Valuation of an Irreversible Investment Involving Agents with Private Information about Stochastic Costs

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Abstract

An investor owns a right to invest in a project that generates positive cash flows when the investment is undertaken. Both the value of the future cash flows and the investment cost follow stochastic processes. Thus, the investment project takes the form of an exchange option of American type. In the paper we analyze this investment project when the investor needs an agent to undertake the investment of the project, and the agent has private information about the investment cost.

In the first part of the paper we assume that there is only one agent having private information, and the problem is analyzed within a principal-agent framework. The investor's problem is to optimize the compensation to the agent. To induce the agent to make the preferred investment decision, the investor needs to leave the agent some information rent.

In the second part we extend the model by assuming that n agents compete about the contract. Each agent has private information, and the competition is organized as an auction. We discuss how competition reduces the information rent and the inefficiency of the chosen investment strategy.

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1 Introduction

The starting point of the analyzes is a standard real option problem: An investor owns a right to invest in a project that generates positive net cash flows when the investment is undertaken. Both the net present value of the future cash flows and the investment cost follow stochastic processes. To maximize the value of the investment project, the investor aims to find the optimal time to exercise the option to invest. Thus, the investment project takes the form of an exchange option of American type.

The purpose of this paper is to analyze how the value of the real option is changed when the investor needs an expert to undertake the investment of the project, and the expert has private knowledge about the investment cost. In the first part of the paper we assume that there is only one agent having private information. Thus, we analyze the optimal stopping problem within a principal-agent framework. In the second part we assume that two or more agents compete about the contract of managing the investment. The problem is analyzed in an auction model.

Both the investor and the agents are value maximizers. If the investor had the same information as the agents, the investment problem would be optimized according to the investor's first-best preferences, and the agent who wins the contract would be compensated by the investment cost, only. However, because of the agents' private information, they can increase their value by signalling investment costs different from their true ones. Alternatively, the agents may have incentives to maximize slack in their organizations, thereby increasing the realized investment cost compared to the necessary cost. The investor's problem is how to compensate the agent in order to maximize the principal's value of the investment project. This problem amounts to finding the optimal trade-off between keeping the compensation to the agent low, and transfer some information rent to induce the winning agent to behave according to the investor's preferences.

The problems presented here are extensions of the problem in Mæland (1999). The main difference in assumptions between the two papers, is that the agent's private information is constant in Mæland (1999), whereas it is driven by a stochastic process in the models presented here.

A conclusion from the model in Mæland (1999), is that the optimal investment trigger is higher in the case of asymmetric information than under full information, and therefore may lead to under-investment. Furthermore, we find an optimal (second-best) compensation function that is increasing and concave in the stochastic value of the future cash flows. Corresponding results are found in the sections below, i.e., when the private information is stochastic.

An application of the problems discussed in Mæland (1999) and in this paper, is the case where a principal owns natural resources (say, a petroleum resource), and needs an agent to manage the investment strategy of the project. The assumption of a constant investment cost is realistic in cases where the investment consists of standard technology. In other cases the investment project has the character of being more like a development project, with new technical solutions, or frequent changes in the design of these. For such investment projects it is more realistic to assume that the agent's private information changes as time passes, as is assumed in the models to be presented below.

In the case where the private information is constant, the agent reports only once, and the investment strategy is based on that report. Thus, the agent is committed to this report. In the case where the private information changes continuously, the agent correspondingly reports continuously. In this paper we want to see how the value of the contract changes when the private information is stochastic. A question we will discuss is whether the value of private information is higher when the agent is not committed to earlier reports.

The investor cannot do better than to compensate the agent based on the optimal compensation function. This means that the investor is better off by entering into a contract as found in this paper, than to sell the option to invest. The reason is market failure because of asymmetric information: If the investor ex ante wants to sell the investment project at a price based on his expectation of the investment cost, the investor knows that if the agent accepts the price, the investor can do better by entering into a contract with the agent. If the investor's price is too high, then the agent will not buy the investment project.

In section 2 we assume that the parties consist of the investor and only one agent having private information. The problem is solved in a principal-agent model where the investor is the principal. To solve the optimization problem the principal has to find an optimal compensation function. In order to optimize the compensation function for all possible functions, we apply the *revelation principle*, see for example Salanié (1994) for a discussion of the principle. By the revelation principle, we reduce the possible set of compensation function to those where lying is not profitable. The reason we can confine attention to truthful contracts, is that for each possible contract between the principal and the agent, we can find a truthful contract with the same outcome.

Bjerksund and Stensland (2000) analyze a dynamic strategy problem within a principal-agent framework, but where the private information is constant. Parts of the analyzes in Mæland (1999) is built on the model in Bjerksund and Stensland (2000), and thus, the results in the two papers are of the same type: As the agent's private information is constant, the cost is reported only once. Second-best dynamic strategies are found. In both models the principal's value is reduced because of an extra cost parameter in the principal's optimization problem, leading to a lower value than under full information.

In Antle, Bogetoft and Stark (1996) and in MacKie-Mason (1985) the private information is privately observed by an agent at certain points in time. Antle *et al.* analyze a two-period model, where the principal decides to make an investment in period one of the two periods, or no investment at all. The only uncertainty in the model is the cost of the investment project, which changes stochastically from one period to another. Thus, the problem can be interpreted as compound options of the European type. The respective investment triggers at each of the two periods are given by two constants. Antle *et al.* find that the incentive effects from private information tend to defer investment because the investment is done at a higher cost under asymmetric information than under full information. On the other hand, increased volatility by postponing investment tend to reduce the value of waiting, thereby leading to earlier investment. The reason is that an inefficient investment trigger in the last period reduces the principal's advantage of delaying the investment to the last period.

MacKie-Mason (1985) models sequential decision problem, where the private information is given in a similar way as Antle *et al.* As in Antle *et al.* the investment trigger, and thereby the compensation, is given by constants, and the private information leads to under-investment. Also in the models to be presented below are under-investment a result of private information.

In section 3 the principal-agent model is extended to a model where two or more agents compete about obtaining the contract. The purpose is to analyze how much the investor's value of the project is increased by competition, and whether the inefficiency in the second-best investment strategy in the principal-agent model is reduced.

The competition is organized as an auction. The incorporation of competition is done by an approach similar to Laffont and Tirole (1987). They assume that the respective agents' private information is constant and formulate their model as a *second-price sealed-bid private-values auction*, also called a *Vickrey auction*. In such an auction, each bidder simultaneously submits a bid, without seeing others' bids, and the contract is given to the bidder who makes the best bid. However, the contract is priced according to the second-best bidder.

It can be shown that in a Vickrey auction, it is a dominant strategy for the bidder to bid according to his true value. Hence, we see a correspondence to the situation of no competition: truth telling is an optimal strategy for the agents participating in a Vickrey auction, as well as in a principal-agent relationship. This resemblance is emphasized in Laffont and Tirole (1987). Furthermore, the resemblance is exploited in the presentation of the auction model in our paper.

In the next section we solve the optimal stopping problem when there is only one agent. The investment problem when n agents compete about a contract is solved in section 3. Section 4 concludes the paper.

2 The principal-agent model

2.1 Problem formulation

We assume that an investor owns a possibility to invest in production of petroleum, and that he needs an agent to manage the investment of the project. We can split the risk of the project into two main parts. The first one is *market uncertainty*, which is uncertainty correlated to the activity in the economy. The second is *technical uncertainty*. An example of technical uncertainty is uncertainty due to new technical solutions of the investment of the project. The agent has private information about the technical uncertainty.

We assume that the investment cost $K(t)$ of the project is a function of an observable variable $C(t)$ and an unobservable variable $\theta(t)$. The variable $C(t)$ represents the part of the cost that is due to market uncertainty, and $\theta(t)$ is the part of the cost due to technical uncertainty. This approach is similar to the cost assumptions in Pindyck (1993), where the cost uncertainty consists of technical uncertainty defined as uncertainty of the physical difficulty of completing a project, and input cost uncertainty, which covers the uncertainty that is external to the agent.

In Pindyck (1993) it is assumed that the technical uncertainty changes only when investment occurs. Our model is, however, not directly comparable to Pindyck's, as we have not taken into consideration that investment takes time. Thus, in our model the technical uncertainty changes even if no investment occurs. We assume that the agent obtains private information about the investment cost from other sources than the investment project in the model. For example, the agent may manage other, similar investment projects as well, continuously receiving private information from these. Another example is that the agent obtains private information about technical innovations.

Formally, the part of the investment cost that is private information to the agent is given by the stochastic process,

$$d\theta(t) = \sigma_2 \theta(t) dB_2(t), \quad \theta_0 = \theta(0), \tag{1}$$

where σ_2 is the volatility parameter, and $B_2(t)$ is a standard Brownian motion. As $\theta(t)$ is a measure of technical uncertainty only, we assume that $\theta(t)$ is independent of market uncertainty.

The observable part of the investment cost may be correlated with capital markets. The risk adjusted process is given by,

$$dC(t) = (r - \delta_c)C(t)dt + \sigma_1 C(t) dB_1(t), \quad c_0 = C(0). \tag{2}$$

The parameter r denotes the risk-free rate, δ_c is the convenience yield parameter of $C(t)$, the volatility parameter is given by σ_1, and $B_1(t)$ is a standard Brownian motion that may be correlated with capital markets.

The cost variables $\theta(t)$ and $C(t)$ are both log-normal processes, and therefore the product of the variables leads to a new log-normal process. We assume that the true cost, $K(t)$ is given by the function $K(t) = C(t)\theta(t)$. To justify this function we think of $\theta(t)$ and $C(t)$ as suitably normalized "indexes" representing the market uncertainty and the technical uncertainty, respectively, and where the product of the indexes leads to the true investment cost. By Ito's Lemma we find that the stochastic process of the true investment cost is given by

$$dK(t) = (r - \delta_c)K(t)dt + \sigma_1 K(t) dB_1(t) + \sigma_2 K(t) dB_2(t), \quad k_0 = K(0). \tag{3}$$

The expected, future net cash flows from the investment project, if it were completed at time t, has a present value of $S(t)$. Information about $S(t)$ is common knowledge. Thus, the principal obtains all the cash flows when the investment is exercised and the contracted compensation is transferred to the agent. The stochastic process $S(t)$ is given by

$$dS(t) = (r - \delta_s)S(t)dt + \sigma_s S(t) dW(t), \quad S(0) \equiv s_0, \tag{4}$$

where δ_s is the convenience yield parameter of $S(t)$, σ_s is its volatility parameter, and $W(t)$ is a standard Brownian motion.

It is assumed that the Wiener processes $W(t)$ and $B_1(t)$ are independent of $B_2(t)$. The Wiener processes $W(t)$ and $B_1(t)$ may be correlated. Furthermore, it is assumed that the parties are well diversified. As the variables $S(t)$ and $C(t)$ are assumed to be spanned by capital markets, all risk can be hedged against. The variable $\theta(t)$ consists only of technical uncertainty privately known to the agent, and is uncorrelated to capital markets. Thus, the uncertainty in $\theta(t)$ is fully diversifiable.

The probability space corresponding to the Brownian motions $W(t)$, $B_1(t)$ and $B_2(t)$, starting at t_0, is given by $(\Omega, \mathcal{F}, Q)$. The agent's information is given by $\mathcal{F}_t^{S,K}$, which is the σ-algebra generated by $\{S(\xi), K(\xi), \xi \leq t\}$. The "twin assets" $S(t)$ and $C(t)$ are priced in complete markets, and the respective stochastic processes therefore can be adjusted to the risk adjusted measure $Q(t)$. The part of the cost with price equal to $\theta(t)$ is uncorrelated with capital markets. As long as the parties are well diversified, as assumed above, the price process of $\theta(t)$ is the same under the (true) P and the (risk adjusted) Q measures.

The principal's information is formalized by the σ-algebra $\mathcal{F}_t^{S,C}$, generated by $\{S(\xi), C(\xi), \xi \leq t\}$. The principal knows the distribution of $\theta(t)$ at any time $t \geq t_0$.

Although we restrict our analysis to the case of geometric Brownian motions, we do not need to make this restriction. The main result (given by (21)) is reached also when we assume that the stochastic processes are given by general Itô processes, and the true investment cost $K(t)$ is given by a function $K(t) = h(C(t), \theta(t))$. However, an assumption we need to make is that the fraction $F(K(t))/f(K(t))$ is increasing in $K(t)$, where $F(\cdot)$ is the cumulative distribution function of $K(t)$, and $f(\cdot)$ is the corresponding density function. The reason we assume that the stochastic processes are log-normal, is that it is then easier and more informative to compare our results to the well-known case of no private information. When the agent has no private information, the option problem is identical to an exchange option of American type, cf. section 2.2.

We assume that the optimal stopping problem is delegated to the agent. Thus, we want to find a compensation function that induces the agent to behave in the preferred way, and at the least cost to the principal. It may seem to be a difficult problem to find an optimal *function*. However, the revelation principle helps us at the task of capturing the set of possible compensation functions. The revelation principle is based on the observation that to each set of implementable *incentive mechanisms*, a contract with the same outcome can also be implemented through a *direct truthful mechanism* where the agent reveals his private information. By an incentive mechanism, we mean the tools the principal employ in order to induce the agent to behave in a certain way. In our model the incentive mechanisms are given by the compensation function and the observable investment strategy. By a direct mechanism we mean the agent's report of his private information.

As the optimal stopping problem is delegated to the agent, the reports from the agent to the principal is just a device of finding the optimal investment strategy, and hence finding the optimal contract. Moreover, given truthful reports, it does not matter which party decides on the investment strategy, as the same outcome will occur. Our aim is to find a compensation function where communication between the principal and the agent is not necessary, and to find a contract that is as good as any contract in which the agent communicates the private information to the principal.

The compensation $X(\cdot)$ is transferred at the time when the investment is made, and may be based on the agent's report at the stopping time. The agent's report is denoted $\hat{K}$.

Define $t_0 = 0$, corresponding to time equal to zero, as the time when the contract is entered into. Furthermore, let $(s, c, k) = (S(t), C(t), K(t))$, and let t be any time equal to or larger than t_0. Then the principal's optimization problem is given by

$$\begin{aligned} V^P(s_0, c_0, t_0) &= \sup\nolimits_{X(\cdot)} v^P(s_0, c_0, t_0) \\ &= \sup\nolimits_{X(\cdot)} E\left[e^{-r\tau}\left(S(\tau) - X\left(S(\tau), C(\tau), \tau; \hat{K}\right)\right)\Big| \mathcal{F}_{t_0}^{S,C}\right], \end{aligned} \tag{5}$$

subject to the agent's optimization function

$$\begin{aligned} V^A(s, c, k, t; \hat{K}) &= \sup\nolimits_{\tau, \hat{K}} v^A(s, c, k, t) \\ &= \sup\nolimits_{\tau, \hat{K}} E\left[e^{-r(\tau - t)}\left(X\left(S(\tau), C(\tau), \tau; \hat{K}\right) - K(\tau)\right)\Big| \mathcal{F}_t^{S,K}\right], \end{aligned} \tag{6}$$

and the participation constraint

$$V^A(s, c, k, t; \hat{K}) \geq 0. \tag{7}$$

The principal's problem in (5) is to find an optimal compensation function, subject to the agent's optimization problem in (6), and the participation constraint in (7). The compensation function $X(\cdot)$ is specified at time t_0, when the parties enter into a contract. Therefore, the principal's optimization problem with respect to the compensation function $X(\cdot)$ must be solved for at (or before) time t_0. On the other hand, the agent's optimization problem is dynamic: At any time $t \geq t_0$, the agent must decide on whether to invest or not, and which report he is to give to the principal at the investment time. The agent decides on the optimal stopping time $\tau_{\hat{K}}$. The optimal stopping time is based on the report given at the stopping time, $\hat{K}$, i.e., the optimal stopping time is defined by

$$\tau_{\hat{K}} = \inf\left\{t \geq t_0 | V^A\left(S(t), C(t), K(t), t; \hat{K}\right) > X\left(S(t), C(t), t; \hat{K}\right) - K(t)\right\}.$$

The agent must choose an optimal stopping time that is consistent with his report $\hat{K}$. Otherwise, the principal will detect that the time the agent chooses to invest is not optimal given the report $\hat{K}$.

We have included t as a variable that may affect the respective value functions of the principal and the agent. The reason is that we take into consideration that the compensation function $X(\cdot)$ may be time dependent.

As the agent continuously obtains new information, application of the revelation principle implies that the agent correspondingly continuously gives new reports to the principal. However, in the formulation of the problem the compensation function is not based on earlier reports. As long as the agent reports costs higher than the costs at which the parties find it optimal to exercise the option to invest, the value of the agent's compensation will not be dependent on the reports.

In section 2.2 we present the investment problem in the case where the agent has no private information. The case of full information is used as a benchmark when we analyze the effects of private information.

2.2 Full information

In the case of full information, i.e., where the principal too observes the stochastic process $K(t)$, the agent's value of the contract is zero. The reason is that the principal has the same information as the agent, in addition to observing the agent's investment strategy. Thus, the principal can design the contract in such a way as to punish the agent if he does not act in the way preferred by the principal. Following this argumentation, the participation constraint is binding, and the optimal transfer function under full information is given by $X(t) = K(t)$ if the investment is made at time t, and zero otherwise.

Under full information the principal's optimization problem is time homogeneous. Therefore, the principal's optimization problem is given by

$$V^P_{sym}(s,k) = \sup_{\tau} E\left[e^{-r\tau} \left(S(\tau) - K(\tau) \right) \middle| \mathcal{F}^{S,K}_t \right], \tag{8}$$

where the subscript *sym* indicates that this is the principal's value of the contract when information is symmetric.

The optimization problem has the form of an exchange option (sometimes called a Margrabe option, as Margrabe (1978) analyzed European options to exchange one asset for another). In the case where both $S(t)$ and $K(t)$ are geometric Brownian options, the option problem in equation (8) is, among others, solved by McDonald and Siegel (1986), Gerber and Shiu (1996) and Øksendal and Hu (1996).

Øksendal and Hu (1996) state all the necessary conditions with respect to the parameters of the problem, needed for the validity of the result. Thus, we choose to present the results as stated in Øksendal and Hu (1996).

The trigger for investment is given by a linear relationship between $S(t)$ and $K(t)$, $S(t) = \mu K(t)$, where μ is a constant. The solution to equation (8) is given by

$$V^P_{sym}(s,k) = \begin{cases} A s^{\lambda} k^{1-\lambda} & \text{when } s \leq \mu k \\ \\ s - k & \text{when } s > \mu k, \end{cases} \tag{9}$$

where

$$A = \frac{1}{\lambda} \left(\frac{\lambda}{\lambda - 1} \right)^{1-\lambda},$$

and

$$\mu = \frac{\lambda}{\lambda - 1}.$$

Furthermore, we have

$$\lambda = \begin{cases} \frac{1}{a}\left[\frac{1}{2}a + \delta_s - \delta_c + \sqrt{\left(\frac{1}{2}a + \delta_s - \delta_c \right)^2 + 2a\delta_c} \right] & \text{if } a > 0 \\ \\ \frac{\delta_c}{\delta_c - \delta_s} & \text{if } a = 0, \end{cases}$$

where $a = \sigma_s^2 - 2\rho\sigma_s\sigma_1 + (\sigma_1^2 + \sigma_2^2)$, and ρ is the correlation coefficient between $W(t)$ and $B_1(t)$. We need to ensure that $\lambda > 1$, which leads to the following restrictions,

$$\begin{cases} \delta_c + \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 \right) \geq \delta_s + \frac{1}{2}\sigma_s^2 & \text{if } a > 0 \\ \\ \delta_c > \delta_s & \text{if } a = 0. \end{cases}$$

The principal's value function in (9) equals the principal's value of the contract. Below we shall see how the principal's contract value, as well as the optimal investment strategy, is changed when the agent has private information about the investment cost.

2.3 The agent's optimization problem

In this section we analyze the agent's optimization problem in equation (6), and characterize the agent's value of private information.

Following Salanié (1994), section 2.1.2, the revelation principle implies that we can "confine attention to mechanisms that are both *direct* (where the agent reports his information) and *truthful* (so that the agent finds it optimal to announce the true value of his information)". The principal's set of tools to induce the agent to behave in a certain way is given by the incentive mechanisms $(X(\hat{K}), \tau_{\hat{K}})$. If this set of mechanisms can be implemented, then we can implement these incentive mechanisms through a direct truthful mechanism, $(X(\hat{K}), \tau_{\hat{K}}, \hat{K})$, where the agent reveals his private information. Thus, we can find a direct truthful mechanism $(X(\hat{K}), \tau_{\hat{K}}, \hat{K})$ with the same outcome as for any incentive mechanisms $(X(\hat{K}), \tau_{\hat{K}})$. Applied to our model, the revelation principle is shown in the appendix, section A.1.

Guess on a trigger function $\psi(c, t; \hat{K})$ such that the option to invest is exercised immediately when $s > \psi(c, t; \hat{K})$, whereas it is optimal to wait when $s \leq \psi(c, t; \hat{K})$. This means that the optimal stopping time is given by

$$\tau = \inf \left\{ t \geq t_0 | S(t) > \psi(C(t), t; \hat{K}) \right\}.$$

Note that the investment strategy is based on the agent's report $\hat{K}$. Furthermore, define $w^A(s, c, k, t; \hat{K})$ as the agent's value function when $s \leq \psi(c, t; \hat{K})$. The agent's value function can then be expressed as

$$v^A(s, c, k, t; \hat{K}) = \begin{cases} w^A(s, c, k, t; \hat{K}) & \text{if } s \leq \psi(c, t; \hat{K}) \\ \\ X(s, c, t; \hat{K}) - k & \text{if } s > \psi(c, t; \hat{K}). \end{cases} \tag{10}$$

At any time t the agent's truth telling condition is given by the first-order condition

$$\left. \frac{\partial v^A(s, c, k, t; \hat{K})}{\partial \hat{K}} \right|_{\hat{K}=k} = \begin{cases} \left. \frac{\partial w^A(s,c,k,t;\hat{K})}{\partial \hat{K}} \right|_{\hat{K}=k} = 0 & \text{if } s \leq \psi(c, t; \hat{K}) \\ \\ \left. \frac{\partial X(s,c,t;\hat{K})}{\partial \hat{K}} \right|_{\hat{K}=k} = 0 & \text{if } s > \psi(c, t; \hat{K}). \end{cases} \tag{11}$$

We now emphasize that the report $\hat{K}$ is dependent on the true investment cost k, i.e., $\hat{K} = \hat{K}(k)$. By the envelope theorem we find that[1]

$$\frac{dv^A(s,c,k,t;\hat{K}(k))}{dk} = \begin{cases} w_k^A(s,c,k,t;\hat{K}(k)) & \text{if } s \leq \psi(c,t;\hat{K}(k)) \\ \\ -1 & \text{if } s > \psi(c,t;\hat{K}(k)), \end{cases} \tag{12}$$

where $w_k^A(s,c,k,t;\hat{K})$ is defined as the derivative of $w^A(s,c,k,t;\hat{K})$ with respect to k.

We may observe that the optimization of the report $\hat{K}$ at the time of investment corresponds to an impulse in an impulse control problem.

From (11) and (12), we see that no contract that depends on the report $\hat{K}$ dominates a contract that is independent of the report. Thus, we find that $X = X(s,c,t)$.

Although dependence of the report $\hat{K}$ does not improve the contract, the agent's private information still is of value. To induce the agent to choose the principal's preferred investment strategy, the agent must be compensated according to the value of his private information.

Let $\psi(c,k,t)$ be the investment strategy when truth telling is optimal. Define $w^A(s,c,k,t)$ as the agent's value function when $s \leq \psi(c,k,t)$ and when truth telling is the optimal strategy. Then the agent's value function, given truth telling, can be written in the form

$$v^A(s,c,k,t) = \begin{cases} w^A(s,c,k,t) & \text{if } s \leq \psi(c,k,t) \\ \\ X(s,c,t) - k & \text{if } s > \psi(c,k,t). \end{cases} \tag{13}$$

Let $\vartheta(s,c,t)$ be the inverse trigger function, i.e., it is optimal to invest when $k < \vartheta(s,c,t)$, and optimal to wait when $k \geq \vartheta(s,c,t)$. We find the agent's value of private information by integrating both sides of the first-order condition in (12) with respect to the private information k. This leads to

$$v^A(s,c,k,t) = \begin{cases} -\int_k^\infty w_u^A(s,c,u,t)du & \text{if } s \leq \psi(c,k,t) \\ \\ \vartheta(s,c,t) - k - \int_{\vartheta(s,c,t)}^\infty w_u^A(s,c,u,t)du & \text{if } s > \psi(c,k,t) \end{cases} \tag{14}$$

Using equations (13) and (14) we find that the compensation function is characterized by

$$X(s,c,t) = \vartheta(s,c,t) - \int_{\vartheta(s,c,t)}^\infty w_u^A(s,c,u,t)du, \tag{15}$$

[1] $\frac{dv^A(s,c,k,t;\hat{K}(k))}{dk} = \frac{\partial v^A(s,k,c,t;\hat{K}(k))}{\partial \hat{K}(k)} \frac{d\hat{K}(k)}{dk} + \frac{\partial v^A(s,c,k,t;\hat{K}(k))}{\partial k}$. The first term on the right-hand side is zero when $\hat{K}(k)$ is optimal.

when $s > \psi(c, k, t)$, and otherwise $X(s, c, t) = 0$. We see that the compensation function is dependent only on variables observable to the principal. However, so far we do not know the investment strategy given by $\vartheta(s, c, t)$, nor the function $w^A(\cdot)$. The investment strategy is optimized by the principal, given the agent's truth telling conditions. This problem is analyzed in the section below.

2.4 The principal's optimization problem

The principal's optimization problem in (5) is rewritten here as,

$$V^P(s_0, c_0, t_0) = \sup_X E\left[e^{-r\tau} g^P(S(\tau), C(\tau), \tau) \middle| \mathcal{F}_{t_0}^{S,C}\right], \tag{16}$$

where

$$g^P(s, c, t) = S(t) - X(s, c, t). \tag{17}$$

The principal's problem is to implement an optimal compensation function. In order to optimize the principal's payoff value with respect to an optimal stopping time, we need to replace the unknown compensation function with known functions. The value of the compensation function $X(\cdot)$ is characterized in (15). Now suppose that the option to invest is exercised at time t, i.e., that $k = \vartheta(s, c, t)$ at time t. Then replace $X(s, c, t)$ in the principal's value of the payoff, by the right-hand side of (15), leading to

$$\begin{aligned} g^P(s, c, t) &= E\left[s - \left(\vartheta(s, c, t) - \textstyle\int_{\vartheta(s,c,t)}^{\infty} w_u^A(s, c, u, t)du\right) \middle| \mathcal{F}_t^{S,C}\right] \\ &= E\left[s - \left(k - \textstyle\int_k^{\infty} w_u^A(s, c, u, t)du\right) \middle| \mathcal{F}_t^{S,C}\right] \\ &= \textstyle\int_0^{\infty} \left(s - k + \int_k^{\infty} w_u^A(s, c, u, t)du\right) f(k|c, t)dk. \end{aligned} \tag{18}$$

The function $f(k|c, t)$ is the probability density of k given the principal's information about k.

By partial integration[2] we find that the right-hand side of (18) can be formulated

[2]Partial integration of $\int_0^{\infty} \int_k^{\infty} w_u^A(s, c, u, t)duf(k|c, t)dk$ leads to

$$\begin{aligned} &\int_0^{\infty} \int_k^{\infty} w_u^A(s, c, u, t)duf(k|c, t)dk \\ &= \left[\textstyle\int_k^{\infty} w_k^A\,(s, c, k, t)\,duF(k|c, t)\right]_0^{\infty} - (-) \textstyle\int_0^{\infty} w_k^A(s, c, k, t)F(k|c, t)dk \\ &= \textstyle\int_0^{\infty} w_k^A(s, c, k, t)F(k|c, t)dk. \end{aligned}$$

by

$$g^P(s,c,t) = \int_0^\infty \left(s - k + w_k^A(s,c,k,t)\frac{F(k|c,t)}{f(k|c,t)} \right) f(k|c,t)dk. \tag{19}$$

The function $F(k|c,t)$ is defined as the cumulative distribution of the investment cost k.

A condition in the agent's optimal stopping problem is that the first-order derivative of v^A must be continuous for all the variables included in the problem[3]. In particular, we need to check that the value function is continuous at the trigger where the investment is exercised. Derivation of the value function in (13) at the trigger where $k = \vartheta(s,c,t)$ leads to the following condition

$$w_k^A(s,c,\vartheta(s,c,t),t) = -1.$$

This condition is often called the "smooth pasting condition" or the "high contact principle". Under the assumption that the investment is made at time t, we use the smooth pasting condition to replace $w_k^A(s,c,k,t)$ in (19) by -1, leading to

$$g^P(s,c,t) = \int_0^\infty \left(s - k - \frac{F(k|c,t)}{f(k|c,t)} \right) f(k|c,t)dk. \tag{20}$$

Thus, we have reformulated the principal's payoff value so that it consists of known functions, only, and at the same time is satisfied with respect to the agent's truth telling condition.

The time zero discounted value of the payoff, under the assumption that the investment is made at time t, is given by

$$\begin{aligned} e^{-rt}g^P(s,c,t) &= e^{-rt}\int_0^\infty \left(s - k - \tfrac{F(k|c,t)}{f(k|c,t)} \right) f(k|c,t)dk \\ &= \int_0^\infty e^{-rt}\left(s - k - \tfrac{F(k|c,t)}{f(k|c,t)} \right) f(k|c,t)dk \\ &= E\left[e^{-rt}\left(s - k - \tfrac{F(k|c,t)}{f(k|c,t)} \right) |\mathcal{F}_t^{S,C} \right]. \end{aligned}$$

The agent's incentive compatibility restriction is now incorporated in the principal's value of the payoff. This means that the agent has no incentives not to report the true investment cost. Thus, the principal finds the optimal investment strategy by

[3] The condition is for instance stated in Øksendal (1998), Theorem 10.4.1, where the verification theorem for variational inequalities for optimal stopping is given.

solving the following optimal stopping problem,

$$\begin{aligned} V^P(s_0,c_0,t_0) &= \sup_\tau E\left[e^{-r\tau}\left(S(\tau)-K(\tau)-\tfrac{F(K(\tau)|C(\tau),\tau)}{f(K(\tau)|C(\tau),\tau)}\right)\Big|\,\mathcal{F}^{S,C}_{t_0}\right] \\ &= \sup_\tau E\left[E\left[e^{-r\tau}\left(S(\tau)-K(\tau)-\tfrac{F(K(\tau)|C(\tau),\tau)}{f(K(\tau)|C(\tau),\tau)}\right)\Big|\,\mathcal{F}^{S,K}_{t_0}\right]\Big|\,\mathcal{F}^{S,C}_{t_0}\right], \end{aligned} \tag{21}$$

as if he knows the unobservable variable k_0.

If we compare the result in (21) to the case where the principal has full information, given by (8), we see that the principal's payoff value is reduced by the fraction $\frac{F(\cdot)}{f(\cdot)}$. The fraction is interpreted as the inefficiency due to asymmetric information. Thus, the relationship between output value $S(t)$ and the investment cost $K(t)$ is not linear as in the case of full information, see section 2.2, equation (9). Also, the inefficiency leads to under-investment: Because of the reduced payoff the stochastic process $S(t)$ must have a higher value, and/or the variables $C(t)$ and $K(t)$ must be lower, to trigger investment in the case of asymmetric information compared to the case of full information.

Let $\tilde{v}^P(s,c,k,t)$ be the principal's value function given that the principal's information is given by $\mathcal{F}^{S,K}_t$,

$$\tilde{v}^P(s_0,c_0,k_0,t_0) = E\left[e^{-r\tau}\left(S(\tau)-K(\tau)-\tfrac{F(K(\tau)|C(\tau),\tau)}{f(K(\tau)|C(\tau),\tau)}\right)\Big|\,\mathcal{F}^{S,K}_{t_0}\right]. \tag{22}$$

We find the optimal investment strategy by optimizing equation (22) with respect to the optimal stopping time. The optimal solution must satisfy the variational inequalities (Øksendal (1998), Theorem 10.4.1):

$$\tilde{v}^P(s_0,c_0,k_0,t_0) \geq \tilde{g}^P(s_0,c_0,k_0,t_0) \tag{23}$$

$$L\tilde{v}^P(s_0,c_0,k_0,t_0) \leq 0 \tag{24}$$

$$\max\left\{L\tilde{v}^P(s_0,c_0,k_0,t_0),\tilde{g}^P(s_0,c_0,k_0,t_0)-\tilde{v}^P(s_0,c_0,k_0,t_0)\right\}=0, \tag{25}$$

where $\tilde{g}^P(s,c,k,t)$ is given by

$$\tilde{g}^P(s,c,k,t) = s-k-\frac{F(k|c,t)}{f(k|c,t)}.$$

The differential operator L which coincides with the generator A of the states $\{s,c,k,t\}$ is given by

$$\begin{aligned} L\tilde{v}^P(s,c,k,t) &= \tfrac{\partial \tilde{v}^P}{\partial t}+(r-\delta_s)s\tfrac{\partial \tilde{v}^P}{\partial s}+\tfrac{1}{2}\sigma_s^2 s^2\tfrac{\partial^2 \tilde{v}^P}{\partial s^2} \\ &\quad +(r-\delta_c)k\tfrac{\partial \tilde{v}^P}{\partial k}+\tfrac{1}{2}(\sigma_1^2+\sigma_2^2)k^2\tfrac{\partial^2 \tilde{v}^P}{\partial k^2} \\ &\quad +(r-\delta_c)c\tfrac{\partial \tilde{v}^P}{\partial c}+\tfrac{1}{2}\sigma_1^2 c^2\tfrac{\partial^2 \tilde{v}^P}{\partial c^2}+\rho\sigma_s\sigma_1 sc\tfrac{\partial \tilde{v}^P}{\partial s\partial c}, \end{aligned} \tag{26}$$

where ρ is the correlation coefficient between the standard Brownian motions $W(t)$ and $B_1(t)$.

There is no analytical solution to the problem in (22), and the optimal investment strategy must be solved numerically. However, we guess that as $\frac{\partial(F(k|c,t)/f(k|c,t))}{\partial k} > 0$ and $\frac{\partial^2(F(k|c,t)/f(k|c,t))}{\partial k^2} > 0$, the optimal investment trigger $\psi^*(c,k,t)$ is strictly increasing and convex in k, i.e., $\frac{\partial \psi^*}{\partial k} > 0$ and $\frac{\partial^2 \psi^*}{\partial k^2} > 0$. Thus, the higher k is, the more severe is the inefficiency in the investment strategy compared to the full information case (remember that in the full information situation the relationship between $S(t)$ and $K(t)$ is linear).

The principal's unconstrained optimization problem in (21) has the same form as the principal's optimization problem when the private information is constant: the principal's expected payoff consists in both cases of the net present value from future expected cash flows minus the true investment cost and the fraction $\frac{F(\cdot)}{f(\cdot)}$. The only difference is that in the case of a constant private information, the fraction is a constant, too.

It may be surprising that the principal's expected value of the payoff is of the same type, because there seems to be an important difference between the case where the agent's private information is stochastic, and the case where it is constant. As the private information does not change in the first case, the agent is committed to the same report during the contracting time. However, when the private information changes stochastically, the agent continuously submits new reports without committing to earlier reports (given that the contract does not depend on reports earlier than the one at the investment time). Intuitively, one may therefore be led to believe that this gives the agent a higher value of his private information compared to the case where the private information is constant. The principal's payoff function above (equation (19)) shows that this is not the case.

One reason for the fact that the agent's value of private information is of the same form whether it is constant or stochastically changing, is that in both cases we find a contract where the investment strategy is delegated to the agent. Thus, communication has no value. For a discussion of the value of communication versus delegation, see Melumad and Reichelstein (1987) and (1989).

As mentioned earlier, the result in (21) is not confined to stochastic processes given by the geometric Brownian motion. We reach the same result if we assume that the processes are given by general Itô processes. However, a restriction we need to make is that $\frac{F(k|c,t)}{f(k|,c,t)}$ is increasing in k.

Observe that in the case where the stochastic processes are given by geometric Brownian motions, we can reformulate the principal's value function to

$$V^P(s_0, c_0, t_0) = \sup_{\tau} E\left[e^{-r\tau} \left(S(\tau) - C(\tau) \left(\frac{F(\theta(\tau), \tau)}{f(\theta(\tau), \tau)} - \theta(\tau) \right) \right) \middle| \mathcal{F}_{t_0}^{S,C} \right]. \quad (27)$$

Thus, we see that there is a linear relationship between $S(t)$ and $C(t)$. This result is consistent to the linear symmetric information case in subsection 2.2. The reason is that both $S(t)$ and $C(t)$ are observable to the principal. However, the part of the investment cost that is not observable to the principal, is not linearly dependent on $S(t)$ and $C(t)$.

2.5 Implementation of the optimal investment strategy

The optimal investment strategy ψ^* found from equation (21) must be implemented into the compensation function. Let $\vartheta^*(s, c, t)$ be the inverse trigger function, i.e., it is optimal to invest when $k < \vartheta^*(s, c, t)$, and optimal to wait when $k \geq \vartheta^*(s, c, t)$. From (15) we find that the optimal compensation function is given by

$$X^*(s, c, t) = \vartheta^*(s, c, t) - \int_{\vartheta^*(s,c,t)}^{\infty} w_u^{A*}(s, c, u, t) du \quad (28)$$

if the investment is made at time t, and $X^*(s, c, t) = 0$ otherwise. The function w^{A*} is the optimal value when the optimal investment strategy is followed.

As $\vartheta^*(s, c, t)$ is the inverse trigger function of $\psi^*(c, k, t)$ (keeping c and t fixed), we guess that it is concavely increasing in s, i.e., $\frac{\partial \vartheta^*}{\partial s} > 0$ and $\frac{\partial^2 \vartheta^*}{\partial s^2} < 0$. Correspondingly, we find that the optimal compensation function is increasing and concave in the output value s. As s gets higher, the agent's value of the contract gets higher. Hence, even if the agent obtains a lower share of the output value as s increases, he will at a certain trigger be induced to exercise the option to invest.

We need to verify that the compensation function in (28) is the optimal and incentive compatible solution for the agent. This can be done using the verification theorem (see for instance Øksendal (1998), Theorem 10.4.1)

3 An auction model: Competition when the agents' private information changes stochastically

3.1 Problem formulation

Now we introduce competition in the model. We assume that n agents compete about the contract. The agents' respective private information is given by independent stochastic processes.

We organize the model as an auction of the Vickrey type. As described in the introduction, in a Vicrey auction each agent simultaneously reports their respective costs, without seeing each other's bids, and the contract is given to the agent who reports the lowest cost. In our problem, the cost variables change continuously and stochastically, and therefore the basic auction model is slightly changed: New reports are given simultaneously and continuously by all the agents participating in the auction, until one or more agents report a cost low enough to trigger investment. At this point in time, the agent with the lowest cost report wins the contract, and invests immediately. The investor offers a predetermined compensation to the agents participating in the auction.

It is not optimal to assign the contract to any of the agents before the time of investment. The reason is that the investment cost is given by a stochastic process that is independent of time. This means that the agent reporting the lowest cost at one point in time, does not necessarily have the lowest cost at a later point in time. Therefore, we assume that all the agents continuously participate in the auction, until a cost is reported that is low enough to trigger immediate investment.

The main difference from the assumptions of the principal-agent model is that there are now n agents competing about the contract. Hereafter the investor will be called the auctioneer (and he is the same person as the principal in last section).

We assume that each agent has private information about the part of the investment cost that is due to technical uncertainty, $\theta^i(t)$. The variable $\theta^i(t)$ is independently distributed between the agents. Each agent i's variable $\theta^i(t)$ is given by the stochastic process

$$d\theta^i(t) = \sigma_2\theta^i(t)dB^i(t), \quad \theta_0^i = \theta^i(0). \tag{29}$$

All the agents face the same volatility parameter σ_2. The investment cost of each agent i is given by the function $K^i(t) = C(t)\theta^i(t)$, with the stochastic process

$$dK^i(t) = (r-\delta_c)K^i(t)dt + \sigma_1 K^i(t)dB_1(t) + \sigma_2 K^i(t)dB_2^i(t), \quad k_0^i = K^i(0). \tag{30}$$

Each agent i receives the compensation X^i. Analogously to the situation where there is no competition, the compensation is paid at the time the investment is made.

The vector of reports is denoted $\hat{K} = (\hat{K}^1, ..., \hat{K}^n)$. At the time the investment is made, agent i has a probability $y^i(\hat{K}(t))$ of winning the action, where $\hat{K}(t) = (\hat{K}^1(t), ..., \hat{K}^n(t))$ is the vector of reports at time t. We make the assumption

$$\textstyle\sum_{i=1}^n y^i(\hat{K}(t)) \leq 1 \quad \text{for any } \hat{K}(t),\, t \geq t_0. \tag{31}$$

Furthermore, we need

$$y^i(\hat{K}(t)) \geq 0 \quad \text{for any } \hat{K}(t),\, t \geq t_0. \tag{32}$$

The optimal stopping time of agent i is denoted $\tau^i_{\hat{K}}$. The subscript $\hat{K}$ indicates that the optimal stopping time of each agent may be dependent on the vector of all the reports at the stopping time.

The set of incentive mechanisms $(X^i(\cdot), \tau^i, y^i(\cdot))$ constitute at each point in time the auctioneer's set of incentive mechanisms.

In the auction model we assume that $(\Omega, \mathcal{F}, Q)$ is the probability space corresponding to the $(2+n)$-dimensional Brownian motion $(W(t), B_1(t), B_2(t))$, where $B_2(t)$ is the n-dimensional Brownian process given by $B_2(t) = B_2^1(t), ..., B_2^n(t)$. Agent i observes the variables $S(t)$, $C(t)$ and $K^i(t)$. Agent i does not observe the vector $K^{-i}(t) = (K^1(t), ..., K^{i-1}(t), K^{i+1}(t), ..., K^n(t))$, but he knows the expectation and the variance of the competitors' private information. We denote $\mathcal{F}_t^{S,K^i}$ as the σ-algebra generated by $\{S(\xi), K^i(\xi), \xi \leq t\}$.

As earlier we denote $s = S(t)$, $c = C(t)$ and $k^i = K^i(t)$.

Agent i's value function is

$$\begin{aligned} & v^i(s, c, k^i, t; \hat{K}^i) \\ = \quad & E\left[e^{-r(\tau^i_{\hat{K}} - t)} \left(X^i\left(S(\tau^i_{\hat{K}}), C(\tau^i_{\hat{K}}), \tau^i_{\hat{K}}; \hat{K}^i\right) - y^i(\hat{K}) K^i(\tau^i_{\hat{K}})\right) \middle| \mathcal{F}_t^{S,K^i}\right]. \end{aligned} \tag{33}$$

The auctioneer's information at time t is given by the σ-algebra $\mathcal{F}_t^{S,C}$. This means that the auctioneer does not observe the vector $K(t) = (K^1(t), ..., K^n(t))$. However, he knows the expectation and the variance of each agent's private information $K^i(t)$. The auctioneer's value function at time t_0 is given by

$$v^P(s_0, c_0, t_0) = E\left[\sum_{i=1}^n e^{-r\tau^i_{\hat{K}}} \left(y^i(\hat{K}) S(\tau^i_{\hat{K}}) - X^i\left(S(\tau^i_{\hat{K}}), C(\tau^i_{\hat{K}}), \tau^i_{\hat{K}}; \hat{K}\right)\right) \middle| \mathcal{F}_{t_0}^{S,C}\right]. \tag{34}$$

We see that the auctioneer's and the agents' respective value functions are similar to the ones in equations (5) and (6).

The auctioneer's optimization problem is given by

$$V^P(s_0, c_0, t_0) = \sup_{X^i(\cdot), y^i(\cdot), \tau^i} v^P(s_0, c_0, t_0), \tag{35}$$

subject to all the n agents' optimization problems, where agent i has the following optimization problem,

$$V^i(s, c, k^i, t; \hat{K}^i) = \sup_{\hat{K}^i} v^i(s, c, t; \hat{K}^i), \tag{36}$$

and the participation constraint of agent i is given by

$$V^i(s, c, k^i, t; \hat{K}^i) \geq 0. \tag{37}$$

In the principal-agent problem, we want to find a contract where the investment decision is delegated to the agent. In the auction model, the investment strategy cannot be delegated. The reason is that the winner of the contract is chosen at the same time as the investment is made.

3.2 The agents' optimal reporting strategies

As in the principal-agent model, truth telling is a dominant strategy in the auction. Thus, we follow the same approach as for the principal-agent model when we analyze each agent's optimal reporting strategy.

Define $g^i(s, c, k^i, t; \hat{K}^i)$ as the value of agent i's payoff if the investment is made at time t, i.e,

$$g^i(s, c, k^i, t; \hat{K}^i) = E\left[X^i(s, c, t; \hat{K}) - y^i(\hat{K})k^i | \mathcal{F}_t^{S,K^i}\right] \tag{38}$$

As in the one-agent case, only reports at the investment time affect the compensation function. Thus, incentive compatibility requires that all the n agents' payoff values satisfies the following first order condition,

$$\left.\frac{\partial g^i(s, c, k^i, t; \hat{K})}{\partial \hat{K}^i}\right|_{\hat{K}^i = k^i} = E\left[X_{\hat{K}^i}(s, c, t; \hat{K}) - y^i_{\hat{K}^i}(\hat{K})k^i | \mathcal{F}_t^{S,K^i}\right] = 0 \tag{39}$$

where the subscripts of $X^i_{\hat{K}^i}$ and $y^i_{\hat{K}^i}$ denote the first-order derivatives of the respective functions.

Let $g^i(s,c,k^i,t)$ be the value function of agent i when truth telling is optimal, and k is defined as the vector of the agents' true investment cost at time t, $k = (k^1, ..., k^n)$. Then we state agent i's payoff value when truth telling is optimal by

$$g^i(s,c,k^i,t) = E\left[X^i(s,c,k,t) - y^i(k)k^i|\mathcal{F}_t^{S,K^i}\right] \tag{40}$$

By the envelope theorem, agent i first order condition is then written as

$$\frac{dg^i(s,c,k^i,t)}{dk^i} = E\left[-y^i(k)|\mathcal{F}_t^{S,K^i}\right]. \tag{41}$$

The truth telling condition in (41) is similar to the truth telling condition in the principal-agent model. The main difference is that the condition in (41) includes the probability $y^i(k)$, which implies that we cannot exclude the possibility that the optimal expected compensation is dependent on agent i or the competing agents' reports.

3.3 The auctioneer's optimization problem

To solve the auctioneer's optimization problem, we incorporate the n agents' truth telling restrictions into the auctioneer's optimization problem in (35)-(37). The approach is similar to the one in the principal-agent model, subsection 2.4.

When the agents' truth telling restrictions are satisfied, the auctioneer's value function can be formulated as

$$v^P(s_0,c_0,t_0) = E\left[\sum_{i=1}^{n} e^{-r\tau_K^i} g_i^P\left(S(\tau_K^i), C(\tau_K^i), \tau_K^i\right)\middle| \mathcal{F}_{t_0}^{S,C}\right], \tag{42}$$

where $g_i^P(s,c,t)$ equals

$$g_i^P(s,c,t) = E\left[y^i(k)s - X^i(s,c,k,t)|\mathcal{F}_t^{S,C}\right]. \tag{43}$$

The function $g_i^P(s,c,t)$ is interpreted as agent i's contribution to the auctioneer's payoff value. We replace the compensation function in (43) with agent i's value of private information given investment at time t. Thus, using equation (40), equation

(43) is reformulated to

$$
\begin{aligned}
g_i^P(s,c,t) &= E\left[y^i(k)s - X^i(s,c,k,t)|\mathcal{F}_t^{S,C}\right] \\
&= E\left[E\left[y^i(k)s - X^i(s,c,k,t)|\mathcal{F}_t^{S,K^i}\right]|\mathcal{F}_t^{S,C}\right] \\
&= E\left[E\left[y^i(k)\left(s-k^i\right) - g^i(s,c,k^i,t)|\mathcal{F}_t^{S,K^i}\right]|\mathcal{F}_t^{S,C}\right] \\
&= E\left[y^i(k)\left(s-k^i\right) - g^i(s,c,k^i,t)|\mathcal{F}_t^{S,C}\right].
\end{aligned}
\tag{44}
$$

Define $Y^i(k^i) = E\left[y^i(k)|\mathcal{F}^{S,K^i}\right]$. Since y^i is linearly dependent on g_i^P we see that we can simplify equation (44) to being dependent on the uncertainty with respect to k^i, only, i.e.,

$$
g_i^P(s,c,t) = E\left[Y^i(k^i)\left(s-k^i\right) - g^i(s,c,k^i,t)|\mathcal{F}_t^{S,C}\right]. \tag{45}
$$

The next step in finding an unconstrained optimization problem for the auctioneer, is to incorporate an expression of agent i's value of private information into the auctioneer's optimization problem. Correspondingly to the definition of $w^A(\cdot)$ in the principal-agent model, we now define $w^i(s,c,k^i,t)$ as the value of the option when agent i does not find it optimal to report a cost low enough to trigger immediate investment. Suppose that the investment is exercised at time t, by agent i with the probability of $Y^i(k^i)$. From the principal-agent model we know that each agent's value of private information can be expressed as

$$
g^i(s,c,k^i,t) = -\int_{k^i}^{\infty} w_u^i(s,c,u,t)du, \tag{46}
$$

when the investment is made at time t.

We insert the expression in (46) into (52), and by some derivations (see the appendix, section A.2) we find that

$$
g_i^P(s,c,t) = E\left[Y^i(k^i)\left(s-k^i-\frac{F(k^i|c,t)}{f(k^i|c,t)}\right)\Bigg|\mathcal{F}_t^{S,C}\right]. \tag{47}
$$

From equation (43) we see that the control variable $y^i(k)$ is linearly dependent on agent i's contribution to the auctioneer's value. This means that the optimal value of $y^i(\cdot)$ is given by 0 or 1. Suppose that $y^i(k) = 1$, i.e., that agent i is the winner of the contract, and that his investment cost at time t is low enough to trigger investment. Then the auctioneer's optimization problem with respect to the investment strategy, is given by

$$\begin{aligned}
&\sup_{\tau^i} E\left[e^{-r\tau^i} \left(S(\tau^i) - K^i(\tau^i) - \frac{F(K^i(\tau^i)|C(\tau^i),\tau^i)}{f(K^i(\tau^i)|C(\tau^i),\tau^i)} \right) \middle| \mathcal{F}_{t_0}^{S,C} \right] \\
&\quad = \sup_{\tau^i} E\left[E\left[e^{-r\tau^i} \left(S(\tau^i) - K^i(\tau^i) - \tfrac{F(K^i(\tau^i)|C(\tau^i),\tau^i)}{f(K^i(\tau^i)|C(\tau^i),\tau^i)} \right) \middle| \mathcal{F}_{t_0}^{S,K^i} \right] \middle| \mathcal{F}_{t_0}^{S,C} \right].
\end{aligned} \tag{48}$$

For simplicity we have suppressed the subscript $\hat{K}$ of τ^i in (48), but the auctioneer's investment strategy is still dependent on the cost reports, which now are ensured to be truth telling. However, note that in the auctioneer's optimization problem in (48), the optimal stopping time τ^i), no longer depends on the vector of all the agents' investment cost variables. Instead, the investment strategy depends on the reports of agent i, only. Thus, the optimization problem is split into n programs, where each program i only depends on the cost reports of agent i.

As the control variable y^i is linear in the auctioneer's payoff values, we find that at any time $t \geq t_0$, the optimal solution of $y^i(k)$ is given by

$$\begin{aligned}
&y^{i*}(k) = 1 \quad \text{if } k^i < \min_{j \neq i} k^j \text{ and } k^i < \vartheta^i(s,c,t),\ t \leq \tau^i \\
&y^{i*}(k) = 0 \quad \text{if } k^i > \min_{j \neq i} k^j,
\end{aligned} \tag{49}$$

where the function $\vartheta^i(s,c,t)$ is the trigger for investment.

The auctioneer's optimization problem with respect to each agent is identical to the principal's optimization problem given by equation (21). This means that the optimal investment strategy is not improved when two or more agents compete about a contract. However, as the winning agent in a competition probably has a lower investment cost than the agent in a principal-agent model, the investment will probably take place at a lower cost. Moreover, if the number of competing agents gets large, the winner's cost level gets close to the lowest possible investment cost, at which point the cumulative distribution $F(\cdot)$ converges to zero. Thus, the inefficiency leading to under-investment is not so severe in the case of competition as in the case of only one agent having private information.

In the auction model presented here we have assumed that the admissible set of investment cost variables of the agents have a lower level of zero and no upper level, as is the admissible values of log-normal variables. If the number of competing agents gets very large, the agent with the lowest cost will approach the lower cost level of zero. This is in most cases not a realistic assumption. Hence, in this case we need to assume that the number of agents participating in the auction is not "too" large.

3.4 Implementation

Let the optimal investment strategy found from optimization of (48) be given by $\psi^{i*}(c, k^i, t)$. This means that based on agent i's cost report, it is optimal to invest immediately when $s > \psi^{i*}(c, k^i, t)$ and wait when $s \leq \psi^{i*}(c, k^i, t)$. Let $\vartheta^{i*}(s, c, t)$ be the inverse trigger, i.e., we invest immediately if $k^i < \vartheta^{i*}(s, c, t)$ and wait if $k \geq \vartheta^{i*}(s, c, t)$. Furthermore, $Y^{i*}(k^i)$ is defined as the optimal $Y^i(k^i)$.

Agent i's optimal compensation function X^{i*} when $s > \psi^{i*}(c, k^i, t)$, is equal to (see the appendix, section A.3 for derivation of the result)

$$X^{i*}(s, c, k^i, t) = Y^{i*}(k^i)k^i + \int_{k^i}^{\vartheta^{i*}(s,c,t)} Y^{i*}(u)du - \int_{\vartheta^{i*}(s,c,t)}^{\infty} w_u^i(s, c, u, t)du, \quad (50)$$

if $s > \psi^{i*}(c, k^i, t)$. Otherwise, $X^{i*}(c, k^i, t) = 0$. Note that, in contrast to the principal-agent model, the expected compensation function X^i is dependent on k^i under competition. This means that the expected compensation is implemented by a direct mechanism, instead of the investment decision being delegated to a winning agent.

So far we have only found each agent's optimal reporting strategy on average, i.e., given the other agents' strategies through the expectation Y^i. We now construct a dominant strategy auction[4] that implements the same investment strategy as the one found from optimizing equation (48). In addition, the dominant strategy auction selects the firm with the lowest investment cost at the time of investment. Let

$$\tilde{X}^{i*}(s, c, k, t) = \begin{cases} \vartheta^{i*}(s, c, t) + \int_{\vartheta^{i*}(s,c,t)}^{k^j} w_u^i(s, c, u, t)du & \text{if } \psi^{i*}(c, k^i, t) < s \leq \psi^{i*}(c, k^j, t) \\ k^j & \text{if } s > \psi^{i*}(c, k^j, t). \end{cases} \quad (51)$$

if $k^i = \min_h k^h$ and $k^j = \min_{h \neq i} k^h$. Otherwise, $\tilde{X}^{i*}(s, c, k^i, t) = 0$.

In the appendix it is shown that $X^{i*}(s, c, k^i, t) = E\left[\tilde{X}^{i*}(s, c, k, t) | \mathcal{F}_t^{S,K^i}\right]$. Hence, we conclude that the contract given by equation (51) is the optimal contract under competition.

By implementing the contract in (51), the agent having the lowest investment cost at the time the investment is exercised, wins the contract. If agent i wins the

[4] A dominant strategy auction is an auction in which each agent has a strategy that is optimal for any strategies of its competitors.

contract, the agent's compensation equals the value of his private information when the distribution is truncated at k^j. Thus, competition for the winner implies that the interval of possible investment cost variables, $(0, \infty)$, is truncated to $(0, k^j)$, where k^j is the second-lowest bid at time t.

If we compare the compensation functions under competition (51) and under no competition (28), we find that the compensation functions are very similar. One difference is that the compensation under competition cannot be higher than k^j. Therefore, the transferred amount is lower under competition. The closer the investment cost reported by the second-best agent is to the winner's report, the lower is the winner's information rent. This implies a higher value to the auctioneer. Another difference between the two models is that the optimal investment strategy under competition is not as inefficient as under no competition. The reason is that the winner of the contract under competition probably has a lower investment cost than the agent's investment cost when there is only one agent.

4 Conclusion

In this paper we study effects of stochastically changing private information on an optimal stopping problem. In the first part of the paper we formulate a principal-agent model where an agent has private information about the investment cost. We find that due to private information, the optimal investment trigger will be higher (i.e., more inefficient) than the optimal investment strategy in the case of full information. Thus, for some intervals of the variables included in the valuation, private information leads to under-investment.

We extend the principal-agent model to the case where there are two or more agents competing about obtaining a contract that gives the winner the right to invest in the project. Each agent has private information about the investment cost of the project, and each agent's investment cost is different. As each agent's investment cost is independent of each other, it is not optimal for the auctioneer to enter into a contract with any of the agents before the point in time when it is optimal to invest. If the auctioneer were restricted to choose an agent before the time of investment, the auctioneer's value of the auction would be lower than in the auction model presented in section 3.

We find that the optimal investment strategies are the same in the auction model and in the principal-agent model. However, the winner's compensation is lower under competition, because the winner's value of private information is truncated to the report given by the agent with the second-lowest investment cost. Thus, the investor's value of the investment project is higher under competition than in the principal-agent model, and the inefficiency leading to under-investment is less severe under competition.

In this version of the paper, we have not implemented the results numerically. However, in order to say something more about how much the investor's and the agents' values are changed because of private information, we need to include numerical analyzes. This will be done in a later version of the paper.

A Appendix

A.1 The revelation principle applied to our model

Applied to the model in this paper, the revelation principle can be shown by the following arguments (we follow the proof used by Salanié (1994), section 2.1.2).

Define a trigger function $\psi(c, t; \hat{K})$ such that the investment strategy is based on the agent's report $\hat{K}$. The option to invest is exercised immediately when $s > \psi(c, t; \hat{K})$, whereas it is optimal to wait when $s \leq \psi(c, t; \hat{K})$. This means that the optimal stopping time is given by

$$\tau = \inf\{t \geq t_0 | S(t) > \psi(C(t), t; \hat{K})\}.$$

Let M be the space of admissible reports, and let $(X(\cdot), \psi(\cdot), M)$ be a set of incentive mechanisms that implement the compensation function X^*, and the investment strategy ψ^*. Moreover, let $\hat{K}^*(k)$ be the optimal report if it is given at time t, so that $X^* = X^*(\hat{K}^*)$. Now consider the direct mechanism (X^*, ψ^*, k). If it were not truthful, then an agent would prefer to announce some k' other than k, and we would have

$$v^A(s, c, k, t; X^*(k'), \psi^*(k')) > v^A(s, c, k, t; X^*(k), \psi^*(k)).$$

But by definition this would imply that

$$v^A(s, c, k, t; X^*(K^*(k')), \psi^*(K^*(k'))) > v^A(s, c, k, t; X^*(K^*(k)), \psi^*(K^*(k)))$$

so that K^* would not be an optimal given that the true investment cost equals k. Hence the direct mechanism (X^*, ψ^*, k) must be truthful, and implement the compensation X^* and the investment strategy ψ^*.

A.2 Derivation of equation (47).

By inserting (46) into (44), we find that

$$\begin{aligned} g_i^P(s,c,t) &= E\left[Y^i(k^i)\left(s-k^i\right)+\textstyle\int_{k^i}^{\infty} w_u^i(s,c,u,t)du \middle| \mathcal{F}_t^{S,C}\right] \\ &= \textstyle\int_0^{\infty}\left\{Y^i(k^i)\left(s-k^i\right)+\int_{k^i}^{\infty} w_u^i(s,c,u,t)du\right\} f(k|c,t)dk. \end{aligned} \tag{52}$$

Partial integration of the $\int_0^{\infty}\int_{k^i}^{\infty} w_u^i(s,c,u,t)du f(k|c,t)dk$ leads to

$$g_i^P(s,c,t) = E\left[Y^i(k^i)\left(s-k^i+w_{k^i}^i(s,c,k^i,t)\frac{F(k^i|c,t)}{f(k^i|c,t)}\right)\middle| \mathcal{F}_t^{S,C}\right]. \tag{53}$$

If agent i's compensation function is to be self-selective, among other requirements, smooth pasting must be satisfied. This means that at the investment time we need to have

$$w_{k^i}^i(s,c,k^i,t) = -Y^i(k^i),$$

where $w_{k^i}^i$ denotes the first-order derivative of w^i with respect to k^i. The smooth pasting condition is found from agent i's value function given by (40). Replace $w_{k^i}^i(s,c,k^i,t)$ by $-Y^i(k^i)$ in (53), and we find the result in (47).

A.3 Derivation of the compensation function under competition

From equation (40) we find that agent i payoff value, given that the optimal investment strategy is implemented, equals

$$g^i(s,c,k^i,t) = X^i(s,c,k^i,t) - Y^i(k^i)k^i,$$

which can be written as $X^i(s,c,k^i,t) = Y^i(k^i) - g^i(s,c,k^i,t)$. The value of $g^i(s,c,k^i,t)$ is found by integration on both sides of the equality in (41), i.e.,

$$\int_{k^i}^{\vartheta^i(s,c,t)} \frac{dg^i(s,c,u,t)}{du}du = -\int_{k^i}^{\vartheta^i(s,c,t)} Y^i(u)du,$$

which leads to

$$\begin{aligned} g^i(s,c,k^i,t) &= \textstyle\int_{k^i}^{\vartheta^i(s,c,t)} Y^i(u)du + g^i\left(s,c,\vartheta^i(s,c,t),t\right) \\ &= \textstyle\int_{k^i}^{\vartheta^i(s,c,t)} Y^i(u)du + \int_{\vartheta^i(s,c,t)}^{\infty} w_u^i(s,c,u,t)du. \end{aligned}$$

Hence, we find that

$$X^i(s,c,k^i,t) = Y^i(k^i)k^i + \int_{k^i}^{\vartheta^i(s,c,t)} Y^i(u)du + \int_{\vartheta^i(s,c,t)}^{\infty} w_u^i(s,c,u,t)du.$$

A.4 Equality between the two approaches of finding the optimal compensation function when n agents compete about a contract

As each agent's true investment cost is independently drawn, the probability $Y^{i*}(k^i)$ is given the principal's optimal choice of the agent to win the contract, equals $\left[1-F(k^i)\right]^{n-1}$, where we define $F(k^i) = F(k^i|c,t)$. Substitution of $Y^{i*}(k^i) = \left[1-F(k^i)\right]^{n-1}$ in (50), leads to

$$\begin{aligned} & X^{i*}(s,c,k^i,t) \\ = \; & k^i\left[1-F(k^i)\right]^{n-1} + \int_{k^i}^{\vartheta^{i*}(s,c,t)} \left[1-F(u)\right]^{n-1} du - \int_{\vartheta^{i*}(s,c,t)}^{\infty} w_u^i(s,c,u,t)du. \end{aligned} \tag{54}$$

We will now show that $X^{i*}(s,c,k^i,t) = E\left[\tilde{X}^{i*}(s,c,k,t)|\mathcal{F}_t^{S,K^i}\right]$. We treat k^j as the first-order statistic in a sample of size $n-1$. Observe that the compensation is zero if k^j is less than k^i, and is given by (51) is k^j is higher than k^i. Then we find $E\left[\tilde{X}^{i*}(s,c,k,t)|\mathcal{F}_t^{S,K^i}\right]$ as follows

$$\begin{aligned} & E\left[\tilde{X}^{i*}(s,c,k,t)|\mathcal{F}_t^{S,K^i}\right] \\ = \; & \int_{k^i}^{\vartheta^{i*}(s,c,t)} k^j d\left(-[1-F(k^j)]^{n-1}\right) \\ & + \int_{\vartheta^{i*}(s,c,t)}^{\infty} \left\{\vartheta^{i*}(s,c,t) + \int_{\vartheta^{i*}(s,c,t)}^{k^j} w_u^i(s,c,u,t)du\right\} d\left(-[1-F(k^j)]^{n-1}\right). \end{aligned} \tag{55}$$

Partial integration of (55) leads to equation (54).

References

[1] Rick Antle, Peter Bogetoft, and Andrew W. Stark. Incentive problems and the timing of investment. Working paper, March 1998.

[2] Petter Bjerksund and Gunnar Stensland. A self-enforced dynamic contract for processing of natural resources. In Michael J. Brennan and Lenos Trigeorgis, editors, *Project Flexibility, Agency, and Competition: New Developments in the Theory and Application of Real Options*. Oxford University Press, 2000.

[3] Hans U. Gerber and Elias S. W. Shiu. Martingale approach to pricing perpetual American options on two stocks. *Mathematical Finance*, 6(3):303–322, July 1996.

[4] Jean-Jacques Laffont and Tirole Jean. Auctioning incentive contracts. *Journal of Political Economy*, 95(5):921–937, 1987.

[5] Jeffrey K. MacKie-Mason. Sequential decision problems and asymmetric information. CREST Working Paper, Dep. of Economics, University of Michigan, September 1985.

[6] Jøril Mæland. Valuation of irreversible investments: Private information about the investment cost. Discussion paper, 1999/12, Norwegian School of Economics and Business Administration, Department of Finance and Management Science, 1999.

[7] William Margrabe. The value of an option to exchange one asset for another. *Journal of Finance*, XXXIII:177–86, 1978.

[8] Robert McDonald and D. Siegel. The value of waiting to invest. *Quarterly Journal of Economics*, (101):707–727, 1986.

[9] Nahum D. Melumad and Stefan Reichelstein. Centralization versus delegation and the value of communication. *Journal of Accounting Research*, 25:1–18, 1987.

[10] Nahum D. Melumad and Stefan Reichelstein. Value of communication in agencies. *Journal of Economic Theory*, 47:334–368, 1989.

[11] Bernt Øksendal. *Stochastic Differential Equations*. Springer, fifth edition edition, 1998.

[12] Bernt Øksendal and Yaozhong Hu. Optimal time to invest when the price processes are geometric Brownian motions. Working Paper no.1/96, Norwegian School of Economics and Business Administration, 1996.

[13] Robert S. Pindyck. Investments of undertain costs. *Journal of Financial Economics*, (34):53–76, August 1993.

[14] Bernard Salanié. *The Economics of Contracts. A Primer*. The MIT Press, 1994.