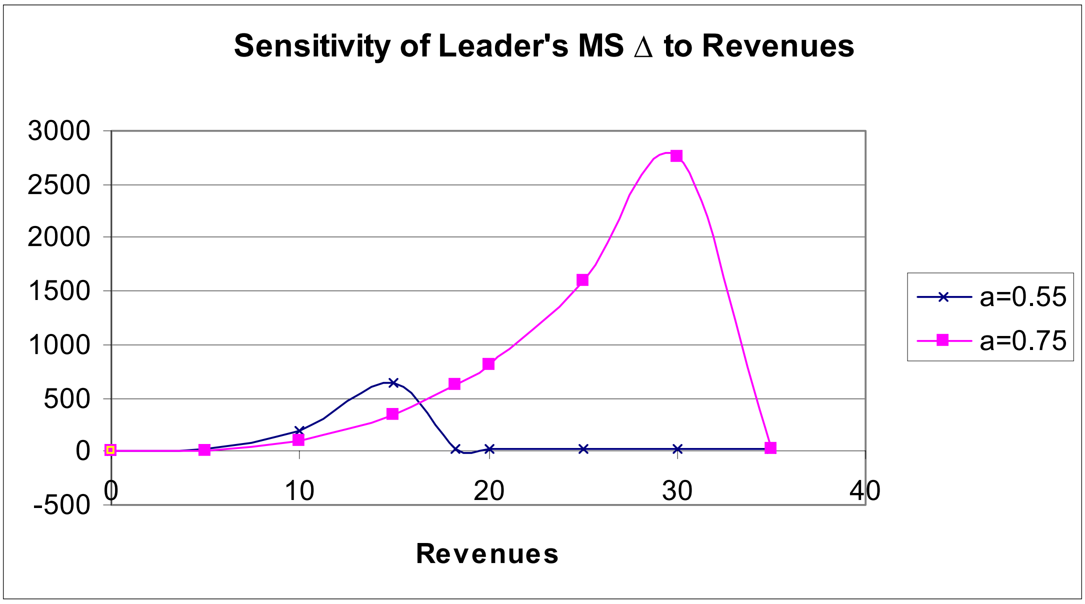
The parameters are: μ=0.02, K=5, r=0.09, σ=0.1, a=0.55 and ρ=1.01

Figure 5- Sensitivity of the Follower's MS Δ to Revenues



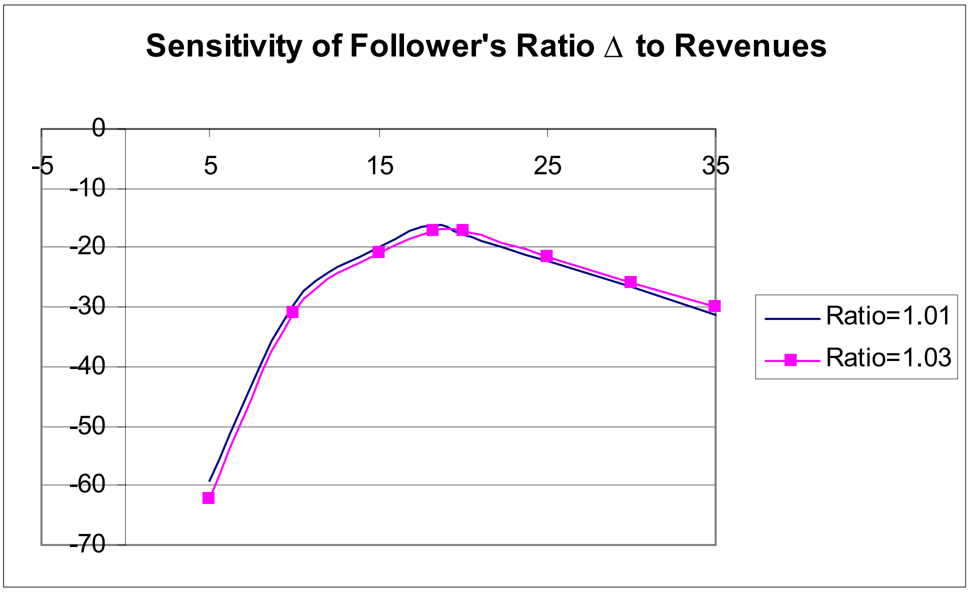
The parameters are: μ=0.02, K=5, r=0.09 and ρ=1.01.

Figure 6- Sensitivity of the Leader's MS Δ to Revenues



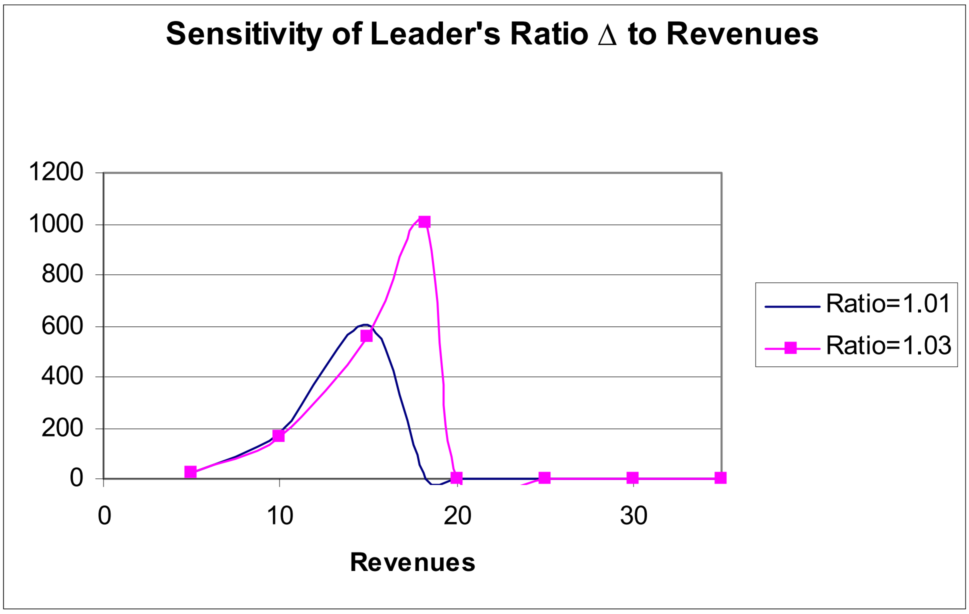
The parameters are: μ=0.02, K=5, r=0.09 and ρ=1.01.

Figure 7- Sensitivity of the Follower’s Ratio Δ to Revenues



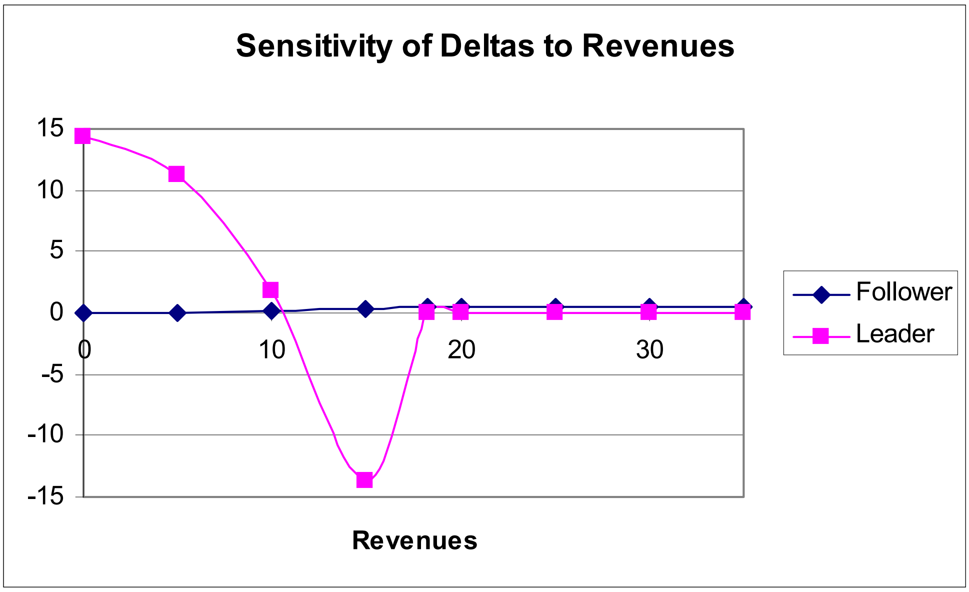
The parameters are: μ=0.02, K=5, r=0.09 and a=0.55

Figure 8- Sensitivity of the Leader’s Ratio Δ to Revenues



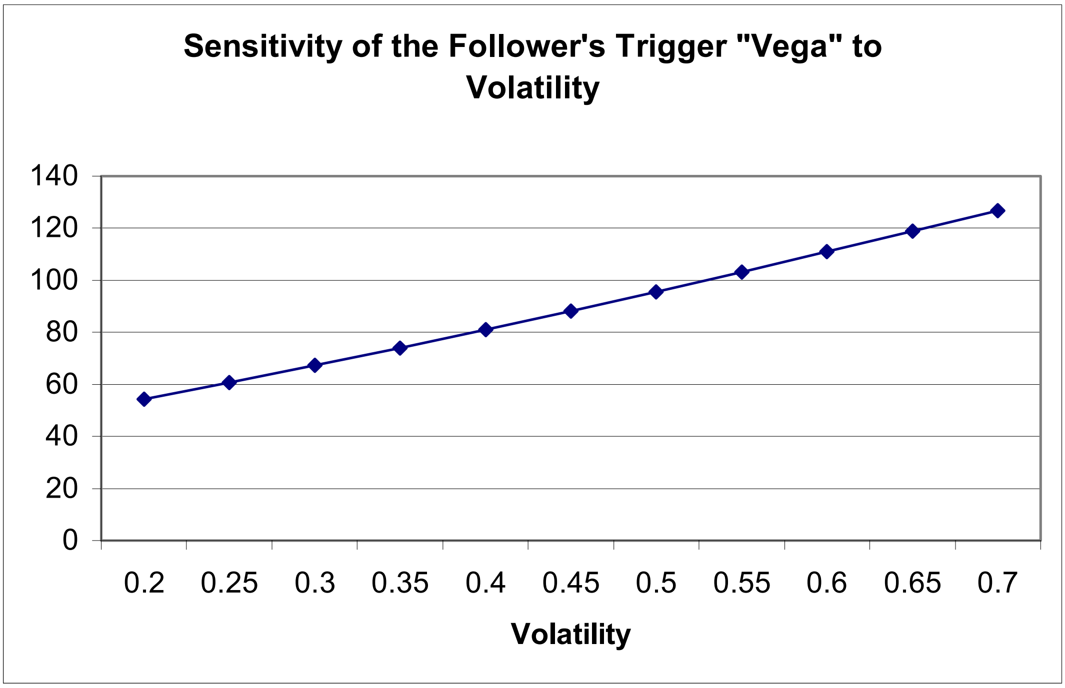
The parameters are: μ=0.02, K=5, r=0.09 and a=0.55

Figure 9 Sensitivity of the Follower's and Leader's Delta to Revenues



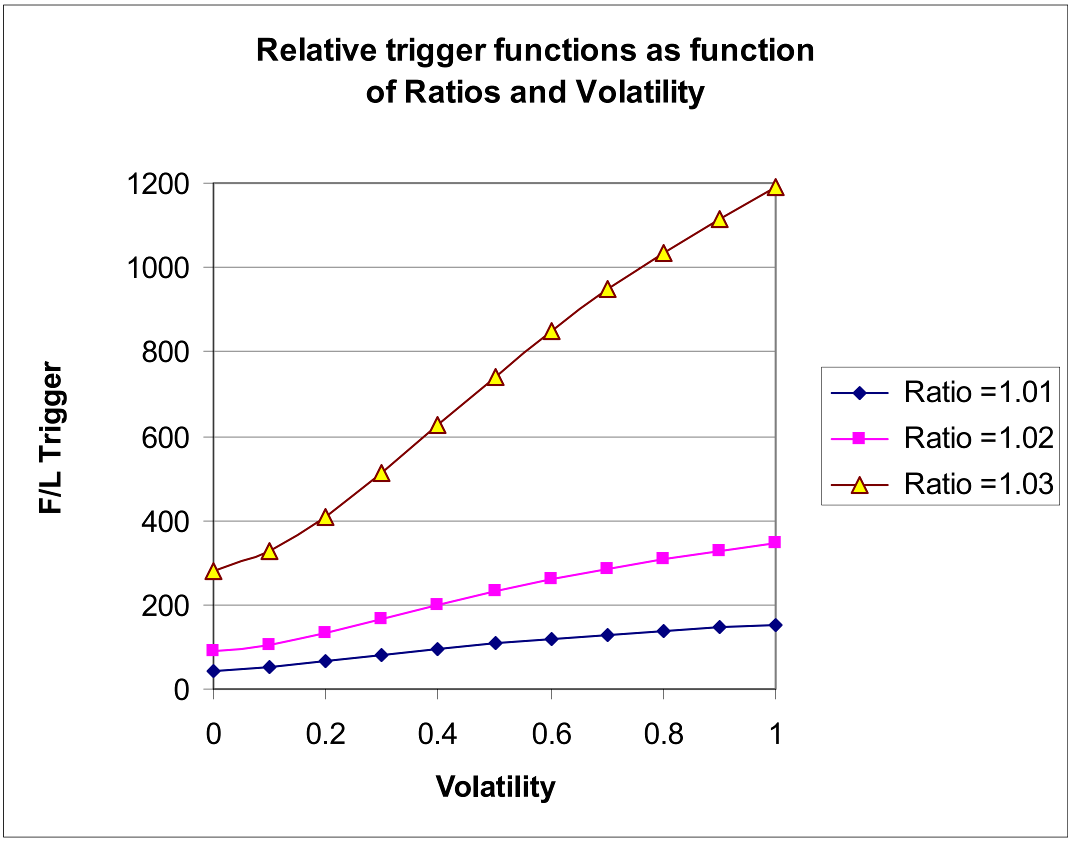
The parameters are: μ=0.02, K=5, r=0.09, a=0.55, ρ=1.01

Figure 10 Sensitivity of the partial derivative of the Follower's trigger function to volatility, as a function of volatility



The parameters are: μ=0.02, K=5, r=0.09, a=0.55, ρ=1.01 and x=2.

Figure 11 Sensitivity of the Follower/Leader Trigger functions to Volatility and the Immigration-death Ratio



The parameters are: μ=0.02, K=5, r=0.09, a=0.55 and x=5.