A Log-transformed Binomial Lattice Extension for Multi-Dimensional Option Problems*

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Abstract

We propose a log-transformed binomial lattice approach for pricing options whose payoff depends on several state variable following a joint diffusion process. Our method extends the log-transformed approach proposed by Trigeorgis [31] to several state variables and improves other known lattice algorithms (Boyle, Evnine and Gibbs [7] and Kamrad and Ritchken [19]). The method we propose is consistent, stable and efficient. We present some applications of our method both to financial and real option pricing problems.

1 Introduction

Contingent claims dependent on more than one state variable can be found in financial economics both in financial and real investment valuation problems. As far as financial contingent claims are concerned, many authors have studied pricing models for contracts on several underlyings (Stulz [28], Johnson [21], Boyle [6] and Boyle, Evnine and Gibbs [7] provide models to price options on the maximum and on the minimum of several asset prices, just to mention a few) or on more than one state variable (e.g., Hull and White [18], Schwartz [27]). For what concerns multi-factor real investment valuation problems, real options have been studied for instance by Brennan and Schwartz [10], Triantis and Hodder [30], Cortazar and Schwartz [12], Geltner, Riddiuogh and Stojanovic [15], Cortazar, Schwartz and Salinas [13], Martzoukos and Trigeorgis [23], Brekke and Schieldrop [8] and others.

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Unfortunately, closed-form solution for the contingent claim valuation problems are rarely available and, quite soon, numerical methods must be employed to price options. Many methods have been proposed to numerically tackle the option pricing problem. They can be divided into three main classes: finite difference methods (first introduced by Brennan and Schwartz [9]), Monte Carlo simulation methods (introduced by Boyle [5]) and lattice methods first proposed by Cox, Ross and Rubinstein [14] (CRR in what follows). In this paper we propose a new binomial lattice approach to price contingent claims whose payoff depends on several state variables following a joint (correlated) geometric Brownian motion.

In the financial literature, many extensions of the CRR algorithm have been proposed to approximate the price of options written on asset prices following a geometric Brownian motion. Rendleman and Bartter [26] and Jarrow and Rudd [20] propose binomial lattice approaches with different choices of parameters for jumps and probabilities. Boyle [6] and Kamrad and Ritchken [19] propose a trinomial lattice algorithm to improve over the other binomial lattice algorithms. The accuracy of the method they propose depends on a choice of a given "stretch parameter" that need to be chosen before implementing the algorithm. Although they improve the accuracy, this is done at the cost of increasing the computational effort. Boyle [6] and Kamrad and Ritchken [19] provide also an extension of their three-state approach for option valuation problems with several underlying assets. Trigeorgis [31] provides a log-transformed binomial lattice algorithm for complex contingent claims depending on a single state variable which presents several improved qualities: it proves to be more efficient than other lattice schemes, keeping the appealing simplicity of the CRR approach. Boyle, Evnine and Gibbs [7] (BEG, from now on) provide a straightforward extension of the CRR approach to several underlyings. This keeps the features of the CRR scheme and can be easily extended to any number of underlyings. As Boyle, Evnine and Gibbs acknowledge, their scheme provides positive probability if the size of the time step is small (i.e., if the number of steps, is large enough). Unfortunately, if the number of underlying assets is high, one can hardly afford a reasonably large number of steps because of the exponential complexity of the lattice approach. So it may happen that for some values of the parameters, the probability of the jumps in BEG's scheme (which is based on CRR's choice of probability) are negative, giving inaccurate estimates of the value of the option. Other lattice approaches have been proposed to cope with different stochastic models (see Nelson and Ramaswamy [25]) or with time-varying variance-covariance structures (see Ho, Stapleton and Subrahmanyan [17]) for the underlying asset prices.

We extend the log-transformed binomial lattice approach proposed by Trigeorgis [31] to several state variables. We obtain an improvement over other known lattice (CRR, BEG and Kamrad and Ritchken) algorithms. The method we propose is: consistent (i.e., the mean and the variance of the approximating stochastic process are the same as the mean and the variance of the diffusion process of the state variables), stable (that is, the approximating errors are not amplified) for a wide choice of parameters and efficient (i.e., the computational cost for accuracy of a given approximation is lower than in other methods).

The paper is organized as follows. In Section 2 we review the log-transformed binomial lattice approach proposed by Trigeorgis [31]. In Section 3 we propose a first extension of the log-transformed binomial lattice approach to a multi-dimensional setting in the spirit of Boyle, Evnine and Gibbs [7]. In Section 4 an improved log-transformed approach is given to overcome the drawbacks presented by the first log-transformed extension. In Section 5 we illustrate some applications of our algorithm to option pricing problems with an increasing number of (up to five) assets.

2 The log-transformed binomial lattice approach

Trigeorgis [31] proposed a log-transformed binomial lattice method for valuing options with one underlying asset to overcome known drawbacks of the classical binomial scheme proposed by Cox, Ross and Rubinstein [14] in approximating the continuous-time Brownian motion of the return of the underlying asset.

Consider an asset whose price X_t follows a diffusion process

$$\frac{dX_t}{X_t} = \alpha dt + \sigma dZ_t \qquad X_0 = x \tag{2.1}$$

under the risk-neutral or martingale probability where α is the risk-neutral drift, σ is the volatility of asset return and dZ_t is the increment of a Gauss-Wiener process. As shown in Harrison and Kreps [16], under the martingale probability the risk-neutral drift can be written as $\alpha = \hat{\alpha} - \mathrm{RP}$, where $\hat{\alpha}$ is the actual drift (i.e., the drift under the empirical probability) and RP is a risk premium. If the security is traded and pays a dividend (or convenience) yield, δ, and r is the risk-free interest rate then the risk-neutral drift becomes $\alpha = r - \delta$.

Given a derivative security (e.g., a financial or real option) whose payoff depends on the price of the asset, X_t, its market price $F(t, X_t)$ must satisfy the following partial differential equation (p.d.e.)

$$\frac{1}{2} F_{xx} X_t^2 + \alpha F_x X_t + F_t - rF = 0$$

and some appropriate boundary conditions, as shown by Black and Scholes [4] and Merton [24]. The boundary conditions can be as simple as $F(T, X_T) = \max\{K - X_t, 0\}$, as in the case of European put option with exercise price K or can be quite complex because of the early exercise feature, as in the case of an American-type put option. An analytic solution is feasible with some derivative securities, like European call and put options, which have a closed-form solution to the above p.d.e.. In other cases, e.g. with American put options or American call options on a dividend paying ($\delta \neq 0$) underlying asset, numerical methods must be employed to evaluate $F(t, X)$.

A first set of numerical option pricing methods tackles the valuation problem by approximating the p.d.e. (e.g., as proposed first in Brennan and Schwartz [9]). A second class of numerical techniques approximates the underlying diffusion process in (2.1) with a suitable discrete-time process $\{\widehat{X}_t\}$ and then estimates $F(t, X)$ with respect to $\widehat{X}$. This family of numerical methods includes the Monte Carlo simulation approach for the valuation of European-like derivatives (see Boyle [5]).[1]

One of the most flexible and appealing methods in this second family of numerical approaches is the binomial lattice approach, first proposed in Cox, Ross and Rubinstein [14] (in what follows, we will refer to this as the CRR approach) and thereafter extended in several directions. According to this method, the continuous-time diffusion $\{X_t\}$ is approximated by a discrete-time process in a time interval $[0, T]$ for valuing an option written on X_t. Usually, T is taken equal to the maturity of the option. By considering $y = \log x$, the process $\{Y_t\}$ has dynamic

$$dY_t = \left(\alpha - \frac{\sigma^2}{2}\right) dt + \sigma dZ_t \qquad Y_0 = y. \tag{2.2}$$

Given n, the interval $[0, T]$ is subdivided in n subintervals of length $\Delta t = T/n$. In the dates $\{0, \Delta t, 2\Delta t, \dots, n\Delta t\}$, the discrete-time process approximating $\{Y_t\}$ proposed in [14] is

$$\widehat{Y}_j = \widehat{Y}_{j-1} + gU_j \text{ for } j = 1, \dots, n, \qquad \widehat{Y}_0 = y \tag{2.3}$$

where $\{U_j\}$ is a family of i.i.d. binomial random variables with

$$U_j = \begin{cases} 1 & \text{with probability } q \\ -1 & \text{w.p. } 1 - q, \end{cases}$$

for each j, $g = \sigma\sqrt{\Delta t}$ and

$$q = \frac{1}{2}\left(1 + \frac{m}{\sigma}\sqrt{\Delta t}\right)$$

[1] Recently, a big effort has been done to extend the pleasant features of Monte Carlo simulation also to the valuation of American-style securities (i.e., contingent claims that can be exercised at any date before maturity). Examples are the model proposed by Tilley [29], Barraquand and Martineau [3], Broadie and Glasserman [11], Longstaff and Schwartz [22].

where $m = \alpha - \sigma^2/2$. Note that the up-step of X_t is $u = e^g = e^{\sigma\sqrt{\Delta t}}$ and the down-step is $d = 1/u = e^{-\sigma\sqrt{\Delta t}}$. Hence, the discrete-time process of the underlying asset price is

$$\widehat{X}_j = \widehat{X}_{j-1} e^{gU_j}$$

for $j = 1, \ldots, n$. This approximation is consistent in the limit (when Δt goes to zero): the mean of the increment is

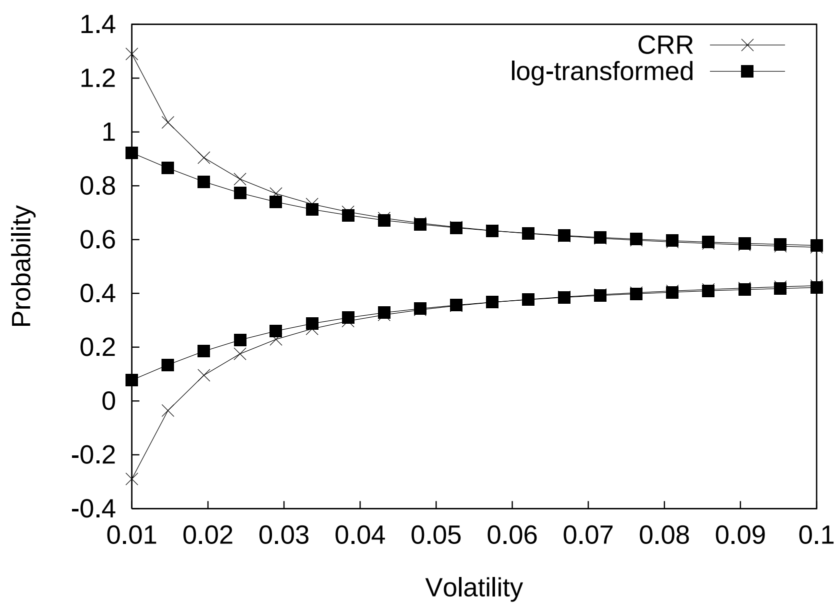
$$\mathbb{E}[\Delta \widehat{Y}_t] = m\Delta t = \mathbb{E}[\Delta Y_t]$$

and the variance

$$\mathrm{Var}[\Delta \widehat{Y}_t] = \sigma^2 \Delta t - m^2 (\Delta t)^2 \leq \sigma^2 \Delta t = \mathrm{Var}[\Delta Y_t].$$

Note that, the variance of the increment of the discrete-time process is generally biased downward by the square of the mean. If the volatility or the number of steps used are small with respect to the (risk-adjusted) drift α, then the method becomes unstable because the probability (and the variance) can turn negative (see Figure 1). A very important feature of the CRR method is that it makes pedagogically clear the relationship between the no-arbitrage (or risk neutral valuation) argument and the hedging argument. Having said that, it does not need to be also the best approach from a numerical viewpoint. Several other approaches have been proposed.

Figure 1: CRR probability and log-transformed (Trigeorgis [31]) probability vs volatility (with $\alpha = 0.05$, $T = 1$ and $\Delta t = 0.1$).



Among other lattice approximations[2] the log-transformed approximation

[2] See Jarrow e Ruud [20], Boyle [6], Kamrad e Ritchen [19], for one state-variable.

has been proposed by Trigeorgis [31] to overcome some of the above flaws presented by the CRR approach. The geometric Brownian motion in (2.2) is approximated by a discrete-time process

$$\widetilde{Y}_j = \widetilde{Y}_{j-1} + hU_j \text{ for } j = 1, \ldots, n, \qquad \widetilde{Y}_0 = y$$

with parameters:

$$\begin{aligned}
\mu &= \frac{m}{\sigma^2} = \frac{\alpha}{\sigma^2} - \frac{1}{2} \\
k &= g = \sigma\sqrt{\Delta t} \\
h &= \sqrt{k^2 + (k^2\mu)^2} \\
p &= \frac{1}{2}\left(1 + \frac{k^2\mu}{h}\right).
\end{aligned}$$

Note that the up step here is $u = e^h$ and $d = 1/u$ and the discrete-time process of the underlying asset price is

$$\widetilde{X}_j = \widetilde{X}_{j-1}e^{hU_j}$$

for $j = 1, \ldots, n$ and where the probabilities of $\{U_j\}$ are now replaced by p and $(1-p)$. A remarkable feature of this method is that, since $h \geq |k^2\mu|$, then $0 \leq p \leq 1$ with no need of external constraints on the parameters. Thus a key feature of the log-transformed approach is that it allows for unconditional stability. This is so because the time unit is k instead of Δt; i.e., time is measured in units of variance. This feature makes the log-transformed approximation consistent at each step n (not just in the limit as $n \to \infty$, as in CRR):

$$\mathbb{E}[\Delta\widetilde{Y}_t] = m\Delta t \qquad \mathrm{Var}[\Delta\widetilde{Y}_t] = \sigma^2\Delta t.$$

For this reason, unlike the CRR approach, it does not explodes for small volatility and/or number of time steps, as can be seen in Figure 1. In addition, as proved by Trigeorgis [31], the CRR approach and the log-transformed approach are equal in the limit.

Trigeorgis [31] presents an extensive numerical comparison with other methods for the American put option case and for a wide choice of parameter values. The log-transformed method generally shown to be more accurate and more efficient than the other methods providing estimate errors with respect to the true value within 1% with much less time steps.[3]

In the subsequent sections of this paper we present two extensions of the log-transformed approach to numerically evaluate options with multiple

[3]Trigeorgis [31], following Barone-Adesi and Whaley [2], considers the value obtained by a finite difference approximation of the Black and Scholes p.d.e. for the American put as the benchmark. Trigeorgis [31] shows that the log-transformed binomial lattice algorithm is as accurate with $n = 50$ time steps as the CRR algorithm with $n = 500$ steps.

underlying assets. Our aim is to generalize to multi-dimensional cases the numerically attractive features obtained in [31] with one underlying asset. The efficiency of the log-transformed method proves to be even more important when evaluating options with many underlyings because memory and computational constraints do not allow for a high number of steps.

3 Extension of the log-transformed binomial approximation to several underlying assets

This section presents first an extension of Trigeorgis' [31] log-transformed binomial approach to several underlying assets along the line of Boyle, Evnine e Gibbs [7]. The basic idea is to approximate a multi-dimensional diffusion process with a lattice so that the first two moments (mean and variance-covariance) of the continuous process match the corresponding moments of the discrete distribution.[4] As we will see, this approximation suffers with the same drawbacks as in CRR (and BEG) scheme because of the presence of correlation. If the multiple underlying assets were uncorrelated, then the multi-dimensional extension of the log-transformed approximation would have the same features as the case one-dimensional case. In the next section we show that, if we are able to change the coordinate system in order to have a set of uncorrelated diffusions, then we can evaluate an option written on multiple assets while preserving many of the positive features of the log-transformed approach. This section proceeds as follows: to give the flavor of the log-transformed approach in the multi-dimensional case we first develop the two-dimensional case and then we describe the three- and the N-dimensional case.

Consider N correlated assets whose price dynamic[5] $X^\top = (X_1, \dots, X_N)$ is a N-dimensional geometric Brownian motions (under the martingale probability):

$$\frac{dX_i}{X_i} = \alpha_i dt + \sigma_i dZ_i \qquad i = 1, 2, \dots, N \tag{3.1}$$

where α_i is the risk-adjusted drift of the i-th asset price and $\mathbb{E}[dZ_i dz_j] = \rho_{ij} dt$, $i \neq j$.

Given a derivative security with maturity T and price F whose pay-off depends on the underlying assets prices, we want to estimate the risk-neutral price of the derivative security. Following the usual argument first introduced by Black and Scholes, the valuation p.d.e. for F in the multi-

[4] As shown in [7], this is the same as approximating the characteristic function of the increments of the diffusion process with a second order Taylor expansion.

[5] The symbol $^\top$ denotes transposition.

dimensional case is

$$\frac{1}{2}\sum_{i=1}^{N}\sum_{j=1}^{N}\rho_{i,j}\sigma_i\sigma_j X_i X_j \frac{\partial^2 F}{\partial x_i \partial x_j} + \sum_{i=1}^{N}\alpha_i X_i \frac{\partial F}{\partial x_i} + \frac{\partial F}{\partial t} - rF = 0$$

with appropriate boundary conditions.

Since in general an analytic solution to this p.d.e. (for given boundary conditions) does not exist, we can obtain a numerical solution by approximating the continuous-time dynamics in (3.1) with a binomial lattice approach.

To illustrate the process more simply, we first show the two-dimensional case, $N = 2$. First, we take the log- of the asset values: $Y_i = \log X_i$, $i = 1, 2$. The dynamic of $Y^\top = (Y_1, Y_2)$ is

$$dY_i = \left(\alpha_i - \frac{1}{2}\sigma_i^2\right) dt + \sigma_i dZ_i, \qquad i = 1, 2. \tag{3.2}$$

Given the time interval $[0, T]$ specified by the maturity of the option, we consider n subintervals of width $\Delta t = T/n$. Following the standard approach (see for instance [14, 7]), we approximate the continuous-time process $\{Y\}$ with the discrete-time one $\{(\widehat{Y}_1, \widehat{Y}_2)\}$. The approximation criterion is the following: the discrete-time process approximates the diffusion if the characteristic function of the first one approximates the characteristic function of the second one. This is equivalent to matching the first two moments of the distributions. The discrete process is

$$\widehat{Y}_i(t) = \widehat{Y}_i(t-1) + h_i U_i(t) \qquad i = 1, 2$$

$t = 1, \ldots, n$ where (U_1, U_2) is a bi-variate i.i.d. binomial random variable:

$$(U_1, U_2) = \begin{cases} (1, 1) & \text{with probability } p_1 \\ (1, -1) & \text{w.p. } p_2 \\ (-1, 1) & \text{w.p. } p_3 \\ (-1, -1) & \text{w.p. } p_4 \end{cases} \tag{3.3}$$

and $\sum_{i=1}^{4} p_i = 1$. Let

$$\begin{aligned} \mu_i &= \alpha_i/\sigma_i^2 - 1/2, \\ k_i &= \sigma_i\sqrt{\Delta t}, \\ h_i &= \sqrt{k_i^2 + (k_i^2\mu_i)^2}, \\ R_{ij} &= k_i k_j/(h_i h_j), \\ M_i &= k_i^2\mu_i/h_i, \end{aligned} \tag{3.4}$$

$i = 1, 2$. Note that $u_i = e^{h_i}$, $d_i = 1/u_i$. Moreover, let

$$\begin{aligned} p_1 = p_{uu} &= (1 + (R\rho + M_1 M_2) + M_1 + M_2)/4, \\ p_2 = p_{ud} &= (1 - (R\rho + M_1 M_2) + M_1 - M_2)/4, \\ p_3 = p_{du} &= (1 - (R\rho + M_1 M_2) - M_1 + M_2)/4, \\ p_4 = p_{dd} &= (1 + (R\rho + M_1 M_2) - M_1 - M_2)/4, \end{aligned} \tag{3.5}$$

where $\rho = \rho_{12}$ and $R = R_{12}$.

With these parameters, the first moments of the increment of the discrete-time process match the first moments of the increment of the continuous-time process for any given time step Δt:

$$\mathbb{E}[\Delta \widehat{Y}_i] = k_i^2 \mu_i = \mathbb{E}[\Delta Y] = \left(\alpha_i - \frac{1}{2}\sigma_i\right)\Delta t,$$

$$\mathrm{Var}[\Delta \widehat{Y}_i] = k_i^2 = \mathrm{Var}[\Delta Y] = \sigma_i^2 \Delta t,$$

$i = 1, 2$ and

$$\mathrm{Cov}[\Delta \widehat{Y}_1, \Delta \widehat{Y}_2] = \rho_{12} k_1 k_2 = \mathrm{Cov}[\Delta Y_1, \Delta Y_2] = \rho_{12}\sigma_1\sigma_2 \Delta t.$$

Hence, the approximation of the bi-variate geometric Brownian motion $\{X\}$ is given by the process $\{(\widehat{X}_1, \widehat{X}_2)\}$ such that

$$\widehat{X}_i(t) = \widehat{X}_i(t-1) e^{h_i U_i}$$

$t = 1, \ldots, n$, $i = 1, 2$, U_i as in (3.3).

Following the same argument, it is easy to extend this analysis to $N > 2$ assets. For example, the three-dimensional case is given by expressions (3.4) for the parameters and the probabilities are lengthy expressions of the form (for brevity, we show only the case with an up-jump for security 1 and 3 and a down-jump for security 2):

$$\begin{aligned} p_{udu} = (1 - (R_{12}\rho_{12} + M_1 M_2) + (R_{13}\rho_{13} + M_1 M_3) \\ -(R_{23} + M_2 M_3) + M_1 - M_2 + M_3)/8. \end{aligned}$$

To develop more compact expressions for the resulting probability in the N-dimensional case, we can observe that, at the end of any time interval of length Δt, starting from a given state, we can be in one of 2^N possible states. Let $s = 1, \ldots, S = 2^N$ be one of these states. We denote

$$\delta_{ij}(s) = \begin{cases} 1 & \text{if both asset prices } i \text{ and } j \text{ jump up or down together} \\ -1 & \text{if the asset prices take antithetic jumps} \end{cases} \tag{3.6}$$

$i, j = 1, \ldots, N$, $i \neq j$ and

$$\delta_i(s) = \begin{cases} 1 & \text{if asset price } i \text{ jumps up} \\ -1 & \text{if asset price } i \text{ jumps down} \end{cases} \tag{3.7}$$

for $i = 1, .., N$. Note that $\delta_{ij}(s) = \delta_i(s)\delta_j(s)$. Given the parameters in (3.4), the two-dimensional case probability can be expressed in the more compact form:

$$p(s) = \frac{1}{4}\left[1 + \delta_{12}(s)(R_{12}\rho_{12} + M_1 M_2) + \delta_1(s)M_1 + \delta_2(s)M_2\right], \tag{3.5'}$$

$s = 1, \ldots, 4$.

More generally, the probabilities for the log-transformed approximation in the N-dimensional case are:

$$p(s) = \frac{1}{S}\left[1 + \sum_{1 \leq i < j \leq N} \delta_{i,j}(s)(R_{ij}\rho_{ij} + M_i M_j) + \sum_{i=1}^{N} \delta_i(s) M_i\right], \tag{3.8}$$

$s = 1, \ldots, S$, where $S = 2^N$.

With these parameters it can be verified that

$$\mathbb{E}[\Delta\widehat{Y}_i] = k_i^2 \mu_i = \left(\alpha_i - \frac{1}{2}\sigma_i\right)\Delta t, \quad i = 1, \ldots, N;$$

$$\operatorname{Var}[\Delta\widehat{Y}_i] = k_i^2 = \sigma_i^2 \Delta t = \quad i = 1, \ldots, N;$$

and

$$\operatorname{Cov}[\Delta\widehat{Y}_i, \Delta\widehat{Y}_j] = \rho_{ij} k_i k_j = \rho_{ij}\sigma_i\sigma_j\Delta t \quad i \neq j.$$

The approximation of the geometric Brownian motion $\{X\}$ is then given by the discrete-time process $\{(\widehat{X}_1, \ldots, \widehat{X}_N)\}$ such that

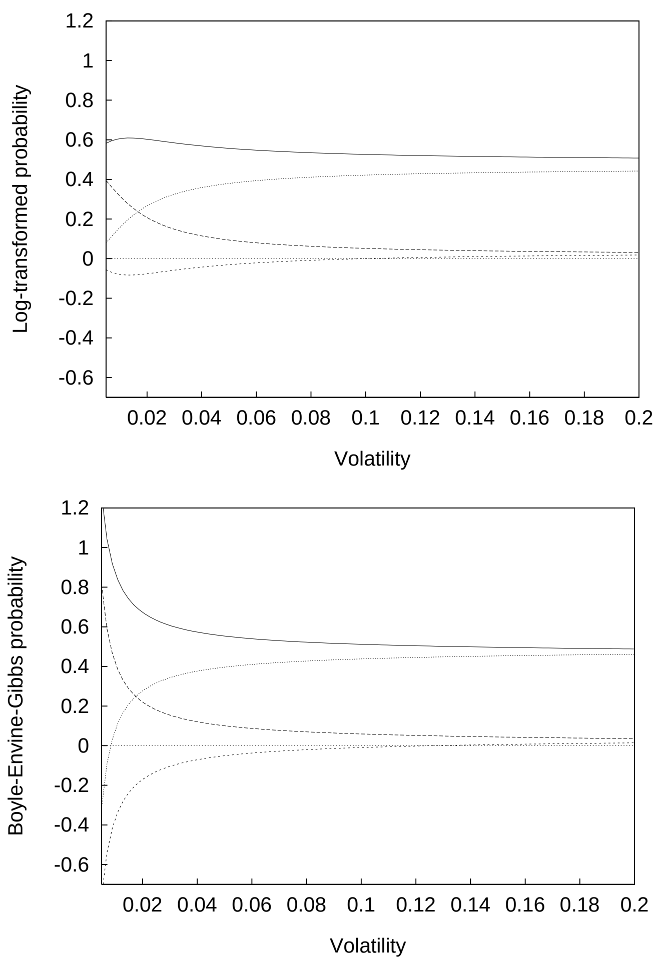
$$X_i(t) = X_i(t-1)e^{h_i U_i(t)} \qquad i = 1, \ldots, N$$

$t = 1, \ldots, n$, with $\{(U_1(t), \ldots, (U_N(t))\}$ being a i.i.d. N-variate binomial process (an N-dimensional extension of (3.3)).

It is easy to see that the options price estimates obtained with the multi-dimensional log-transformed algorithm are the same as the one given by the BEG's algorithm when the width of the time step shrinks to zero. Actually, if $\Delta t \longrightarrow 0$, then by Equation (3.4) we obtain $h_i = \sigma_i\sqrt{\Delta t}$, $R_{ij} = 1$, and the probabilities in (3.8) converge to the BEG's probabilities (see [7, Equation (15)]).

Although this probability distribution is an improvement over the BEG's extension, which it follows, still for some values of the parameters, it can have negative values. In Figure 2 we compare log-transformed probabilities with BEG probabilities for the same parameters. Obviously the correlation plays an important role in the ability to maintain a positive probability in all states. If the asset returns are uncorrelated, then the log-transformed probability is always strictly positive. In the following section, we exploit this fact to provide an alternative, improved, log-transformed binomial lattice approach for the N-dimensional case.

Figure 2: Log-transformed probabilities (above) and Boyle-Evnine-Gibbs [7] probabilities (below) vs volatility of the first underlying asset (σ_1) in a two-dimensional case (with $\alpha_1 = 0.05 = \alpha_2$, $\sigma_2 = 0.3$, $\rho = 0.9$, $T = 1$ and $\Delta t = 0.1$).



4 An improved extension of the log-transformed lattice approximation

In this section, we propose an improved log-transformed binomial extension based on the idea of transforming the basis to have uncorrelated assets; that is, we change the basis of $\mathbb{R}^N$, the market space generated by the N-dimensional diffusion of the asset returns $Y^\top = (Y_1, \ldots, Y_N)$, so that the price of the derivative security is dependent on an N-dimensional diffusion $y^\top = (y_1, \ldots, y_N)$ obtained by a change of basis such that its components y_i are uncorrelated. Note that if we change the basis of the market space we have to change also the payoff function accordingly: denoting by $\Pi(Y)$ the payoff of the contingent claim, and W the matrix representing the change of basis, the expression of the payoff with respect to the new basis is $\widehat{\Pi}(y) = \Pi(Wy)$. The dynamics of the returns y can then be approximated by a suitable log-transformed binomial lattice that overcomes the problems presented at the end of the previous section. In particular this method maintains the stability feature of the approach presented in the one-dimensional case by Trigeorgis [31] for a wider choice of parameters than the log-transformed extension presented in the previous section.

The economic rationale of the approach based on a change of basis is the following. We want to price the contingent claim with payoff $\Pi(Y)$, where Y are the returns of the assets traded in the market, in a risk neutral setting. If the financial market is complete[6], we can generate N portfolios with the original assets: we denote $w_i^\top = (w_{i1}, \ldots, w_{iN})$ the i-th portfolio, $i = 1, \ldots, N$, where w_{ij} is the position in the j-th asset in portfolio i. We can see these portfolios as new synthetic assets spanning the (same) market space. Any contingent claim which is redundant with respect to the original assets is redundant also with respect to these synthetic assets. The N portfolios we generate are selected so as to have uncorrelated returns. The contingent claim to be priced is dependent on the returns of the synthetic assets and is denoted by $\widehat{\Pi}(y)$. Since the risk structure of the market is unchanged,[7] we can price the claim according to a risk-neutral approach with respect to a martingale probability derived by the original one by a simple change of basis.

[6] A market is complete if the number of non-perfectly correlated traded assets is equal to the number of the sources of uncertainty. In a complete market any risk can be hedged with a suitable portfolio strategy and hence a contingent claim can be valued by replication if there are no arbitrage opportunities. Moreover, in a complete market there is only one equivalent martingale probability. The contingent claim can be valued with a risk neutral approach by taking expectation of its terminal payoff with respect to the martingale probability. The argument here presented remains valid also in the more general case of valuation of a contingent claim, which is redundant with respect to the asset span, according to the (unique) equilibrium martingale probability.

[7] The market spanned by the synthetic assets is the same as the assets span generated by the primitive securities. The only thing that changes is the representation of returns.

Let

$$dY_i = \left(\alpha_i - \frac{1}{2}\sigma_i^2\right) dt + \sigma_i dZ_i, \qquad i = 1, \dots, N$$

be the log- of the dynamics of the underlying prices $X^\top = (X_1, \dots, X_N)$ given in Equation (3.1) or, with a vector notation,

$$dY = adt + bdZ$$

where $a^\top = (a_1, \dots, a_N)$, with $a_i = \alpha_i - \sigma_i^2/2$, $dZ^\top = (dZ_1, \dots, dZ_N)$ such that[8]

$$dZdZ^\top = \begin{pmatrix} 1 & \rho_{12} & \cdots & \rho_{1d} \\ \rho_{12} & 1 & \cdots & \rho_{2d} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{1d} & \rho_{2d} & \cdots & 1 \end{pmatrix} dt = \Sigma dt$$

and

$$b = \begin{pmatrix} \sigma_1 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \sigma_N \end{pmatrix}.$$

In what follows, we hypothesize that dY has a time-independent covariance matrix. The covariance matrix of dY is

$$dYdY^\top = bdZdZ^\top b^\top = b\Sigma b^\top dt = \Omega dt.$$

Hence, Ω is independent on time if Σ and b are independent on time.

By definition, Ω is a symmetric positive definite square matrix. Hence, there exists a $N \times N$ matrix W such that $W^\top W = \mathcal{I}_N$, with $\mathcal{I}_N$ the N-dimensional identity matrix, such that $W^\top \Omega W = \Lambda$, where Λ is the diagonal N-dimensional matrix (λ_i), with $\lambda_i > 0$.[9] The matrix W represents a change of basis in $\mathbb{R}^N$ and, from an economic viewpoint, it is a set of N portfolios of the original assets.

We denote $y = W^\top Y$ the returns of the synthetic securities obtained by linear combinations of the original securities spanning the financial market. The diffusion of y is

$$dy = Adt + BdZ$$

[8] With the usual rules: $dtdZ_i = 0, (dt)^2 = 0$, $dZ_i dZ_j = \rho_{ij} dt$.

[9] Given the square matrix Ω, the columns of W are the eigenvectors and the diagonal of Λ are the eigenvalues of Ω, that is $\Omega W = \Lambda W$.

where $A = W^\top a$ and $B = W^\top b$. The covariance matrix of dy is

$$dy dy^\top = \Lambda dt;$$

that is, the components of $y^\top = (y_1, \ldots, y_N)$ are uncorrelated: $dy_i dy_j = 0$ whenever $i \neq j$ and $(dy_i)^2 = \lambda_i dt$.

Let $\Pi(X(t)) = \Pi(X_1(t), \ldots, X_N(t))$ be the payoff of the contingent claim. According to the change of variable in (3.2) the payoff is

$$\Pi(X_1(0)e^{Y_1(t)}, \ldots, X_N(0)e^{Y_N(t)}).$$

We make the derivative security dependent on $y = W^\top Y$ by changing the payoff function as follows:

$$\widehat{\Pi}(y(t)) = \Pi\left(X_1(0)e^{(Wy(t))_1}, \ldots, X_N(0)e^{(Wy(t))_N}\right)$$

where $(Wy(t))_i$ is the i-the component of $Y(t) = Wy(t)$.

The risk-neutral price of $\widehat{\Pi}$, denoted $\widehat{F}$, is equal to the risk neutral price of Π, F:

$$\widehat{F}(y(t)) = e^{r(T-t)}\mathbb{E}_y\left[\widehat{\Pi}(y(T))\right] = e^{r(T-t)}\mathbb{E}_Y\left[\Pi(Y(T))\right] = F(Y(t)) \qquad (4.1)$$

where $\mathbb{E}_y[\cdot]$ denotes the risk neutral expectation with respect to ν_y, the martingale probability of the process $\{y\}$, and $\mathbb{E}_Y[\cdot]$ is the expectation w.r.t. ν_Y, the martingale probability of the process $\{Y\}$.[10] Note that, although we have phrased the argument for an European-like contingent claim, the above is true also for an American-like contingent claim as confirmed by the numerical results presented in Section 5.[11]

The economic intuition behind Equation (4.1) is that a change of basis does not change the risk structure of the market. The geometric intuition is the following: since the covariance matrix Ω is time-independent, the measure ν_Y is invariant under a change of basis. Given this result, we can numerically evaluate $\widehat{F}(y)$ by approximating the martingale probability ν_y.

To illustrate this alternative log-transformed binomial technique, we present the two-dimensional case. Suppose

$$dY_i = a_i dt + \sigma_i dZ_i, \qquad i = 1, 2$$

[10] Equation (4.1) can be derived as follows

$$\begin{aligned} F(Y) &= \int e^{-rT}\Pi(Y_T)\nu_Y(dY_T) \\ &= \int e^{-rT}\Pi(Wy_T)\nu_Y(Wdy_T) \\ &= \int e^{-rT}\widehat{\Pi}(y_T)\nu_y(dy_T) = \widehat{F}(y) \end{aligned}$$

where $\nu_y(dy) = \nu_Y(Wdy_T)$ and integration is over the support of Y_t and y_T respectively.

[11] A formal proof of this statement would follow the argument presented in Amin and Khanna [1] for the BEG algorithm.

where $\rho = \rho_{12}$ and

$$\Sigma = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix} \qquad \text{and} \qquad b = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}.$$

Since

$$\Omega = \begin{pmatrix} \sigma_1^2 & \sigma_1\sigma_2\rho \\ \sigma_1\sigma_2\rho & \sigma_2^2 \end{pmatrix}$$

we get $\Lambda = (\lambda_i)$, a two-dimensional diagonal matrix, where

$$\lambda_{1,2} = \frac{1}{2}\left(\sigma_1^2 + \sigma_2^2 \mp \sqrt{\sigma_1^4 + 2(1-2\rho^2)\sigma_1^2\sigma_2^2 + \sigma_2^4}\right),$$

and

$$W = \begin{pmatrix} \left(\frac{\lambda_1}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right) / (\rho c_1) & \left(\frac{\lambda_2}{\sigma_1\sigma_2} - \frac{\sigma_2}{\sigma_1}\right) / (\rho c_2) \\ 1/c_1 & 1/c_2 \end{pmatrix}$$

where

$$c_i = \sqrt{1 + \frac{\left(\lambda_i - \sigma_2^2\right)^2}{\rho^2\sigma_1^2\sigma_2^2}}.$$

The processes of the returns of the synthetic securities are

$$dy_i = A_i dt + B_{i1} dZ_1 + B_{i2} dZ_2 \qquad i = 1, 2$$

where $B = (B_{ij}) = W^\top b$ and $A = W^\top a$. We approximate $\{y\}$ with a discrete process: given the time interval $[0, T]$ and n, we consider subintervals of width $\Delta t = T/n$. The discrete process is $\widetilde{y}^\top = (\widetilde{y}_1, \widetilde{y}_2)$ with dynamic

$$\widetilde{y}_i(t) = \widetilde{y}_i(t-1) + \ell_i U_i(t) \qquad i = 1, 2 \tag{4.2}$$

$t = 1, \ldots, n$ where (U_1, U_2) is a bi-variate i.i.d. binomial random variable with distribution as in (3.3).

By assigning the parameters

$$\begin{aligned} \kappa_i &= A_i \Delta t \\ \ell_i &= \sqrt{\lambda_i \Delta t + \kappa_i^2} \\ L_i &= \kappa_i / \ell_i \end{aligned} \tag{4.3}$$

$i = 1, 2$ and probabilities

$$p(s) = \frac{1}{4}\left(1 + \delta_{12}(s) L_1 L_2 + \delta_1(s) L_1 + \delta_2(s) L_2\right) \quad s = 1, \ldots, 4, \tag{4.4}$$

for the discrete-time process, we have the following:

$$\begin{aligned} \mathbb{E}[\Delta\widetilde{y}_i] &= \kappa_i = A_i\Delta t & i &= 1,2 \\ \mathrm{Var}[\Delta\widetilde{y}_i] &= \ell_i^2 - \kappa_i^2 = \lambda_i\Delta t & i &= 1,2 \\ \mathrm{Cov}[\Delta\widetilde{y}_1, \Delta\widetilde{y}_2] &= 0. \end{aligned}$$

Hence, this discrete process is consistent with the continuous process for any time step (not just in the limit). Note that the up-steps here are $u_i = e^{\ell_i}$ and the down-steps $d_i = 1/u_i$.

If the correlations $\rho_{ij} = 0$ for all $i \neq j$, then the improved log-transformed approach is the same as the log-transformed method proposed in Section 3, because, in this case, matrix W reduces to the identity matrix $\mathcal{I}_N$.

To illustrate the above, consider the following numerical example.

Example 1. Let the parameters of the process (3.1) be $\alpha_1 = 0.05$, $\alpha_2 = 0.08$, $\sigma_1 = 0.02$, $\sigma_2 = 0.3$, $\rho = 0.9$ and $dt = 0.1$. Given these parameters, the covariance matrix is

$$\Omega = \begin{pmatrix} 0.0004 & 0.0054 \\ 0.0054 & 0.09 \end{pmatrix}.$$

The eigenvalues of Ω are $\lambda_1 = 0.0001$ and $\lambda_2 = 0.0903$ and

$$W = \begin{pmatrix} -0.9982 & 0.0599 \\ 0.0599 & 0.9982 \end{pmatrix}.$$

With this change of basis, the parameters of the process in (4.2) are

$$A = \begin{pmatrix} -0.0476 \\ 0.0379 \end{pmatrix}, \qquad B = \begin{pmatrix} -0.0199 & 0.0179 \\ 0.0012 & 0.2994 \end{pmatrix},$$

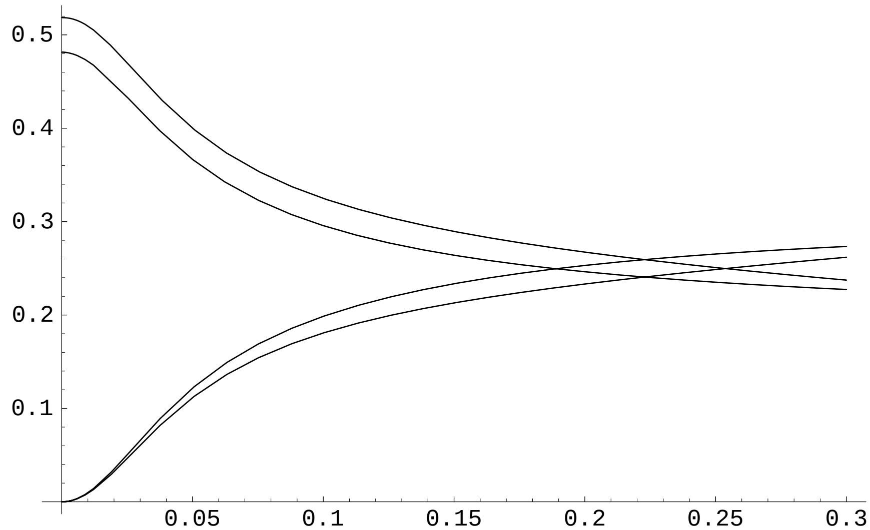
$\ell_1 = 0.0055$ and $\ell_2 = 0.0951$. Since $L_1 = -0.8658$ and $L_2 = 0.0399$, the resulting probabilities of the process in (4.2) are:

$$\begin{aligned} p_{uu} &= p_1 = 0.0349 \\ p_{ud} &= p_2 = 0.0322 \\ p_{du} &= p_3 = 0.4850 \\ p_{dd} &= p_4 = 0.4479. \end{aligned}$$

The above can be easily generalized to the N-dimensional case: concerning the matrices Λ and W, instead of having exact expressions as in the two-dimensional case, in the N-dimensional case they are be computed numerically (with fair accuracy). The discrete process $\widetilde{y}^\top = (\widetilde{y}_1, \ldots, \widetilde{y}_N)$ such that

$$\widetilde{y}_i(t) = \widetilde{y}_i(t-1) + \ell_i U_i(t) \qquad i = 1, \ldots, N$$

Figure 3: Log-transformed probability (p_1, p_2, p_3, p_4) vs volatility of the first underlying asset (σ_1) in a two-dimensional case (with $\alpha_1 = 0.05$, $\alpha_2 = 0.08$, $\sigma_2 = 0.3$, $\rho = 0.9$, $T = 1$ and $\Delta t = 1/10$).



has parameters as in (4.3) for $i = 1, \ldots, N$ and probabilities:

$$p(s) = \frac{1}{S}\left(1 + \sum_{1 \le i < j \le N} \delta_{ij}(s) L_i L_j + \sum_{i=1}^{N} \delta_i(s) L_i\right) \tag{4.5}$$

with $s = 1, \ldots, S$, where $S = 2^N$ is the number of states at the end of any time step. This approximates the first two moments of the continuous-time diffusion y:

$$\begin{aligned}
\mathbb{E}[\Delta\widetilde{y}_i] &= \kappa_i = A_i \Delta t && i = 1, \ldots, N \\
\operatorname{Var}[\Delta\widetilde{y}_i] &= \ell_i^2 - \kappa_i^2 = \lambda_i \Delta t && i = 1, \ldots, N \\
\operatorname{Cov}[\Delta\widetilde{y}_i, \Delta\widetilde{y}_j] &= 0 && i \neq j
\end{aligned}$$

for any time step.

It can be easily proved that the probabilities of the improved log-transformed approximation are positive and lower than one for any parameter values in the two-dimensional case, we observe that Equation (4.4) can be written also as

$$p(s) = \frac{1}{S}\left(1 + \delta_1(s) L_1\right)\left(1 + \delta_2(s) L_2\right).$$

Since $|L_i| < 1$ for all i, from Equation (4.3), then $0 < p(s) < 1$ for all $s = 1, \ldots, S$. Note also that the same argument applies for the first kind of log-transformed approximation if, in Equation (3.8), the correlation parameters are all zero.

The fact that the probability of the improved log-transformed approach is always positive makes the method unconditionally stable in the two dimensional case. Numerical experiments suggest that the probabilities are positive (and the algorithm stable) for a wider choice of parameters than the BEG and the log-transformed algorithms presented above.

5 Applications

The numerical procedure to evaluate a contingent claim dependent on several underlying assets according to the log-transformed approximation is similar to the BEG algorithm (see [7]): first, a multi-dimensional binomial tree of future asset prices is generated according to the parameters in (3.4) for the log-transformed or (4.3) for the improved log-transformed until the time horizon is attained. Next, if the option is European, we need only to evaluate the payoff on the leaves of the tree and properly averaging and discounting (according to a risk-neutral valuation approach) to obtain the current ($t = 0$) market price of the contingent claim. If the option is American, we can easily take into account the early exercise feature in a backward dynamic programming fashion by comparing, at each node of the lattice, the current payoff with the continuation value obtained by applying the risk-neutral approach. Also the case of discrete dividend-like payments can be accommodated (see Trigeorgis [31] for details). For an American contingent claim, the efficiency is improved (i.e., the computational effort is reduced, leaving accuracy unchanged) if, instead of generating the whole lattice at the outset, at each time step t, only asset prices for t and $t+1$ are generated, in order to apply the dynamic programming valuation.

In this section we provide the results of some numerical experiments to illustrate the accuracy and efficiency of the log-transformed methods we proposed compared to the BEG's scheme. Note that the number of operations and the amount of computing time needed by the three methods are the same.[12]

The first example is related to a financial option pricing problem. We use the numerical results presented in Boyle, Evnine e Gibbs [7] as a benchmark. The second example is a real option valuation problem presented in Martzoukos and Trigeorgis [23], involving four sources of uncertainty. We provide numerical valuations and compare them with those presented in [23], obtained with the BEG's scheme. The third case deals with the option on the maximum of five assets presented in Broadie and Glasserman [11]. Note that no accurate value is offered in the literature for this case. Moreover, the most accurate results presented by Broadie and Glasserman in this case are downward biased because, instead of valuing an American option, they

[12] The routine for diagonalizing the covariance matrix comes into play only once, at the beginning of the valuation algorithm, and so it does not affect computational complexity.

evaluate a Bermudan option with four exercise dates. Hence, we benchmark our results also against the ones obtained by Least-Squares Monte Carlo (LSM) algorithm proposed by Longstaff and Schwartz [22] with a higher number of dates in which the option can be exercised.

5.1 European and American options on the max, the min and the average of three assets

As for the first example, we evaluate four European options on three underlying assets: an option on the maximum, an option on the minimum, an option on the geometric average and an option on the arithmetic average of the prices of the assets. The underlying assets do not pay dividends. The parameter of the valuation problem are:

initial value:	$X_i(0) = 100$	$i = 1, 2, 3$
drift:	$\alpha_i = r = 0.1$	$i = 1, 2, 3$
volatility:	$\sigma_i = 0.2$	$i = 1, 2, 3$
correlation:	$\rho_{ij} = 0.5$	$i \neq j, i, j = 1, 2, 3$
maturity:	$T = 1$	
strike price:	$K = 100$	

Table 1 shows the estimates of the option prices given in [7] and the relative absolute errors with respect to the accurate value. For the option on the arithmetic average, since in [7] an accurate value is not presented, we take as accurate the one value obtained by Richardson extrapolation (RE) with four points ($n = 20, 40, 60, 80$). As suggested by Boyle, Evnine and Gibbs, RE is a practical method to obtain accurate approximations of exact values avoiding unnecessary computing. In particular, we use a four point RE, that is, we fit option values (as a function of $1/n$) with a cubic polynomial.

To comment the results, the numerical estimates of the option prices given by the three approaches (BEG, LT1 and LT2[13]) converges as the number of steps grows. As far as accuracy is concerned, generally, both the log-transformed approaches are more accurate that the BEG' scheme. As for the options on the max and on the min, either LT1 or LT2 seems to have the same accuracy for as few as 20 time steps as the BEG's approach for much higher number of steps. The only exception is the case of the put option on the arithmetic average of the prices of the underlying assets: here the log-transformed approaches are dominated by the BEG approach. One possible reason for this to happen is our choice for the accurate value.

[13]LT1 stands for the first log-transformed approach (Section 3) and LT2 for the improved log-transformed approach (Section 4).

5.2 Bets of product standards with four underlyings

The second numerical example is related to real option valuation and is drawn from Martzoukos and Trigeorgis [23]. This example permits to see the influence of various parameter values on the accuracy of the numerical methods proposed in this work.

In particular, there is a firm which is considering the development of two product standards in consumer electronic industry in a given time horizon. The standard that finally will prevail is uncertain. If the firm invests in both technologies, it acquires an option on the best of two assets (product standards). Each underlying asset of this option is the market value of the resulting cash flows if that standard prevails. Moreover, the underlying assets are correlated. The cost of introducing each standard is the strike price of the option. Also the strike prices for the two technologies are stochastic and correlated with the other state variables. Hence, we have evaluated an American-like option to invest in both technologies with payoff

$$\max\{V_1 - C_1, V_2 - C_2, 0\}$$

where V_i is the market value of i-th underlying (i.e., the value of cash flows obtained by product standard i), C_i is the cost to introduce the standard i, $i = 1, 2$. These variables are assumed to follow correlated geometric Brownian motions. Besides the above described case (see Table 2), we have evaluated different versions of the investment problem by considering several features of the opportunity of the firm. In particular, we have evaluated also the impact of higher volatility, lower correlation, longer maturity of the option, different investment scale on the option value and for a completely different choice of parameters with respect to the base case. The results[14] show that the LT2 algorithms is, for the same number of steps, far better accurate than the BEG and the LT1 algorithms and, when correlations $\rho_{ij} = 0$ for all $i \neq j$, the numerical results obtained with LT1 and LT2 approximations are the same.

The base case parameters are

initial asset value:	$V_i(0) = 100$	$i = 1, 2$
initial cost value:	$C_i(0) = 100$	$i = 1, 2$
drift:	$\alpha_i = r = 0.07$	$i = 1, \dots, 4$
volatility:	$\sigma_i = 0.2$	$i = 1, \dots, 4$
correlation:	$\rho_{ij} = 0.5$	$i \neq j, i, j = 1, \dots, 4$
maturity:	$T = 2$	

To compare numerical results with exact solutions, we consider also the case of non-stochastic development costs for both technologies and $C_1 = C =$

[14] The numerical results for the variations on the base case though not included in this paper for the sake of brevity, are available from the Authors on request.

C_2. With this choice of parameters the problem has an analytic solution: if both the dividend yields are zero ($\delta_{V_1} = 0 = \delta_{V_2}$), then the model reduces to the European[15] option on the maximum of two risky assets and the solution formula has been provided by Stulz [28]; if at least one of the dividend yields is not zero, the extension of Stulz' formula for the European option on the maximum of two assets is in Martzoukos and Trigeorgis [23].[16]

By inspection of the numerical results, it can be easily assessed that the improved log-transformed approach is the most efficient one in all cases. The accuracy of the improved log-transformed with 12 or 24 steps is comparable to the one given by the other methods with a much higher number of steps. Moreover, if we compare the numerical estimates of option prices with exact prices (see Table 3), then we can see that the accuracy of the improved log-transformed approach is much greater than the one offered by the other methods for the same number of steps.

In Figure 4 we describe the convergence of the three binomial lattice approaches for an option on the max of two asset prices both of the European and the American type. It is easy to see that the convergence rate of the improved log-transformed approach is faster than the other two methods and that the rate of convergence of the LT1 approach is the same as the BEG algorithm.

5.3 American options on five assets

This example is drawn from Broadie and Glasserman [11]. We estimate the price of an American option on the maximum of five assets. In the article [11, Tables 5 and 6]), Broadie and Glasserman (BG, thereafter) provide confidence bounds for the price estimate of this option by means of a numerical approach based on simulated trees with a small number of dates where early exercise is allowed (in their examples, there are only 4 dates) and with a

[15]Note that, if both the dividend yields are zero, the American option and the European option are the same.

[16]For convenience, we report the valuation formula for the European option price $C_{\max}$ on the maximum of two assets with $\delta_{V_i} = \delta_i \neq 0$, $\sigma_{V_i} = \sigma_i$, $i = 1, 2$:

$$C_{\max} = C_{\mathrm{BS}}\left(V_1, C, \delta_1, r, T\right) + C_{\mathrm{BS}}\left(V_2, C, \delta_2, r, T\right) - C_{\min}\left(V_1, V_2, C, \delta_1, \delta_2, r, T\right) \qquad (5.1)$$

where C_{BS} is the price of the European call option according to Black and Scholes formula, and $C_{\min}$ is price of European option price on the minimum of two assets:

$$\begin{aligned} C_{\min} = {} & V_2 e^{-\delta_2 T} \mathcal{N}\left[d_1 + \sigma_2\sqrt{T}, \left(\log(V_1/V_2) + (\delta_2 - \delta_1 - \sigma^2/2)T\right)/(\sigma\sqrt{T}), \rho_1\right] \\ & + V_1 e^{-\delta_1 T} \mathcal{N}\left[d_2 + \sigma_1\sqrt{T}, \left(\log(V_2/V_1) + (\delta_1 - \delta_2 - \sigma^2/2)T\right)/(\sigma\sqrt{T}), \rho_2\right] \\ & - C e^{-rT} \mathcal{N}\left[d_1, d_2, \rho_{12}\right] \end{aligned}$$

where $\mathcal{N}[a, b, \rho]$ is the bivariate cumulative Normal distribution integrated up to a and b for two variables with correlation ρ; $\rho_1 = (\rho_{12}\sigma_1 - \sigma_2)/\sigma$; $\rho_2 = (\rho_{12}\sigma_2 - \sigma_1)/\sigma$; $\sigma = \left(\sigma_1^2 + \sigma_2^2 - 2\rho_{12}\sigma_1\sigma_2\right)^{1/2}$, $d_i = \left[\log(V_i/C) + (r - \delta_i - \sigma_i^2/2)T\right]/(\sigma_i\sqrt{T})$, $i = 1, 2$.

large number of branches (50) at each node. With respect to this, BG do not value truly American options: actually, they price Bermudan options with four exercise dates. Moreover, although they present accurate values with two underlying assets (based on Kamrad and Ritchen algorithm [19]), in the five asset case there is no such value. Hence, no assessment of the relative error is available from BG [11] for the five asset case.[17]

The main flaw in the BG's approach is that the value it provides is always downward biased as far as truly American options are concerned, because they compute sub-optimal values when early exercise is allowed in any date before maturity. Actually, their confidence range is always lower than the true confidence range for an American option. This is apparent especially when they improve their estimates by a Control Variate technique: their confidence bounds shrinks around a sub-optimal value for the American option, though optimal for a Bermudan option with four early exercise dates.

In order to overcome this flaw and to assess accuracy of the log-transformed algorithms, first we present numerical results for the two-dimensional case [11, Table 3] based on the Least-Squares Monte Carlo[18] approach (LSM) proposed by Longstaff and Schwartz [22] and we benchmark our results over both the results provided by LSM and the confidence bounds given Broadie and Glasserman to establish accuracy of our method. Next, we proceed with the five-dimensional case, where there are no known results, providing the values based on the LSM algorithm. Then we show that the log-transformed algorithm provide accurate estimates for the American option price on five assets with a fair number of steps.

[17] In [11, page 1339] they note that "relative errors are not reported because the true value is unknown. [. . .] With $k = 5$ [i.e., with five assets] the computations are prohibitive for n as small as 50. And even if the computations could be done, the resulting value would not be very accurate." Later, in the Conclusions, they suggest to use parallel machines or network of workstations to evaluate truly American options.

[18] The LSM algorithm is suited to properly evaluate the early exercise feature of the American-style securities. It is based on the Bellmann dynamic programming approach. In order to determine the exercise date (*stopping time*) of the option, at any time step they compare the payoff from immediate exercise with the expected value of the option one step ahead (*continuation value*). To estimate the continuation value, they simulate several paths of the underlying asset prices and approximate the continuation value (i.e., the expected value of the future payoff of the option calculated with respect to the conditional probability) with a suited polynomial of the asset prices. At each time step, the continuation value is estimated by regressing (by the least-squares method) the present value of the payoff of the option in the subsequent steps on a polynomial of the realizations of the underlying asset prices in the current step. Once the stopping time are determined for each path, the value of the option in computed by averaging the present value of the payoff obtained by applying, for each path, the above determined stopping rule. As observed by Longstaff and Schwartz [22], an approximation error of the continuation value in the LSM algorithm produces a downward biased option price estimate.

Concerning numerical experiments, the case parameters are

initial asset values:	$X_i(0) = 100$	$i = 1, \ldots, 5$
drift:	$\delta_i = 0.1$	$i = 1, \ldots, 5$
volatility:	$\sigma_i = 0.2$	$i = 1, \ldots, 5$
correlation:	$\rho_{ij} = 0.3$	$i \neq j, i, j = 1, \ldots, 5$
risk-free rate:	$r = 0.05$	
maturity:	$T = 1$	
strike:	$K = 100$	

and the option payoff is $\max\{X_1 - K, \ldots, X_5 - K, 0\}$.

To comment the results, by inspection of Table 5.3 we observe that in the case with two assets the lattice algorithms gives results within BG confidence bounds.[19] Also the results obtained by the LSM algorithm are within that bounds, though downward biased as observed in Longstaff and Schwartz [22]. Generally, the option prices provided by the LT1 and LT2 lattice algorithms are closed to the value given by BEG lattice algorithm, though the LT2 algorithm converges faster than the other lattice approaches to that value, as already observed in Figure 4. For the five asset case, the results obtained by the three lattice algorithms are almost the same and sometimes ($S_0 = 130$) are outside BG confidence intervals. In Figure 5.3 we presents the convergence rate for the three binomial lattice methods and compare them to the confidence bounds. As can be easily seen, the value obtained by lattice methods is always closer to the high estimator than to the low estimator in the BG's algorithm. Notably, the LT2 method provides options values very close to the most accurate ones with very few steps (with as few as 10 steps the difference from the value obtained with 27 steps is less that 0.1). Also in this case, the LSM algorithm provides downward biased option price estimates, though always within the confidence bounds. As for the computational cost of the three lattice methods, the log-transformed algorithms proves to be less time-consuming than the LSM.[20] Though the BG confidence bounds can be obtained very fastly, we should benchmark the efficiency of our approach with the BG's algorithm with a higher number of exercise dates to obtained results not affected by the flaw of sub-optimality with respect to American exercise feature. In that case, the efficiency of BG's algorithm would decline sharply.

[19] This is not the case if we consider the confidence bounds by BG improved with the control variate technique, as can be checked by comparing our results with [11, Table 6].

[20] One computation with LSM with five assets, $n = 50$ time steps and 50 000 paths takes about 10 minutes, whereas the LT algorithms takes 12 minutes with $n = 26$, on a standard PC with CPU speed 866 MHz and RAM of 320 MB. Nevertheless, we think that for problems with dimension higher than five the LSM algorithm is the most suited to evaluate American options.

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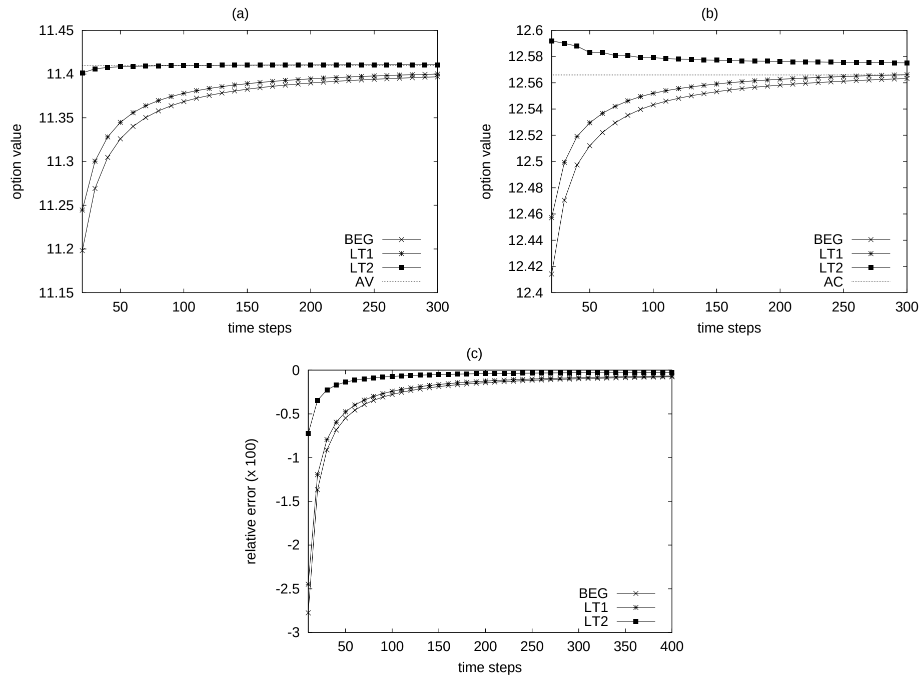
Figure 4: Convergence of the three binomial algorithms for European and American options on the max of two assets.



The payoff is max $\{V_1 - C, V_2 - C\}$ with $C_1 = C_2 = C = 100)$ and $\delta_i = 0.1,\ i = 1, 2$ and the other parameters are the same as in the base case.
a) European option. AC $= 11.41$ is the exact value from the analytic formula (see Martzoukos and Trigeorgis [23] or Equation (5.1).
b) American option. $AC = 12.566$ is the value obtained by the BEG algorithm with $n = 500$.
c) Relative errors for the three binomial algorithms for the case with $\delta_i = 0,\ i = 1, 2$ (the other parameters are the same as above). The exact value from the analytic formula in Stulz [28] is AC $= 26.61$. The relative error is

$$\left| \frac{F(n) - AC}{AC} \right| \cdot 100$$

where $F(n)$ is the value of the option computed with n time steps.

Table 1: European Call and Put Options on the max, on the min and on the arithmetic average of three stochastic underlying prices.

n	BEG	LT1	LT2	BEG	LT1	LT2
MAX		Call			Put	
20	22.281	22.311	22.685	0.919	0.935	0.951
	(1.725%)	(1.591%)	(0.058%)	(1.816%)	(0.064%)	(1.595%)
40	22.479	22.495	22.679	0.925	0.933	0.941
	(0.851%)	(0.779%)	(0.032%)	(1.175%)	(0.276%)	(0.575%)
60	22.544	22.555	22.677	0.928	0.933	0.938
	(0.565%)	(0.516%)	(0.023%)	(0.855%)	(0.314%)	(0.259%)
80	22.576	22.585	22.676	0.929	0.933	0.937
	(0.423%)	(0.385%)	(0.019%)	(0.748%)	(0.327%)	(0.104%)
RE[1]	26.673	22.672	22.673	0.936	0.933	0.933
AV[2]	22.672			0.936		

MIN		Call			Put	
20	5.226	5.266	5.241	7.24	7.264	7.403
	(0.438%)	(0.323%)	(0.144%)	(2.202%)	(1.879%)	(0.001%)
40	5.237	5.257	5.244	7.323	7.335	7.405
	(0.229%)	(0.147%)	(0.095%)	(1.081%)	(0.919%)	(0.023%)
60	5.241	5.254	5.245	7.35	7.359	7.405
	(0.152%)	(0.093%)	(0.070%)	(0.716%)	(0.599%)	(0.031%)
80	5.243	5.253	5.246	7.364	7.370	7.406
	(0.114%)	(0.068%)	(0.057%)	(0.527%)	(0.440%)	(0.034%)
RE[1]	5.249	5.249	5.248	7.403	7.406	7.406
AV[2]	5.249			7.403		

ARITH		Call			Put	
20	12.06	12.103	12.062	2.566	2.593	2.550
	(0.199%)	(0.156%)	(0.181%)	(0.000%)	(1.038%)	(0.633%)
40	12.072	12.094	12.079	2.567	2.581	2.564
	(0.099%)	(0.081%)	(0.045%)	(0.039%)	(0.568%)	(0.068%)
60	12.076	12.091	12.083	2.567	2.576	2.568
	(0.066%)	(0.055%)	(0.012%)	(0.039%)	(0.405%)	(0.062%)
80	12.078	12.089	12.084	2.567	2.574	2.569
	(0.050%)	(0.041%)	(0.000%)	(0.039%)	(0.318%)	(0.106%)
RE[1]	12.084	12.083	12.085	2.566	2.567	2.569
AV[2]	12.084			2.566		

The payoff of the option on the arithmetic average is $\max\{(X_1+X_2+X_3)/3-K, 0\}$.
Case parameters: $X_i(0) = 100$, $\alpha_i = r = 0.1$, $\sigma_i = 0.2$, $i = 1,2,3$, $\rho_{ij} = 0.5$, $i \neq j$, $i,j = 1,2,3$, $T = 1$ and $K = 100$. BEG is the result from Boyle, Evnine and Gibbs [7] algorithm; LT1 from the log-transformed approach and LT2 is from the improved log-transformed approach.

In parentheses, the relative absolute error with respect to the accurate value.

[1] RE = Richardson extrapolation for $n = 20, 40, 60, 80$.

[2] AV = accurate value from [7].

Table 2: Option to choose the best of two product standards, each with stochastic cost: four stochastic assets.

	European			American		
n	BEG	LT1	LT2	BEG	LT1	LT2
12	15.503	15.502	15.797	16.217	16.219	16.467
	(1.631%)	(1.636%)	(0.237%)	(1.597%)	(1.587%)	(0.078%)
24	15.641	15.644	15.786	16.360	16.362	16.483
	(0.753%)	(0.738%)	(0.167%)	(0.727%)	(0.717%)	(0.019%)
36	15.684	15.686	15.779	16.404	16.405	16.485
	(0.482%)	(0.471%)	(0.123%)	(0.463%)	(0.455%)	(0.030%)
48	15.704	15.706	15.775	16.425	16.426	16.485
	(0.353%)	(0.344%)	(0.097%)	(0.337%)	(0.330%)	(0.030%)
RE[1]	15.760	15.760	15.760	16.484	16.483	16.482

Base case parameters: $V_i(0) = 100$, $C_i(0) = 100$, $\alpha_i = r = 0.07$, $\sigma_i = 0.2$, $\rho_{ij} = 0.5$ and $T = 2$. MT[2] = 15.76 for European and MT[2] = 16.48 for American option.

In parentheses, the relative absolute error with respect to the accurate value.

[1] RE = Richardson extrapolation for $n = 12, 24, 36, 48$.

[2] MT = results from Martzoukos and Trigeorgis [23].

Table 3: Option to choose the best of two product standards (on "dividend" paying assets) with non-stochastic development costs as known benchmarks.

	$\delta_i = 0.1$ European			American		
n	BEG	LT1	LT2	BEG	LT1	LT2
12	11.056	11.131	11.388	12.324	12.394	12.601
	(3.107%)	(2.443%)	(0.189%)	(1.925%)	(1.366%)	(0.277%)
24	11.234	11.272	11.404	12.442	12.478	12.596
	(1.545%)	(1.206%)	(0.054%)	(0.987%)	(0.702%)	(0.242%)
36	11.293	11.319	11.407	12.489	12.513	12.587
	(1.027%)	(0.799%)	(0.025%)	(0.612%)	(0.419%)	(0.169%)
48	11.322	11.342	11.408	12.510	12.528	12.583
	(0.768%)	(0.596%)	(0.015%)	(0.446%)	(0.301%)	(0.140%)
RE[1]	11.411	11.411	11.410	12.549	12.549	12.583

	$\delta_i = 0$ European		
n	BEG	LT1	LT2
12	25.998	26.072	26.452
	(2.298%)	(2.020%)	(0.595%)
24	26.307	26.346	26.534
	(1.138%)	(0.993%)	(0.285%)
36	26.408	26.434	26.560
	(0.759%)	(0.661%)	(0.188%)
48	26.458	26.478	26.572
	(0.570%)	(0.496%)	(0.141%)
RE[1]	26.608	26.608	26.607

The parameters are the same as in the base case but the costs are non stochastic and $C_1 = C_2 = C = 100$). The problem collapses on the valuation of an option on the max of two assets with a given strike price C: $\max\{V_1 - C, V_2 - C\}$. We consider both the case with $\delta = 0$ and $\delta_i = 0.1$, $i = 1, 2$. AC[2] $= 11.41$ for the European and AC[2] $= 12.566$ for the American option with $\delta_i = 0.1$; AC[2] $= 26.61$ for both the European and American option when $\delta_i = 0$.

In parentheses, the relative absolute error with respect to the accurate value.

[1] RE = Richardson extrapolation for $n = 12, 24, 36, 48$.

[2] As far as the American option with $\delta = 0.1$ is concerned, AC is the value obtained by the BEG algorithm with $n = 500$; for the European options, AC is the exact value from analytic formula (see Stultz [28] for the case $\delta = 0$ and Martzoukos and Trigeorgis [23] or Equation (5.1) for the case $\delta \neq 0$).

Table 4: American option on the maximum of five assets (Broadie and Glasserman [11, Table 5]).

S_0	L&S	BEG	LT1	LT2	Bounds BG
70	0.541	0.544	0.553	0.568	[0.536,0.581]
80	2.633	2.761	2.782	2.790	[2.578,2.746]
90	7.678	8.009	8.037	8.019	[7.674,8.069]
100	15.756	16.178	16.202	16.214	[15.634,16.319]
110	25.740	26.210	26.226	26.237	[25.359,26.276]
120	36.519	36.984	36.996	37.026	[36.121,37.107]
130	47.422	48.000	48.012	48.051	[46.785,47.888]

The payoff is $\max\{X_1 - K, \ldots, X_5 - K, 0\}$. The parameters are $r = 0.05$, $T = 1$, $K = 100$, $X_i(0) = 100$, $\sigma_i = 0.2$, $\rho_{ij} = 0.3$ and $\delta_i = 0.1$ for all i.

The upper and lower bounds in the last column are represented by the 90% confidence interval of the distribution of the estimate of the option price from Broadie and Glasserman [11].

BEG, LT1, LT2: average of the value obtained with 25 and 26 steps. We do that because the numerical results for the BEG and LT1 algorithms oscillate; the LT2 algorithm produce a monotonic path towards the (unknown) asymptotic option value.

L&S: estimate of the option value using the LSM algorithm (see Longstaff and Schwartz [22]) with $n = 50$ time steps and 50 000 paths; we approximate the continuation value of the option by regressing data on a 5 degree polynomial including all mixed terms up to second degree.

Table 5: American option on the maximum of two assets (Broadie and Glasserman [11, Table 3]) as known benchmarks.

S_0	L&S	BEG	LT1	LT2	Bounds BG
70	0.237	0.245	0.245	0.245	[0.234,0.263]
80	1.259	1.305	1.306	1.305	[1.191,1.281]
90	4.081	4.216	4.218	4.215	[3.938,4.200]
100	9.475	9.628	9.630	9.636	[9.075,9.644]
110	17.210	17.347	17.349	17.350	[16.558,17.461]
120	26.388	26.544	26.545	26.548	[25.515,26.599]
130	36.346	36.453	36.453	36.457	[35.221,36.583]

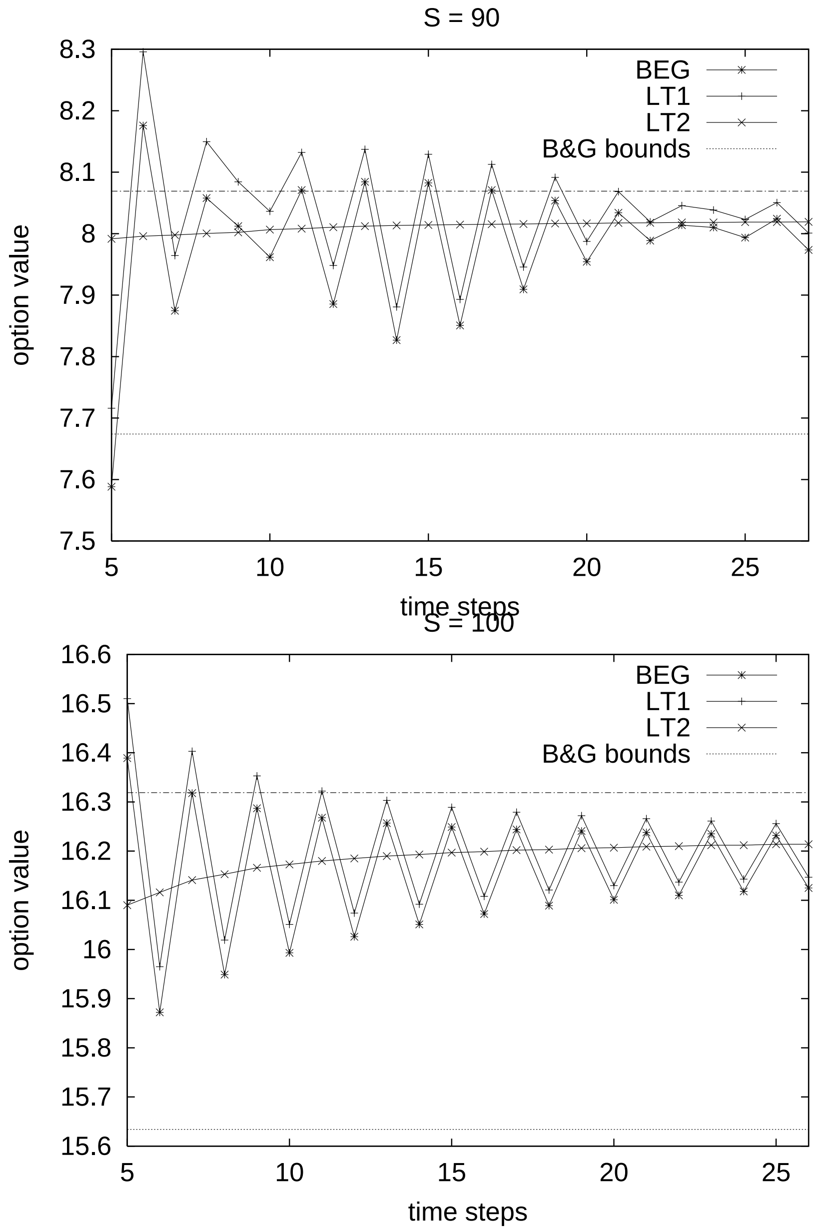
The parameters are $r = 0.05$, $T = 1$, $K = 100$, $X_i(0) = 100$, $\sigma_i = 0.2$, $\rho = 0.3$ and $\delta_i = 0.1$ for all i.

The upper and lower bounds in the last column are represented by the 90% confidence interval of the distribution of the estimate of the option price by Broadie and Glasserman [11].

BEG, LT1, LR2 with $n = 300$ time steps.

L&S: estimate of the option value using the LSM algorithm (see Longstaff and Schwartz [22]) with $n = 50$ time steps and 100 000 paths; we approximate the continuation value of the option by regressing data on a 5 degree polynomial including all mixed terms up to second degree.

Figure 5: Convergence paths for the three binomial algorithms for an American option on the max of five assets (Broadie and Glasserman [11]). The stright lines in each plot represents the upper and lower bounds given by Broadie and Glasserman (Table 5) obtained by simulation.



The parameters are $r = 0.05$, $T = 1$, $K = 100$, $X_i(0) = 90, 100, 110$ respectively, $\sigma_i = 0.2$, $\rho_{ij} = 0.3$ and $\delta_i = 0.1$ for all i.