Options on the M Best of N Risky Assets[1]

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Abstract

This paper develops the existing methodology to determine a firm s growth option value. Instead of modeling the growth option value as a sum of individual growth options, it is modeled as an option on the M best of N investment opportunities. This option analogy facilitates the inclusion of exercise capacity constraints and interdependencies between individual investment opportunities to the valuation. Johnson s (1987) equation for the valuation of an option on the maximum and minimum of N risky assets is generalized. Option portfolio value maximizing portfolio composition is simulated with a portfolio of N similar non-correlated risky assets.

I INTRODUCTION

A small technology-based start-up firm may have a number of growth areas that it could enter, but only limited resources for a full-scale entry. The existence of multiple growth opportunities in the presence of exercise capacity constraints makes the valuation of a firm s growth option value methodologically challenging. The most common capacity constraints include the time needed for fund raising, the amount of funding that the company is able to raise, and the ability to hire people to form an organization to work on the investment opportunities. In some cases, the option exercise capacity constraints may be potentially alleviated by a firm s ability to abandon some of its existing, worse performing operations.

There is no previous research that would link the growth option value of a firm to its option exercise capacity. Determining the growth option value of a firm as a sum of individual growth opportunities provides in many cases a highly optimistic value. It is optimal for a firm to develop a number of opportunities in order to see which of the potential growth areas start growing. That does not mean, however, that the firm eventually would have the necessary exercise capacity to exercise all of the options that become attractive. Instead of a sum of options analogy, a more appropriate analogy for a firm s growth option value is to model it as an option on the best, second best, third best, and so on, of all the investment opportunities generated.

When valuing a firm it is necessary to recognize that a firm is a combination of assets and options that are linked to each other. Dixit and Pindyck (1994) show that firms can have several investments on-hold at different stages of the investment process, waiting for the critical exercise boundary. Some of the potential investments are optimally deferred while other, exceeding a critical value or price threshold, are exercised. In principle, assuming no exercise constraints, a firm could develop an infinite amount of options that would make the firm s sum of options value go to infinity.[1] In the presence of exercise constraints, however, a firm has to make choices. At each point in time, a firm has to choose the best investment opportunity, the second best, and so on, until all the attractive investment opportunities are exhausted or the option exercise capacity of the firm is reached.

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1 Dixit and Pindyck (1994) overcome this problem by assuming decreasing marginal productivity of capital investment. A capacity constraint is a different way of saying that there is a decreasing marginal productivity of capital investment.

This paper advances the application of option pricing theory in capital budgeting by providing an analogy of a firm s option value as a value of an option on the M best of N investment opportunities available. The analogy expands Myers (1977) option analogy of a firm s equity. The insight of Myers (1977) was to show that the equity of the firm is in essence an option on the future cash flows of the firm. This paper expands the analogy from Myers one option case to a multi-option case. The potential future cash flows of the firm are separated into different cash flow streams that can be attributed to N different investment opportunities. From these N investment opportunities, the firm has an option to undertake M, as determined by the exercise capacity. The analogy also expands the treatment of real option interdependencies from a two-dimensional setting (Childs 1995; Childs et al. 1998; Childs and Triantis, 1999) to a multi-dimensional setting. In the terminology of Childs and Triantis (1999), the capacity constraint refers to a budget constraint.

The analogy is made operational by generalizing Johnson s (1987) equation for the valuation single asset options to the valuation of multiple asset options. In order to demonstrate the characteristics of the new option analogy, this paper provides an analysis of the option value maximizing portfolio composition with a portfolio of similar non-correlated assets. First, the optimal portfolio composition is determined as a function of M and N. Optimal N with respect to a given M is approximated analytically. Second, numerical sensitivity analysis is provided to simulate the option portfolio value with different parameter values. Changes in the initial asset values, asset volatility, time to maturity, or the price for increasing N are all shown to alter the optimal portfolio composition. The analysis shows also that the optimal N is an approximately linear function of M when N is large relative to M. Propositions are provided for further empirical testing of the portfolio analogy. The portfolio analogy would imply that the optimal N/M is different in firms and industries with different asset volatility and research and development times.

The rest of this paper is organized as follows. Section II shows the derivation of Johnson s option valuation equations. Section III derives equations to extend the valuation to the option on the M best of N risky assets. Section IV provides an analysis of the option value maximizing portfolio composition with a portfolio of similar non-correlated assets. Finally, Section V provides a conclusion.

II OPTION ON THE MAXIMUM OR MINIMUM OF SEVERAL ASSETS

To provide a basis to value an option on the M best of N risky assets, it is useful to start by briefly reviewing Johnson s (1987) expansion to Stulz (1982). Based on the intuition derived from Margrabe (1978) and Cox and Ross (1976), Johnson (1987) values a call on the maximum of a number of risky assets as follows:

$$\begin{aligned} c_{Max,N} = {} & S_1 N_n\left(d_1\left(S_1, X, \sigma_1^2\right), d_1^{'}\left(S_1, S_2, \sigma_{12}^2\right), \ldots, d_1^{'}\left(S_1, S_n, \sigma_{1n}^2\right), \rho_{112}, \rho_{113}, \ldots\right) \\ & + S_2 N_n\left(d_1\left(S_2, X, \sigma_2^2\right), d_1^{'}\left(S_2, S_1, \sigma_{12}^2\right), \ldots, d_1^{'}\left(S_2, S_n, \sigma_{2n}^2\right), \rho_{212}, \rho_{223}, \ldots\right) \\ & + \ldots \\ & + S_n N_n\left(d_1\left(S_n, X, \sigma_n^2\right), d_1^{'}\left(S_n, S_1, \sigma_{1n}^2\right), \ldots, d_1^{'}\left(S_n, S_{n-1}, \sigma_{n-1n}^2\right), \rho_{n1n}, \rho_{n2n}, \ldots\right) \\ & - Xe^{-rT}\left(1 - N_n\left(-d_2\left(S_1, X, \sigma_1^2\right), -d_2\left(S_2, X, \sigma_2^2\right), \ldots, -d_2\left(S_n, X, \sigma_n^2\right), \rho_{12}, \rho_{13}, \ldots\right)\right) \end{aligned} \tag{1}$$

where

$$\sigma_{ij}^2 = \sigma_i^2 - 2\rho_{ij}\sigma_i\sigma_j + \sigma_j^2, \tag{2}$$

$$d_1^{'}\left(S_i, S_j, \sigma_{ij}^2\right) = \frac{\log\frac{S_i}{S_j} + \frac{1}{2}\sigma_{ij}^2 T}{\sigma_{ij}\sqrt{T}}, \tag{3}$$

$$d_1\left(S_i, X, \sigma_i^2\right) = \frac{\log\frac{S_i}{X} + \left(r + \frac{1}{2}\sigma_i^2\right)T}{\sigma_i\sqrt{T}} \text{ and } d_2\left(S_i, X, \sigma_i^2\right) = \frac{\log\frac{S_i}{X} + \left(r - \frac{1}{2}\sigma_i^2\right)T}{\sigma_i\sqrt{T}}, \tag{4}$$

Correlation coefficients in (1) can be determined by recognizing that

$$Cov\left(\log S_i^*, \log\frac{S_i^*}{S_j^*}\right) = Var\left(\log S_i^*\right) - Cov\left(\log S_i^*, \log S_j^*\right) = \sigma_i^2 - \rho_{ij}\sigma_i\sigma_j = \sigma_i\sigma_{ij}\rho_{iij}$$

$$Cov\left(\log\frac{S_i^*}{S_k^*}, \log\frac{S_i^*}{S_j^*}\right) = \sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k = \rho_{ijk}\sigma_{ij}\sigma_{ik}$$

$$\rho_{iij} = \frac{\sigma_i - \rho_{ij}\sigma_j}{\sigma_{ij}}, \quad \rho_{ijk} = \frac{\sigma_i^2 - \rho_{ij}\sigma_i\sigma_j - \rho_{ik}\sigma_i\sigma_k + \rho_{jk}\sigma_j\sigma_k}{\sigma_{ij}\sigma_{ik}} \tag{5}$$

In Johnson s equations, $c_{Max,\,N}$ is the value of a call option on the maximum of n risky assets. S_i indexed from 1 to n refers to values of $N=n$ underlying risky assets. S^* is the value of an underlying asset at expiration. X is the exercise price. It is assumed to be the same for all assets. T is the time to maturity of a European option. N_i is the i-variate standard cumulative normal, r is the risk-free rate of return, σ_i^2 is the variance of the rate of return on the ith asset, and ρ_{ij} is the correlation coefficient for the returns on the ith and jth assets. Johnson makes the assumptions of no dividends, geometric Brownian motion, and zero risk-free rate.

The intuition underlying Johnson s valuation equation is that each asset price S_i and the exercise price X are multiplied by a probability term. Based on Cox and Ross (1976), Johnson recognizes that the probability term that should be the multiplier of the exercise price is the probability that an option will be exercised. In the basic Black and Scholes (1973) call option valuation model, the Cox and Ross intuition means that $N(d_2)$ is the probability that the option is exercised, i.e. $Prob(S^*>X)$. In a multiple asset case, the probability that an option is exercised is the probability that at least one of the assets has a value higher than the exercise price, i.e. $1-Prob\left(S_1^*,S_2^*,\ldots,S_n^*<X\right)$. This is the multiplier in a multiple asset case. With multiple asset values following the geometric Brownian motion, the probability multiplier becomes (Johnson, 1987)

$$1-Prob\left(S_1^*\cap S_2^*\cap\ldots\cap S_n^*<X\right)=1-N_n\left(-d_2\left(S_1,X,s_1^2\right),\ldots,-d_2\left(S_n,X,s_n^2\right),r_{12},r_{13},\ldots\right)$$

For the underlying assets S_1-S_n, Johnson derives the probability expressions using Margrabe s (1978) intuition to change the numeraire in order to transform a call option into a put option. By changing the numeraire, it is trivial to note, based on a similar reasoning as above, that the positive term of a put is the risk neutral joint probability

$$Prob\left(\frac{X}{S_i^*}<1\cap\frac{S_1^*}{S_i^*}<1\cap\frac{S_2^*}{S_i^*}<1\cap\ldots\cap\frac{S_n^*}{S_i^*}<1\right)=Prob\left(S_i^*>X\cap S_i^*>S_1^*\cap S_i^*>S_2^*\cap\ldots\cap S_i^*>S_n^*\right)$$

When the interest rate is zero, Johnson shows that this joint probability is the same as

$$N_n\left(d_1\left(S_i,X,s_i^2\right),d_1'\left(S_i,S_1,s_{i1}^2\right),\ldots,d_1'\left(S_i,S_n,s_{ni}^2\right),r_{i1i},r_{i2i},\ldots\right)$$

Using the joint probabilities an equivalent expression for equation (1) becomes

$$\begin{aligned}c_{Max,N}=\;&S_1Prob\left(S_1^*>X\cap S_1^*>S_2^*\cap S_1^*>S_3^*\cap\ldots\cap S_1^*>S_n^*\right)\\&+S_2Prob\left(S_2^*>X\cap S_2^*>S_1^*\cap S_2^*>S_3^*\cap\ldots\cap S_2^*>S_n^*\right)\\&+\ldots\\&+S_nProb\left(S_n^*>X\cap S_n^*>S_1^*\cap S_n^*>S_2^*\cap\ldots\cap S_n^*>S_{n-1}^*\right)\\&-Xe^{-rT}\left(1-Prob\left(S_1^*,S_2^*,\ldots,S_n^*<X\right)\right)\end{aligned}\tag{1b}$$

Equation (1b) is used next in generalizing Johnson (1987) to M best of N risky assets.

III OPTIONS ON THE M BEST OF N RISKY ASSETS

In order to determine the value of options on the M best of N risky assets, the algorithm of Johnson can be applied recursively. The conditional probabilities of exercise have to be taken into account if the assets correlate. The procedure for valuing options on the M best of N risky assets is first outlined in the general case starting with a capacity constraint of M=2. The procedure is then expanded to M=3 and M=*m*. The complexity of the procedure increases as M increases.

The option on the two best of N risky assets can be modeled as two options: an option on the maximum and an option on the second best conditional on which maximum was chosen. The logic of expanding from one option on the maximum of N risky assets to two options is based on the multiplication rule of probability calculus, Bayesian conditional probabilities, and a recursive application of Johnson s equations, (1)-(5). It is possible to use the probabilities in a similar way than in Johnson s equations to determine the probability that asset S_i was the maximum and that the option on the maximum was exercised.

$$
\begin{aligned}
c_{Max,N,M=2}(S) = {} & Prob\left(S_1^* > X \cap S_1^* > S_2^* \cap S_1^* > S_3^* \cap \ldots \cap S_1^* > S_n^*\right) c_{Max,N-1}\left(S_2,\ldots,S_n \middle| S_1\right) \\
& + Prob\left(S_2^* > X \cap S_2^* > S_1^* \cap S_2^* > S_3^* \cap \ldots \cap S_2^* > S_n^*\right) c_{Max,N-1}\left(S_1,S_3\ldots,S_n \middle| S_2\right) \\
& + \ldots \\
& + Prob\left(S_n^* > X \cap S_n^* > S_1^* \cap S_n^* > S_3^* \cap \ldots \cap S_n^* > S_{n-1}^*\right) c_{Max,N-1}\left(S_1,\ldots,S_{n-1} \middle| S_n\right) \\
& + c_{Max,N}(S)
\end{aligned}
\tag{6}
$$

Equation (6) shows the logic for generalizing Johnson s equations. The value of a call on the two best of N assets is the sum of the option on the maximum of N risky assets plus the probability weighed options on the maximum of N-1 risky assets. The values of options on the maximum on N-1 assets are determined depending on which asset was the maximum of N assets. Since this asset is not known beforehand, probability weights have to be used and all the possible combinations calculated. In order to determine the value of an option on the two best of N risky assets, Johnson s equations have to be evaluated altogether N+1 times. With N similar underlying non-correlated assets, and assuming that the exercise of the first option takes place, the probabilities in equation (6) reduce to 1/N. Options on the maximum of all the different combinations of N-1 assets yield identical values. With N identical non-correlated assets, equation (6) becomes equation (7).

$$
c_{Max,N,M=2}(S_1,\ldots,S_n) = c_{Max,N}(S_1,\ldots,S_n) + c_{Max,N-1}(S_2,\ldots,S_n)
\tag{7}
$$

Writing out equation (6) according to Johnson provides the equation (8a) below to determine the value of an option on the two best of N risky assets. The derivation of input correlation matrices is provided in Appendix 1. The more detailed derivation of equation (8a) is provided in Appendix 2.

$$c_{Max,N,M=2}(S) = \tag{8a}$$

(1st term when S_1 has already been picked out as the first maximum)

$$S_2 N_n\left(-d_1'\left(S_2, S_1, s_{21}^2\right), d_1\left(S_2, X, s_2^2\right), d_1'\left(S_2, S_3, s_{23}\right), ..., d_1'\left(S_2, S_n, s_{2n}\right), -r_{221}, -r_{231}, ..., r_{223}, ...\right) +$$

$$S_3 N_n\left(-d_1'\left(S_3, S_1, s_{31}^2\right), d_1\left(S_3, X, s_3^2\right), d_1'\left(S_3, S_2, s_{32}\right), ..., d_1'\left(S_3, S_n, s_{3n}\right), -r_{331}, -r_{321}, ..., r_{332}, ...\right) +$$

$$+ ...$$

$$S_n N_n\left(-d_1'\left(S_n, S_1, s_{n1}^2\right), d_1\left(S_n, X, s_n^2\right), d_1'\left(S_n, S_2, s_{n2}\right), ..., d_1'\left(S_n, S_{n-1}, s_{nn-1}\right), -r_{nn1}, -r_{n21}, ..., r_{nn2}, ...\right)$$

$$- \; Xe^{-rT}\left(N_n\left(d_1\left(S_1, X, s_1^2\right), d_1'\left(S_1, S_2, s_{12}^2\right), ..., d_1'\left(S_1, S_n, s_{1n}^2\right), r_{112}, r_{113}, ...\right)\right.$$

$$\left. - N_n\left(d_2\left(S_1, X, s_1^2\right), -d_2\left(S_2, X, s_2^2\right) ..., -d_2\left(S_n, X, s_n^2\right), r_{12}, r_{13}, ...\right)\right)$$

$$+$$

(Nth term when S_n has already been picked out as the first maximum)

$$S_1 N_n\left(-d_1'\left(S_1, S_n, s_{12}^2\right), d_1\left(S_1, X, s_1^2\right), d_1'\left(S_1, S_2, s_{31}\right), ..., d_1'\left(S_1, S_{n-1}, s_{1n-1}\right), -r_{11n}, -r_{12n}, ..., r_{112}, ...\right) +$$

$$+ ...$$

$$S_{n-1} N_n\left(-d_1'\left(S_{n-1}, S_n, s_{n-1n}^2\right), d_1\left(S_{n-1}, X, s_{n-1}^2\right), d_1'\left(S_{n-1}, S_1, s_{n-11}\right), ..., d_1'\left(S_{n-1}, S_{n-2}, s_{-1n-2}\right), -r_{n-1n-1n}, ...\right)$$

$$- \; Xe^{-rT}\left(N_n\left(d_1\left(S_n, X, s_n^2\right), d_1'\left(S_n, S_1, s_{n1}^2\right), ..., d_1'\left(S_n, S_{n-1}, s_{nn-1}^2\right), r_{nn1}, r_{nn2}, ...\right)\right.$$

$$\left. - \left(N_n\left(d_2\left(S_n, X, s_n^2\right), -d_2\left(S_1, X, s_2^2\right) ..., -d_2\left(S_{n-1}, X, s_{n-1}^2\right), r_{n1}, r_{n2}, ...\right)\right)\right)$$

$$+ c_{Max,N}\left(S_1, ..., S_n\right)$$

where the parameters are defined as earlier in connection with equations (1) - (5). It is possible to use, for example, the Boyle-Tse (1990) algorithm to numerically solve equation (7). Solving numerically equation (8a) would require an algorithm that applies a more general multi-normal integral estimation algorithm (e.g. Drezner, 1992 or Genz, 1992). To simplify the equation, the following notation is adopted:

- Probability that S_i is the maximum of N assets S_1 S_n

$$Prob\left(S_i^* > X \cap S_i^* > S_1^* \cap S_i^* > S_3^* \cap ... \cap S_i^* > S_n^*\right) \equiv Prob\left(S_i^{Max}(N)\right)$$

- Conditional probability that S_i is the maximum when asset $S_{M(j)}$ has already been picked out

$$Prob\left(S_{M(j)}^* > S_i^* \cap S_i^* > X \cap S_i^* > S_2^* \cap S_i^* > S_3^* \cap ... \cap S_i^* > S_n^*\right) \equiv Prob\left(S_i^{Max}(N-1) \middle| S_{M(j)}\right)$$

- *M(j)* is used to denote the index of the asset that is chosen as the *j*th maximum asset

Using this simplified notation equation (8a) can be transformed into equation (8b).

$$c_{Max,N,M=2}(S)=\sum_{M(1)=1}^{N}\Bigg[\sum_{\substack{i=1\\ i\neq M(1)\\ j\subset[1,N],\,j\neq i\\ k\subset[1,N],\,k\neq M(1)}}^{N}\Big[S_i\Big[Prob\big(S_{M(1)}>S_i\cap S_i>X\cap S_i>S_j\big)\Big]\Big]$$

$$-Xe^{-rT}\Big[Prob\big(S_{M(1)}\big)-Prob\big(S_{M(1)}>X\cap X>S_k\big)\Big]\Bigg]+c_{Max,N}\left(S_1,\ldots,S_n\right) \qquad (8b)$$

Expanding from two to three assets can be carried out following a similar line of reasoning than from the one asset case to the two asset case. The detailed derivation of this expansion is provided in Appendix 2. Using the same notation as in (8b), the value of an option on the three best of N risky assets becomes

$$c_{Max,N,M=3}(\mathrm{S})=\sum_{M(1)=1}^{n}\Big[\sum_{\substack{M(2)=1\\ M(1)\neq M(2)}}^{n}\Big[\sum_{\substack{M(3)=1\\ M(i)\neq M(j)}}^{n}Prob\big(S_{M(3)}^{Max}(N-2)\cap S_{M(2)}^{Max}(N-1)\cap S_{M(1)}^{Max}(N)\big)S_{M(3)} \qquad (9)$$

$$-Xe^{-rT}\Big(Prob\big(S_{M(1)}^{Max}(N)\big)Prob\big(S_{M(2)}^{Max}(N-1)\big)-Prob\big(S_{M(1)}>S_{M(2)}\cap S_{M(2)}>X\cap X>S_{M(3)},\ldots,S_n\big)\Big)\Big]\Big]$$

$$+c_{Max,N,M=2}(\mathrm{S})$$

where $\boldsymbol{S}$ is the set of assets, $S_1,\ \ ,S_n$, from which the maximum is exercised if exercise is optimal. Continuing with the adopted notation, the first main proposition of this paper, regarding the generalized recursive equation for determining the value of an option on the M best of N risky assets, becomes

Proposition 1 *The value of an option on the M best of N risky assets can be determined from the equation*

$$c_{Max,N,M=m}\left(S_1,\ldots,S_n\right)= \qquad (10)$$

$$\sum_{M(1)=1}^{n}\Big[\sum_{\substack{M(2)=1\\ M(1)\neq M(2)}}^{n}\Big[\sum_{\substack{M(m)=1\\ M(i)\neq M(j)}}^{n}Prob\big(S_{M(m)}^{Max}(N-(m-1))\cap\ldots\cap S_{M(2)}^{Max}(N-1)\cap S_{M(1)}^{Max}(N)\big)S_{M(m)}$$

$$-Xe^{-rT}\Big(Prob\big(S_{M(1)}^{Max}(N)\big)\ldots Prob\big(S_{M(m)}^{Max}(N-(m-1))\big)-Prob\big(S_{M(1)}>S_{M(2)}\cap\ldots\cap S_{M(m-1)}>X\cap X>S_{M(m)},\ldots,S_n\big)\Big)\Big]\Big]$$

$$+c_{Max,N,M=m-1}\left(S_1,\ldots,S_n\right)\ \text{where}\ c_{Max,N,M=0}\left(S_1,\ldots,S_n\right)\equiv 0$$

Each increase in M makes it necessary to add a new nested loop into the equation. When M=1, the equation reduces into Johnson s basic equation. When M=2, the Johnson s equation is applied N+1 times with different parameter values. When M=3, Johnson s equation is evaluated N(N-1)+N+1 times, and so on. The total computational complexity is a function of M and N.

IV GROWTH OPTION PORTFOLIO VALUE WITH N SIMILAR, INDEPENDENT ASSETS

In practice, it may in some cases be possible to assume that the N assets in a portfolio have relatively similar characteristics. In the following, we can assume that (1) *there is a portfolio of assets possessing same initial asset values, variances, time-to-maturity, and exercise price;* (2) *the assets in the portfolio are independent;* (3) *each unit of exercise capacity M provides an identical option to exercise any one of the assets in the asset portfolio;* (4) *the exercise price is zero;* (5) *increasing M or N incurs a constant unit cost*, and (6) *at least M-1 assets are in the money at the time of exercise.* These assumptions are necessary for transforming equation (10) into equation (11).

$$c_{Max,N,M=m}\left(S_1,\ldots,S_n\right)=c_{Max,N}\left(S_1,\ldots,S_n\right)+c_{Max,N-1}\left(S_2,\ldots,S_n\right)+\ldots+c_{Max,N-m}\left(S_{m+1},\ldots,S_n\right) \quad (11)$$

The first assumption is a precondition for using equation (11). In practice, when the correlation matrix and the variances of individual assets in the portfolio can be estimated, it is always possible to use the more general equation. All the N assets in the asset portfolio are assumed to be independent of each other. The assumption of independence does not remove, however, the fact that when option exercise is considered the assets compete for the overall exercise capacity of the firm. Similarly to the first assumption, the assumption of independence is not necessary when using equation (10). The third assumption is necessary for defining the relationship between N and M. Another possibility would be that exercising different projects requires different amounts of exercise capacity. Incorporating this extension to the analysis would require an additional variable to define the number of units of exercise capacity that is required by each asset if exercised. The extension is deferred at this point. The assumption of zero exercise price would not be necessary when using the option valuation equation (10), but the assumption is required when using equation (11) and the Boyle and Tse (1990) algorithm. Assumption five introduces the costs of expanding M and N to the analysis. Without costs for expanding M and N, the value of the option on the M best of N assets would increase to infinity as M and N increase. There would be an incentive to maximize both and no optimal portfolio composition would emerge. The sixth assumption is a necessary requirement that ensures that the probability weights of each recursive step converge to 1 and, as a result, equations (6) and (10) converge to equations (7) and (11).

With the six assumptions the computational complexity decreases significantly. More importantly, however, equation (11) enables also the simulation of the option portfolio value using the Boyle and Tse (1990) algorithm. Using equation (11) and alternating the values of N and M and the other underlying parameters provides new intuition for option portfolio decision making. In general, when M is very small compared to N, increasing N contributes relatively little to the overall option value of a company s option portfolio. Increasing M in a situation where N is large relative to M can contribute significantly to the option value.

In the N similar, non-correlated assets case, we can approximate the optimal N/M analytically. Starting from the case where M=1, we can determine the optimal N from the Johnson s equation. Using the notation of Johnson, the resulting maximization problem can be formulated as follows:

$$
\begin{aligned}
&Max\left(c_{Max,N,M=1}\left(S_1,\ldots,S_n\right)-NP_N\right)= \\
&Max\Big[S_1N_n\left(d_1\left(S_1,X,s_1^2\right),d_1^{'}\left(S_1,S_2,s_{12}^2\right),\ldots,d_1^{'}\left(S_1,S_n,s_{1n}^2\right),r_{112},r_{113},\ldots\right) \\
&\qquad +S_2N_n\left(d_1\left(S_2,X,s_2^2\right),d_1^{'}\left(S_2,S_1,s_{12}^2\right),\ldots,d_1^{'}\left(S_2,S_n,s_{2n}^2\right),r_{212},r_{223},\ldots\right) \\
&\qquad +\ldots \\
&\qquad +S_nN_n\left(d_1\left(S_n,X,s_n^2\right),d_1^{'}\left(S_n,S_1,s_{1n}^2\right),\ldots,d_1^{'}\left(S_n,S_{n-1},s_{n-1n}^2\right),r_{n1n},r_{n2n},\ldots\right) \\
&\qquad -Xe^{-rT}\left(1-N_n\left(-d_2\left(S_1,X,s_1^2\right),-d_2\left(S_2,X,s_2^2\right),\ldots,-d_2\left(S_n,X,s_n^2\right),r_{12},r_{13},\ldots\right)\right)-NP_N\Big]
\end{aligned}
$$

where

$$s_{ij}^2=s_i^2-2r_{ij}s_is_j+s_j^2=2s_i^2$$

$$d_1^{'}\left(S_i,S_j,s_{ij}^2\right)=\frac{\log\frac{S_i}{S_j}+\frac{1}{2}s_{ij}^2T}{s_{ij}\sqrt{T}}=\frac{1}{2}s_{ij}\sqrt{T}=s_i\sqrt{\frac{T}{2}}$$

$$d_1\left(S_i,X,s_i^2\right)=\frac{\log\frac{S_i}{X}+\frac{1}{2}s_i^2T}{s_i\sqrt{T}}=\infty \text{ and } d_2\left(S_i,X,s_i^2\right)=\frac{\log\frac{S_i}{X}-\frac{1}{2}s_i^2T}{s_i\sqrt{T}}=\infty$$

$$Cov\left(\log S_i^*,\log\frac{S_i^*}{S_j^*}\right)=Var\left(\log S_i^*\right)-Cov\left(\log S_i^*,\log S_j^*\right)=s_i^2-r_{ij}s_is_j=s_i^2$$

$$Cov\left(\log\frac{S_i^*}{S_k^*},\log\frac{S_i^*}{S_j^*}\right)=s_i^2-r_{ij}s_is_j-r_{ik}s_is_k+r_{jk}s_js_k=s_i^2$$

$$r_{iij}=\frac{s_i-r_{ij}s_j}{s_{ij}}=\frac{1}{\sqrt{2}},\quad r_{ijk}=\frac{s_i^2-r_{ij}s_is_j-r_{ik}s_is_k+r_{jk}s_js_k}{s_{ij}s_{ik}}=\frac{1}{\sqrt{2}}$$

The maximization problem can further be simplified to

$$Max\Big[NS_iN_n\left(\infty,s_i\sqrt{\frac{T}{2}},\ldots,s_i\sqrt{\frac{T}{2}};\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\ldots\right)-NP_N\Big]=$$

$$Max\Big[NS_i\int_{-\infty}^{\infty,s_i\sqrt{\frac{T}{2}},\ldots,s_i\sqrt{\frac{T}{2}}}\det(2p\,\Sigma)^{-\frac{1}{2}}e^{-\frac{1}{2}\left(x\Sigma^{-1}x\right)}dx-NP_N\Big],\text{ where } \Sigma=\left|\begin{matrix}1 & s^2 & s^2\\ s^2 & 1 & s^2\\ s^2 & s^2 & 1\end{matrix}\right|$$

Solving the multi-normal integral analytically is not possible even in the simplified case. Fortunately, we can overcome the difficulty of integration by replicating the integral with a logarithmic function

$$NS_i \int_{-\infty}^{\infty, s_i\sqrt{\frac{T}{2}},\dots,s_i\sqrt{\frac{T}{2}}} \det(2p\,\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}\left(x\Sigma^{-1}x\right)} dx = NS_i\left(\frac{a(s,T)}{N}+\frac{b(s,T)}{N}\log N\right) = S_i a(s,T)+S_i b(s,T)\log N$$

where *a* and *b* are functions of s and *T*, but constants with respect to N. It can be numerically shown that the fit of the logarithmic function with the integral is close to 100 percent, see Appendix 4. When the logarithmic function is inserted into the place of the integral, the maximization problem becomes

$$Max\left[NS_i \int_{-\infty}^{\infty, s_i\sqrt{\frac{T}{2}},\dots,s_i\sqrt{\frac{T}{2}}} \det(2p\,\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}\left(x\Sigma^{-1}x\right)} dx - NP_N\right] = Max\left[NS_i\left(\frac{a(s,T)}{N}+\frac{b(s,T)}{N}\log N\right) - NP_N\right]$$

When determining the option value maximizing N, it is necessary to note that the function is continuous and concave. We can take the first derivative with respect to N and the option value maximizing N becomes:

$$\frac{\partial\left[\left(S_i a(s,T)+S_i b(s,T)\log(N)\right)-NP_N\right]}{\partial N} = \frac{S_i b(s,T)}{N} - P_N = 0 \quad => \quad N = \frac{S_i b(s,T)}{P_N}$$

where *a* and *b* are functions of s and *T* and not dependent on N. S_i is the initial asset value. Since all the assets have the same initial asset value we can leave the index away. P_N is the price of increasing one unit of N and the other parameters are defined as earlier in this paper. When M is increased, determining the optimal N follows a similar kind of logic even though the equations are somewhat more complex.

$$Max\left(c_{Max,N,M=M}\left(S_1,\dots,S_n\right)-NP_N\right) = Max\left(\left(MSa(s,T)+Sb(s,T)\left(\log(N)+\dots+\log\left(N-(M-1)\right)\right)\right)-NP_N\right)$$

$$\frac{\partial\left[\left(Ma(s,T)+Sb(s,T)\left(\log(N)+\dots+\log\left(N-(M-1)\right)\right)\right)-NP_N\right]}{\partial N} = \frac{1}{N}+\dots+\frac{1}{N-M+1} = \frac{P_N}{Sb(s,T)}$$

As long as the initial parameters defining the option portfolio stay the same, *b* stays the same, and the optimal N with respect to M can be determined numerically from the equation above. When M is relatively small and N is relatively large, we get an approximately linear relationship for N and M.

$\frac{1}{N}+\frac{1}{N-1}+\dots+\frac{1}{N-M+1} \approx \frac{M}{N} = \frac{P_N}{Sb(s,T)}$, that is, $\frac{N}{M}_{OPT} \approx \frac{Sb(s,T)}{P_N}$ as in the case of M=1.

Based on the above analytical solution to the optimal option portfolio problem, it is possible to put forward the second main proposition of this paper.

Proposition 2 *The relationship of the optimal number of assets N to a given exercise capacity M is a function of the initial asset values, asset variance, time to maturity, and the unit price of N. With a given set of initial parameter values, the optimal N is an approximately linear function of M.*

$$\frac{N}{M}_{OPT} \approx \frac{Sb(s,T)}{P_N} \Leftrightarrow N_{OPT} \approx \frac{Sb(s,T)}{P_N} M$$

where N_{OPT} is the optimal number of risky assets in the portfolio, M is the exercise capacity, S is the initial value of an asset, σ is the variance of an asset, P_M is the price of one unit of M, P_N is the price of one unit of N, and T is the time to maturity of the option. The price of exercise capacity expansion does not affect the position of the option value maximizing N in relation to M.

Since the multi-normal integral is positive and an increasing function of N, s, and T, also b has to be positive and an increasing function of s and T, as shown in Appendix 5. Consequently, the partial differentials with respect to different option portfolio parameters become

$$\frac{\partial\left(\frac{N_{OPT}}{M}\right)}{\partial S} \approx \frac{\partial \frac{Sb(s,T)}{P_N}}{\partial S} = \frac{b(s,T)}{P_N} > 0$$

$$\frac{\partial\left(\frac{N_{OPT}}{M}\right)}{\partial s} \approx \frac{\partial \frac{Sb(s,T)}{P_N}}{\partial s} = \frac{1}{P_N}\frac{S\partial b(s,T)}{\partial s} > 0$$

$$\frac{\partial\left(\frac{N_{OPT}}{M}\right)}{\partial T} \approx \frac{\partial \frac{Sb(s,T)}{P_N}}{\partial T} = \frac{1}{P_N}\frac{S\partial b(s,T)}{\partial T} > 0$$

$$\frac{\partial\left(\frac{N_{OPT}}{M}\right)}{\partial P_N} \approx \frac{\partial \frac{b(s,T)}{P_N}}{\partial P_N} = -\frac{Sb(s,T)}{P_N^2} < 0$$

Since the above results are approximations, it is useful to supplement them with numerical analysis. The purpose of the numerical analysis is to demonstrate the above results regarding the optimal portfolio composition and its sensitivity to the initial portfolio parameters.

The numerical analysis of the option portfolio valuation is divided into two separate parts. At the first stage, the basic shape of Johnson s (1987) option value curve as a function of the number of assets in the portfolio is examined. At the second stage, M is allowed to deviate from 1, the costs for expanding N are included into the analysis, and an analysis is carried out to determine the resulting optimal portfolio composition.

The numerical analysis with Johnson s equation (1987) shows that the relationship between the number of assets N and the value of an option on the maximum of N assets is concave. If increasing N can be carried out without costs, the first derivative of the option value function with respect to N is continuously positive and the second derivative is negative. Five option value curves are provided in Appendix 5. When a constant per unit cost for increasing N is introduced, the first derivative of the option value function with respect to N can reach zero and become negative. With a constant unit cost for N an option portfolio value maximizing portfolio size can always be found.

The approximate linearity, shown through the analytical derivation, is clearly evident also in the numerical analysis when N is large in relation to M. Five simulations with five sets of asset portfolio parameter values, N ranging from 2 to 1000 and M ranging from 1 to 10, were carried out. In each simulation, the optimal N/M was determined. The curves of optimal N with respect to M are shown in Figure 1. Due to the approximate linearity, as shown analytically earlier, the curves resemble straight lines.

[INSERT FIGURE **1** HERE]

Figure 1. The option value maximizing N and the exercise capacity M. The changing parameter values are shown in the figure.

Changes in optimal portfolio size with respect to a given exercise capacity are demonstrated in Figure 1 as a result of changes in initial asset values, asset variance, and the time to maturity. A proportional change in the initial asset values is directly transferred in same proportion as a change in the optimal N/M as already analytically shown in *Proposition 2*. The magnitude of effects when changing the variance and the time to maturity has to be examined by examining parameter *b*. A numerical analysis of the sensitivity of parameter *b* with respect to asset variance and the time to maturity is provided in Appendix 5.

The results of the numerical analysis are in line with the option pricing theory and provide numerical support for the analytical results derived earlier

- Increasing the asset variance increases the optimal portfolio N in relation to M.
- Increasing the time to maturity increases the optimal portfolio N in relation to M.
- Increasing the initial asset values increases the optimal portfolio N in relation to M.
- Increasing the price of N decreases the optimal portfolio N in relation to M.

The results of the simulation analysis imply that for firms in volatile industries it could be optimal to develop and hold a proportionately large portfolio of opportunities in relation to the exercise capacity of the firm. In volatile industries, the asset variance can be expected to be on average higher than on average in less volatile industries. A higher asset variance increases the optimal portfolio N with respect to M. As a consequence, the firms operating in volatile industries could be expected to build and hold larger portfolios of growth options. Firms operating in less volatile industries could be expected build and hold a smaller optimal option portfolio in relation to the exercise capacity of the firm.

Following a similar line of reasoning, it is possible to distinguish between industries where the research and development times are generally longer and industries where the research and development times are shorter. Longer term research and development work, such as common in the pharmaceutical industry, means a longer time to option exercise. According to the earlier results on option portfolios, longer times to option exercise mean higher optimal N/M. Accordingly, firms operating in industries where it is necessary to develop longer term options into the future could be expected to have a larger option portfolio in relation to the firm s option exercise capacity when compared to firms that develop options with a shorter time frame.

V CONCLUSION

This paper provides a new option analogy for the valuation of a firm s growth opportunities. A firm can be regarded as a portfolio of assets and options that are linked to each other. At each point in time, a firm has the option to choose the M best of all the N growth opportunities available for it. Making N assets available requires investments in option building and development. The exercise capacity constraint M determines at each point in time how many of the N underlying risky assets a firm could choose to activate. Both N and M affect the overall firm value. To advance portfolio level growth option analysis, the present paper provides a new method for deciding how much to invest in option creation (N) and how much to invest in option exercise capability (M) to maximize a firm s option portfolio value.

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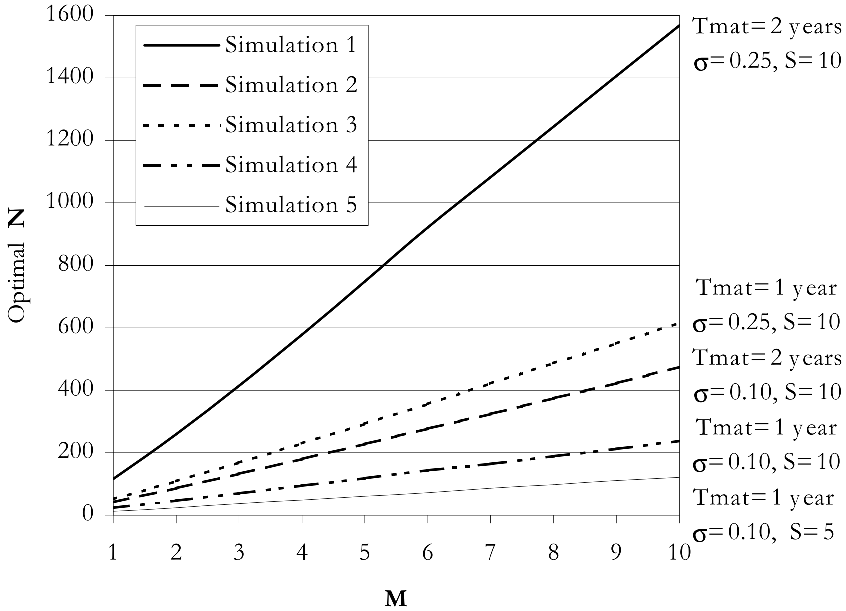

Figure 1. The option value maximizing N and the exercise capacity M. The changing parameter values are shown in the figure.

APPENDIX 1 DERIVATION OF INPUT CORRELATION MATRICES

Johnson (1987) derives the correlation coefficients for multi-normal equations as follows:

$$Cov\left(\log S_i^*, \log\frac{S_i^*}{S_j^*}\right) = Var\left(\log S_i^*\right) - Cov\left(\log S_i^*, \log S_j^*\right) = s_i^2 - r_{ij} s_i s_j = s_i s_{ij} r_{iij}$$

$$=> \qquad r_{iij} = \frac{s_i - r_{ij} s_j}{s_{ij}}$$

$$Cov\left(\log\frac{S_i^*}{S_k^*}, \log\frac{S_i^*}{S_j^*}\right) = Var\left(\log S_i^*\right) - Cov\left(\log S_i^*, \log S_j^*\right) - Cov\left(\log S_k^*, \log S_i^*\right) + Cov\left(\log S_k^*, \log S_j^*\right)$$

$$= s_i^2 - r_{ij} s_i s_j - r_{ik} s_i s_k + r_{jk} s_j s_k = r_{ijk} s_{ij} s_{ik}$$

$$=> \; r_{ijk} = \frac{s_i^2 - r_{ij} s_i s_j - r_{ik} s_i s_k + r_{jk} s_j s_k}{s_{ij} s_{ik}}$$

If the order of the underlying assets S_i is changed, the calculation of correlation coefficients follows a similar line of reasoning, but the sign of the correlation coefficient is reversed.

$$Cov\left(\log S_i^*, \log\frac{S_j^*}{S_i^*}\right) = -Var\left(\log S_i^*\right) + Cov\left(\log S_i^*, \log S_j^*\right) = -s_i^2 + r_{ij} s_i s_j = s_i s_{ij} r_{iji}$$

$$=> \; -r_{iij} = r_{iji}$$

$$Cov\left(\log\frac{S_i^*}{S_k^*}, \log\frac{S_j^*}{S_i^*}\right) = -Var\left(\log S_i^*\right) + Cov\left(\log S_i^*, \log S_j^*\right) + Cov\left(\log S_k^*, \log S_i^*\right) - Cov\left(\log S_k^*, \log S_j^*\right)$$

$$= -s_i^2 + r_{ij} s_i s_j + r_{ik} s_i s_k - r_{jk} s_j s_k = r_{jik} s_{ij} s_{ik}$$

$$=> \; -r_{ijk} = r_{jik}$$

The correlation coefficients that are provided as input for a multi-normal equation can be determined accordingly. For example, when the primary parameters of the multi-normal equation are defined as a vector

$$d_1\left(S_1, X, s_1^2\right), d_1^{'}\left(S_1, S_2, s_{12}^2\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^2\right)$$

then the correlation coefficients **P** of a multi-normal function,

$$N_n\left(d_1\left(S_1, X, s_1^2\right), d_1^{'}\left(S_1, S_2, s_{12}^2\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^2\right), P\right) = \Pr ob\left(S_1 > X \cap S_1 > S_2 \cap \ldots \cap S_1 > S_n\right),$$

are the correlation coefficients between all pairs of the primary components. In this case the correlation coefficients would be

$$\mathrm{P} = \begin{bmatrix} r_{112} & & & \\ r_{113} & r_{123} & & \\ r_{114} & r_{124} & r_{134} & \\ r_{115} & r_{125} & r_{135} & r_{145} \\ \ldots & & & \end{bmatrix}$$

Changing the multi-normal probability into

$$N_n\left(d_1\left(S_1, X, s_1^2\right), -d_1^{'}\left(S_1, S_2, s_{12}^2\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^2\right), P\right) = \Pr ob\left(S_1 > X \cap S_2 > S_1 \cap \ldots \cap S_1 > S_n\right)$$

changes the required correlation coefficient matrix accordingly

$$\mathrm{P}^{'} = \begin{bmatrix} -r_{112} & & & \\ r_{113} & -r_{123} & & \\ r_{114} & -r_{124} & r_{134} & \\ r_{115} & -r_{125} & r_{135} & r_{145} \\ \ldots & & & \end{bmatrix}$$

APPENDIX 2 AN OPTION ON THE M BEST OF N RISKY ASSETS

VALUATION OF AN OPTION ON TWO BEST OF N RISKY ASSETS

Starting from equation (6), it is possible to derive an analytical solution to value an option on the (M=2) two best of N risky assets

$$\begin{aligned} c_{Max,N,M=2}(S) = {} & Prob\left(S_1^* > X \cap S_1^* > S_2^* \cap S_1^* > S_3^* \cap ... \cap S_1^* > S_n^*\right) c_{Max,N-1}\left(S_2,...,S_n \middle| S_1\right) \\ & + Prob\left(S_2^* > X \cap S_2^* > S_1^* \cap S_2^* > S_3^* \cap ... \cap S_2^* > S_n^*\right) c_{Max,N-1}\left(S_1, S_3...,S_n \middle| S_2\right) \\ & + ... \\ & + Prob\left(S_n^* > X \cap S_n^* > S_1^* \cap S_n^* > S_3^* \cap ... \cap S_n^* > S_{n-1}^*\right) c_{Max,N-1}\left(S_1,...,S_{n-1} \middle| S_n\right) \\ & + c_{Max,N}(S) \end{aligned} \qquad (6)$$

Writing out the first probability term according to Johnson (1987) we get

$$N_n\left(d_1\left(S_1, X, s_1^2\right), d_1^{'}\left(S_1, S_2, s_{12}^2\right), ..., d_1^{'}\left(S_1, S_n, s_{1n}^2\right), r_{112}, r_{113}, ...\right)$$

The probability multipliers in the second part of the first term of equation (6) are conditional probabilities. The conditional probabilities are derived according to Johnson (1987) and the Bayesian rules of conditional probability. The probability terms of the second part are shown below

$$\frac{S_2 Prob\left(S_1 > S_2 \cap S_2 > X \cap S_2 > S_3 ... \cap S_2 > S_n\right) +}{Prob\left(S_1 > X \cap S_1 > S_2 ... \cap S_1 > S_n\right)}$$

$$\frac{S_3 Prob\left(S_1 > S_3 \cap S_3 > X \cap S_3 > S_2 ... \cap S_3 > S_n\right) +}{Prob\left(S_1 > X \cap S_1 > S_2 ... \cap S_1 > S_n\right)}$$

$$...$$

$$\frac{S_n Prob\left(S_1 > S_n \cap S_n > X \cap S_n > S_2 ... \cap S_n > S_{n-1}\right)}{Prob\left(S_1 > X \cap S_1 > S_2 ... \cap S_1 > S_n\right)}$$

$$- \quad X e^{-rT}\left(1 - \frac{Prob\left(S_1 > X \cap S_2 < X ... \cap S_n < X\right)}{Prob\left(S_1 > X \cap S_1 > S_2 ... \cap S_1 > S_n\right)}\right)$$

By further deconstructing the second part of the first term and by deriving the appropriate correlation matrices (as shown also in Appendix 1), we get

$$\frac{S_2 N_n\left(-d_1^{'}\left(S_2, S_1, s_{21}^{2}\right), d_1\left(S_2, X, s_2^{2}\right), d_1^{'}\left(S_2, S_3, s_{23}\right), \ldots, d_1^{'}\left(S_2, S_n, s_{2n}\right), -r_{221}, -r_{231}, \ldots, r_{223}, \ldots\right)}{N_n\left(d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^{2}\right), r_{112}, r_{113}, \ldots\right)} +$$

$$\frac{S_3 N_n\left(-d_1^{'}\left(S_3, S_1, s_{31}^{2}\right), d_1\left(S_3, X, s_3^{2}\right), d_1^{'}\left(S_3, S_2, s_{32}\right), \ldots, d_1^{'}\left(S_3, S_n, s_{3n}\right), -r_{331}, -r_{321}, \ldots, r_{332}, \ldots\right)}{N_n\left(d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^{2}\right), r_{112}, r_{113}, \ldots\right)} +$$

$$\ldots$$

$$\frac{S_n N_n\left(-d_1^{'}\left(S_n, S_1, s_{n1}^{2}\right), d_1\left(S_n, X, s_n^{2}\right), d_1^{'}\left(S_n, S_2, s_{n2}\right), \ldots, d_1^{'}\left(S_n, S_{n-1}, s_{nn-1}\right), -r_{nn1}, -r_{n21}, \ldots, r_{nn2}, \ldots\right)}{N_n\left(d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^{2}\right), r_{112}, r_{113}, \ldots\right)}$$

$$- \; Xe^{-rT}\left(1 - \frac{N_n\left(d_2\left(S_1, X, s_1^{2}\right), -d_2\left(S_2, X, s_2^{2}\right) \ldots, -d_2\left(S_n, X, s_n^{2}\right), r_{12}, r_{13}, \ldots\right)}{N_n\left(d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^{2}\right), r_{112}, r_{113}, \ldots\right)}\right)$$

By multiplying the two parts of the first term of (6), the probability terms partially cancel each other out and the first term becomes

$$\begin{aligned}
&S_2 N_n\left(-d_1^{'}\left(S_2, S_1, s_{21}^{2}\right), d_1\left(S_2, X, s_2^{2}\right), d_1^{'}\left(S_2, S_3, s_{23}\right), \ldots, d_1^{'}\left(S_2, S_n, s_{2n}\right), -r_{221}, -r_{231}, \ldots, r_{223}, \ldots\right) + \\
&S_3 N_n\left(-d_1^{'}\left(S_3, S_1, s_{31}^{2}\right), d_1\left(S_3, X, s_3^{2}\right), d_1^{'}\left(S_3, S_2, s_{32}\right), \ldots, d_1^{'}\left(S_3, S_n, s_{3n}\right), -r_{331}, -r_{321}, \ldots, r_{332}, \ldots\right) + \\
&+ \ldots \\
&S_n N_n\left(-d_1^{'}\left(S_n, S_1, s_{n1}^{2}\right), d_1\left(S_n, X, s_n^{2}\right), d_1^{'}\left(S_n, S_2, s_{n2}\right), \ldots, d_1^{'}\left(S_n, S_{n-1}, s_{nn-1}\right), -r_{nn1}, -r_{n21}, \ldots, r_{nn2}, \ldots\right) \\
&- \; Xe^{-rT} \left(N_n\left(d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}^{2}\right), r_{112}, r_{113}, \ldots\right)\right. \\
&\qquad \left. - N_n\left(d_2\left(S_1, X, s_1^{2}\right), -d_2\left(S_2, X, s_2^{2}\right) \ldots, -d_2\left(S_n, X, s_n^{2}\right), r_{12}, r_{13}, \ldots\right)\right)
\end{aligned}$$

In a similar way, the second term becomes

$$\begin{aligned}
&S_1 N_n\left(-d_1^{'}\left(S_1, S_2, s_{12}^{2}\right), d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_3, s_{31}\right), \ldots, d_1^{'}\left(S_1, S_n, s_{1n}\right), -r_{112}, -r_{132}, \ldots, r_{113}, \ldots\right) + \\
&+ \ldots \\
&S_n N_n\left(-d_1^{'}\left(S_n, S_2, s_{n1}^{2}\right), d_1\left(S_n, X, s_n^{2}\right), d_1^{'}\left(S_n, S_1, s_{n2}\right), \ldots, d_1^{'}\left(S_n, S_{n-1}, s_{nn-1}\right), -r_{nn2}, -r_{n12}, \ldots, r_{nn1}, \ldots\right) \\
&- \; Xe^{-rT} \left(N_n\left(d_1\left(S_2, X, s_1^{2}\right), d_1^{'}\left(S_2, S_1, s_{12}^{2}\right), \ldots, d_1^{'}\left(S_2, S_n, s_{1n}^{2}\right), r_{221}, r_{223}, \ldots\right)\right. \\
&\qquad \left. - N_n\left(d_2\left(S_2, X, s_1^{2}\right), -d_2\left(S_1, X, s_2^{2}\right) \ldots, -d_2\left(S_n, X, s_n^{2}\right), r_{21}, r_{31}, \ldots\right)\right)
\end{aligned}$$

and the nth term of equation (6) is

$$\begin{aligned}
&S_1 N_n\left(-d_1^{'}\left(S_1, S_n, s_{12}^{2}\right), d_1\left(S_1, X, s_1^{2}\right), d_1^{'}\left(S_1, S_2, s_{31}\right), \ldots, d_1^{'}\left(S_1, S_{n-1}, s_{1n-1}\right), -r_{11n}, -r_{12n}, \ldots, r_{112}, \ldots\right) + \\
&+ \ldots \\
&S_{n-1} N_n\left(-d_1^{'}\left(S_{n-1}, S_n, s_{n-1n}^{2}\right), d_1\left(S_{n-1}, X, s_{n-1}^{2}\right), d_1^{'}\left(S_{n-1}, S_1, s_{n-11}\right), \ldots, d_1^{'}\left(S_{n-1}, S_{n-2}, s_{-1n-2}\right), -r_{n-1n-1n}, \ldots\right) \\
&- Xe^{-rT} \left(N_n\left(d_1\left(S_n, X, s_n^{2}\right), d_1^{'}\left(S_n, S_1, s_{n1}^{2}\right), \ldots, d_1^{'}\left(S_n, S_{n-1}, s_{nn-1}^{2}\right), r_{nn1}, r_{nn2}, \ldots\right)\right. \\
&\qquad \left. - \left(N_n\left(d_2\left(S_n, X, s_n^{2}\right), -d_2\left(S_1, X, s_2^{2}\right) \ldots, -d_2\left(S_{n-1}, X, s_{n-1}^{2}\right), r_{n1}, r_{n2}, \ldots\right)\right)\right)
\end{aligned}$$

As a consequence, equation (6) as a whole can be expressed as follows

$$S_2 N_n\left(-d_1^{'}\left(S_2,S_1,s_{21}^{2}\right),d_1\left(S_2,X,s_2^{2}\right),d_1^{'}\left(S_2,S_3,s_{23}\right),...,d_1^{'}\left(S_2,S_n,s_{2n}\right),-r_{221},-r_{231},...,r_{223},...\right)+$$
$$S_3 N_n\left(-d_1^{'}\left(S_3,S_1,s_{31}^{2}\right),d_1\left(S_3,X,s_3^{2}\right),d_1^{'}\left(S_3,S_2,s_{32}\right),...,d_1^{'}\left(S_3,S_n,s_{3n}\right),-r_{331},-r_{321},...,r_{332},...\right)+$$
$$+...$$
$$S_n N_n\left(-d_1^{'}\left(S_n,S_1,s_{n1}^{2}\right),d_1\left(S_n,X,s_n^{2}\right),d_1^{'}\left(S_n,S_2,s_{n2}\right),...,d_1^{'}\left(S_n,S_{n-1},s_{nn-1}\right),-r_{nn1},-r_{n21},...,r_{nn2},...\right)$$
$$-\ Xe^{-rT}\left(N_n\left(d_1\left(S_1,X,s_1^{2}\right),d_1^{'}\left(S_1,S_2,s_{12}^{2}\right),...,d_1^{'}\left(S_1,S_n,s_{1n}^{2}\right),r_{112},r_{113},...\right)\right.$$
$$\left.-N_n\left(d_2\left(S_1,X,s_1^{2}\right),-d_2\left(S_2,X,s_2^{2}\right)...,-d_2\left(S_n,X,s_n^{2}\right),r_{12},r_{13},...\right)\right)$$

$$+$$

$$S_1 N_n\left(-d_1^{'}\left(S_1,S_n,s_{12}^{2}\right),d_1\left(S_1,X,s_1^{2}\right),d_1^{'}\left(S_1,S_2,s_{31}\right),...,d_1^{'}\left(S_1,S_{n-1},s_{1n-1}\right),-r_{11n},-r_{12n},...,r_{112},...\right)+$$
$$+...$$
$$S_{n-1}N_n\left(-d_1^{'}\left(S_{n-1},S_n,s_{n-1n}^{2}\right),d_1\left(S_{n-1},X,s_{n-1}^{2}\right),d_1^{'}\left(S_{n-1},S_1,s_{n-11}\right),...,d_1^{'}\left(S_{n-1},S_{n-2},s_{-1n-2}\right),-r_{n-1n-1n},...\right)$$
$$-\ Xe^{-rT}\left(N_n\left(d_1\left(S_n,X,s_n^{2}\right),d_1^{'}\left(S_n,S_1,s_{n1}^{2}\right),...,d_1^{'}\left(S_n,S_{n-1},s_{nn-1}^{2}\right),r_{nn1},r_{nn2},...\right)\right.$$
$$\left.-\left(N_n\left(d_2\left(S_n,X,s_n^{2}\right),-d_2\left(S_1,X,s_2^{2}\right)...,-d_2\left(S_{n-1},X,s_{n-1}^{2}\right),r_{n1},r_{n2},...\right)\right)\right)$$

$$+c_{Max,N}\left(S_1,...,S_n\right)$$

Adopting the following notation

- Probability that S_i is the maximum with N assets $S_1 \quad S_n$

$$Prob\left(S_i^* > X \cap S_i^* > S_1^* \cap S_i^* > S_3^* \cap ... \cap S_i^* > S_n^*\right) \equiv Prob\left(S_i^{Max}(N)\right)$$

- Conditional probability that S_i is the maximum when asset $S_{M(j)}$ has been picked out already

$$Prob\left(S_{M(j)}^* > S_i^* \cap S_i^* > X \cap S_i^* > S_2^* \cap S_i^* > S_3^* \cap ... \cap S_i^* > S_n^*\right) \equiv Prob\left(S_i^{Max}(N-1)\middle|S_{M(j)}\right)$$

- *M(j)* is used to denote the index of the asset that was chosen as the *j*th maximum asset

the above equation can be expressed in a simplified recursive form

$$\sum_{M(1)=1}^{N}\left[\sum_{\substack{i=1\\ i\neq M(1)\\ j\subset[1,N],\,j\neq i\\ k\subset[1,N],\,k\neq M(1)}}^{N}\left[S_i\left[Prob\left(S_{M(1)} > S_i \cap S_i > X \cap S_i > S_j\right)\right]\right]\right.$$

$$\left.-Xe^{-rT}\left[Prob\left(S_{M(1)}\right)-Prob\left(S_{M(1)} > X \cap X > S_k\right)\right]\right]$$

$$+c_{Max,N}\left(S_1,...,S_n\right)$$

VALUATION OF AN OPTION ON THREE AND M BEST OF N RISKY ASSETS

Before generalizing to *M=m (M=m<N)*, the equation for the valuation of an option on two best of N risky assets is expanded to value an option on the three best of N *(N>3)* assets. Expanding from two to three assets can be carried out following a similar line of reasoning than from the one asset case to two assets. When determining the value of the option on 3 or M best of N risky assets, an additional round of summation has to be included each time. Using the adopted probability notation, the value of an option on the three best of N risky assets becomes

$$
\begin{aligned}
c_{Max,N,M=3}\left(S_1,\dots,S_n\right) = {} & Prob\left(S_1^{Max}(N)\right)Prob\left(S_2^{Max}(N-1)\middle|S_1\right)c_{Max,N-2}\left(S_3,\dots,S_n\middle|S_1,S_2\right) \\
& + Prob\left(S_1^{Max}(N)\right)Prob\left(S_3^{Max}(N-1)\middle|S_1\right)c_{Max,N-2}\left(S_2,S_{3,}\dots,S_{n-1}\middle|S_1,S_3\right) \\
& + \dots \\
& + Prob\left(S_1^{Max}(N)\right)Prob\left(S_n^{Max}(N-1)\middle|S_1\right)c_{Max,N-2}\left(S_2,S_{3,}\dots,S_{n-1}\middle|S_1,S_n\right) \\
& + Prob\left(S_2^{Max}(N)\right)Prob\left(S_1^{Max}(N-1)\middle|S_2\right)c_{Max,N-2}\left(S_{3,}\dots,S_n\middle|S_2,S_1\right) \\
& + Prob\left(S_2^{Max}(N)\right)Prob\left(S_3^{Max}(N-1)\middle|S_2\right)c_{Max,N-2}\left(S_1,S_4,\dots,S_n\middle|S_2,S_3\right) \\
& + \\
& + Prob\left(S_2^{Max}(N)\right)Prob\left(S_n^{Max}(N-1)\middle|S_2\right)c_{Max,N-2}\left(S_1,S_3,\dots,S_{n-1}\middle|S_2,S_n\right) \\
& + \\
& + \\
& + Prob\left(S_n^{Max}(N)\right)Prob\left(S_1^{Max}(N-1)\middle|S_n\right)c_{Max,N-2}\left(S_2,\dots,S_{n-1}\middle|S_n,S_1\right) \\
& + Prob\left(S_n^{Max}(N)\right)Prob\left(S_2^{Max}(N-1)\middle|S_n\right)c_{Max,N-2}\left(S_1,S_3\dots,S_{n-1}\middle|S_n,S_2\right) \\
& + \\
& + Prob\left(S_n^{Max}(N)\right)Prob\left(S_{n-1}^{Max}(N-1)\middle|S_n\right)c_{Max,N-2}\left(S_1,\dots,S_{n-2}\middle|S_n,S_{n-1}\right) \\
& + c_{Max,N,M=2}\left(S_1,\dots,S_n\right)
\end{aligned}
$$

The probability terms can be simplified by noting that the first term of (9a) can be written out as follows

$$
\sum_{i=3}^{n} Prob\left(S_1^{Max}(N)\right)\frac{Prob\left(S_2^{Max}(N-1)\cap S_1^{Max}(N)\right)}{Prob\left(S_1^{Max}(N)\right)}\frac{Prob\left(S_i^{Max}(N-2)\cap S_2^{Max}(N-1)\cap S_1^{Max}(N)\right)}{Prob\left(S_2^{Max}(N-1)\cap S_1^{Max}(N)\right)}S_i
$$

$$
-Xe^{-rT}\left(Prob\left(S_1^{Max}(N)\right)Prob\left(S_2^{Max}(N-1)\right)-Prob\left(S_1>S_2\cap S_2>X\cap X>S_3,\dots S_n\right)\right)
$$

$$
=\sum_{i=3}^{n} Prob\left(S_i^{Max}(N-2)\cap S_2^{Max}(N-1)\cap S_1^{Max}(N)\right)S_i
$$

$$-Xe^{-rT}\left(Prob\left(S_1^{Max}(N)\right)Prob\left(S_2^{Max}(N-1)\right)-Prob\left(S_1>S_2\cap S_2>X\cap X>S_3,...,S_n\right)\right)$$

Generalizing the simplification to the whole equation, the equation to determine the value of an option on the three best of N risky assets can be presented as follows

$$c_{Max,N,M=3}(S_1,...,S_n)=\sum_{M(1)=1}^{n}\left[\sum_{\substack{M(2)=1\\M(1)\neq M(2)}}^{n}\left[\sum_{\substack{M(3)=1\\M(i)\neq M(j)}}^{n}Prob\left(S_{M(3)}^{Max}(N-2)\cap S_{M(2)}^{Max}(N-1)\cap S_{M(1)}^{Max}(N)\right)S_{M(3)}\right.\right.$$

$$\left.\left.-Xe^{-rT}\left(Prob\left(S_{M(1)}^{Max}(N)\right)Prob\left(S_{M(2)}^{Max}(N-1)\right)-Prob\left(S_{M(1)}>S_{M(2)}\cap S_{M(2)}>X\cap X>S_{M(3)},...,S_n\right)\right)\right]\right]$$

$$+c_{Max,N,M=2}(S_1,...,S_n)$$

It is easy to see from the equation above how the generalization of the recursion to the general case of M best of N risky assets should be made. The general recursive equation to value M best of N risky assets becomes

$$c_{Max,N,M=m}(S_1,...,S_n)=$$

$$\sum_{M(1)=1}^{n}\left[\sum_{\substack{M(2)=1\\M(1)\neq M(2)}}^{n}\left[\sum_{\substack{M(m)=1\\M(i)\neq M(j)}}^{n}Prob\left(S_{M(m)}^{Max}(N-(m-1))\cap...\cap S_{M(2)}^{Max}(N-1)\cap S_{M(1)}^{Max}(N)\right)S_{M(m)}\right.\right.$$

$$\left.\left.-Xe^{-rT}\left(Prob\left(S_{M(1)}^{Max}(N)\right)...Prob\left(S_{M(m)}^{Max}(N-(m-1))\right)-Prob\left(S_{M(1)}>S_{M(2)}\cap...\cap S_{M(m-1)}>X\cap X>S_{M(m)},...,S_n\right)\right)\right]\right]$$

$$+c_{Max,N,M=m-1}(S_1,...,S_n)$$

APPENDIX 3 ASSUMPTIONS MADE IN THE NUMERICAL ANALYSIS

The assumptions made in the numerical analysis are stated more formally in this appendix to provide an understanding of the underlying assumptions and their extent. Assumptions one, two, four, and six are not required when using the general equation (11) for determining the value of an option on M best of N assets.

Assumption 1
There are N different underlying assets in the asset portfolio. The assets in the asset portfolio are similar in terms of their initial values, variances, time-to-maturity, and exercise price

The first assumption is a precondition for using equation (11) in the numerical analysis. The assumption would not be necessary when using the more general valuation equation (10). In practice, when the correlation matrix and the variances of individual assets in the portfolio can be estimated, it is always possible to use the more general equation. Unfortunately, in the more general case the efficient Boyle and Tse (1990) approximation algorithm is not directly applicable. Instead, general multi-normal function integration algorithms have to be used (see e.g. Genz, 1992; Drezner, 1992). Solving the multi-normal functions numerically in the general case is often relatively time consuming when N is large.

Assumption 2
The assets in the portfolio are independent of each other.

All the N assets in the asset portfolio are assumed to be independent of each other. The underlying drift processes of the assets are assumed non-correlated. The assumption of independence does not remove the fact that when option exercise is considered the assets compete for the overall exercise capacity of the firm. Similarly to Assumption 1, the assumption of independence is not necessary when using equation (10), but the independence of assets is required when using equation (11) and the Boyle and Tse (1990) algorithm.

Assumption 3
The exercise capacity is M. Each unit of exercise capacity provides an identical option to exercise any one of the N assets in the asset portfolio.

The third assumption is necessary for defining the relationship between N and M. Another possibility would be that exercising different projects requires different amounts of exercise capacity. For example, one asset N_1 could require four units of M while another asset N_2 would require only one unit of M. Incorporating this extension to the analysis would require an additional variable to define the number of units of exercise capacity that is required by each asset if exercised. The extension is deferred at this point.

Assumption 4
The exercise price when exercising the option on the M best of N assets is zero.

The fourth assumption of zero exercise price is a necessary requirement when using the Boyle and Tse algorithm in determining the numerical value of an option on the maximum or minimum of a number of underlying assets. Similarly to Assumption 1 and Assumption 2, the assumption of zero exercise price is not needed when using the option valuation equation (10), but the assumption is required when using equation (11) and the Boyle and Tse (1990) algorithm.

Assumption 5
Increasing N incurs a constant unit cost. Increasing M incurs a constant unit cost.

Assumption five introduces the costs of expanding M and N to the analysis. Without costs for expanding M and N, the value of the option on the M best of N assets would increase to infinity as M and N increase. There would be an incentive to maximize both. Having costs for increasing M and N is realistic. Even if in practice the costs for increasing M and N would not be constant, it would be relatively easy to replace the constants by appropriate cost functions. This expansion is not done, however, as the shapes of the cost functions would bring additional complexity that would reduce the clarity of the presentation.

Assumption 6
At least M-1 assets are in the money at the time of exercise.

The sixth assumption is a necessary requirement when using equation (7) in a two-asset case or equation (11) in the multi-asset case. The assumption ensures that the probability weights of each recursive step converge to 1 and, as a consequence, equations (6) and (10) converge to equations (7) and (11). Assumption six is not necessary in the more general option portfolio valuation of equations (6) and (10).

APPENDIX 4 LOGARITHMIC APPROXIMATION

Solving the multi-normal integral analytically is not possible even in the simplified case of similar, independent assets. Fortunately, the difficulty of integration can be overcome by approximating the multi-normal integral with a logarithmic function of N in the manner shown in equation

$$NS_i \int_{-\infty}^{\infty, s_i\sqrt{\frac{T}{2}}, \ldots, s_i\sqrt{\frac{T}{2}}} \det(2p\,\Sigma)^{-\frac{1}{2}} e^{-\frac{1}{2}\left(x\Sigma^{-1}x\right)} dx = NS_i\left(\frac{a(s,T)}{N} + \frac{b(s,T)}{N}\log N\right) = S_i a(s,T) + S_i b(s,T)\log N$$

where *a* and *b* are functions of s and *T*, but constants with respect to N and the other parameters are as defined in the paper. Figure 2 provides five sets of Johnson s (1987) equations solved numerically with the Boyle and Tse (1990) algorithm. For each curve, the algorithm was executed 999 times as the portfolio sizes ranged from 2 to 1000. A logarithmic curve of the form *Sa + Sb log N* was fitted for each curve. The high goodness of fit statistics provide numerical support for the logarithmic approximation.

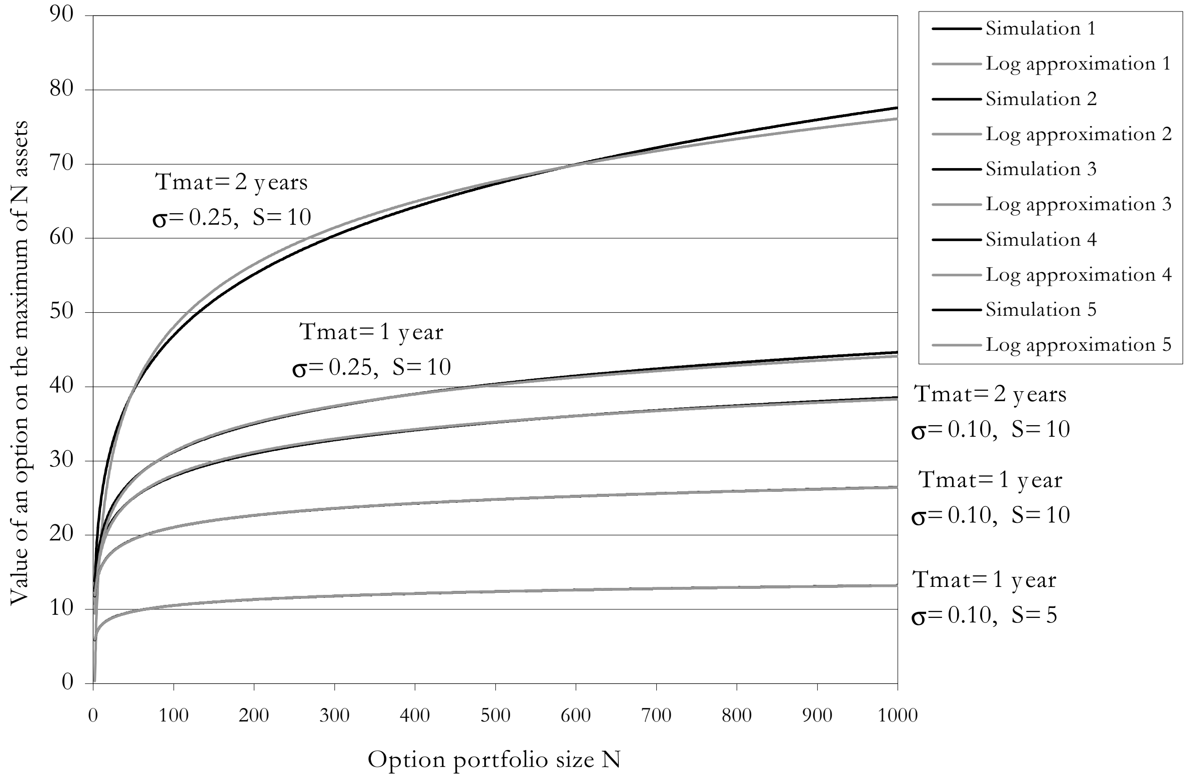

Figure 2. The value of an option on the maximum of N assets as a function of N. The input parameters of the valuation equation are: risk-free rate R_f=0.05, initial asset value S ranging from 5 to 10, volatility of each underlying asset σ ranging from 0.10 to 0.25, time to maturity, Tmat, ranging from 1 year to 2 years, and strike price X=0. The curve marked with Simulation is the value curve. The curve marked with Logarithmic is a curve fitted to the numerical data. The goodness of fit statistics of curve fitting imply 98.88 % fit for Simulation 1, 99.84 % fit for Simulation 2, 99.99 % fit for Simulation 3, 99.99 % fit for Simulation 4 and 99.99 % fit for Simulation 5.

APPENDIX 5 THE LOGARITHMIC APPROXIMATION PARAMETER *b*

The logarithmic approximation multiplier *b* is of special interest due to its relation to the optimal N with respect to M as shown in Proposition 2. We know that a multi-normal integral function is a positive and increasing function of N, s, and *T*. As a consequence, also the parameter *b* of an approximating logarithmic function has to be a positive and increasing function of s and *T*. This can be relatively easily verified numerically. The effects of changing portfolio parameters s and the time to maturity are studied in relation to the parameter *b*. For numerical analysis, the Boyle and Tse (1990) algorithm was executed 49.900 times and logarithmic curve fitting was carried out 100 times. The resulting effects of variance and the time to maturity are shown in Figure 3. The figure provides the results of curve fitting analysis with 10 x 10 different variance - time to maturity combinations. As shown in Figure 3, increasing variance increases the multiplier parameter *b*. The effect of the time to maturity is similar to the effect of variance. Increasing the time to maturity from 1 year to 10 years increases the parameter *b* significantly.

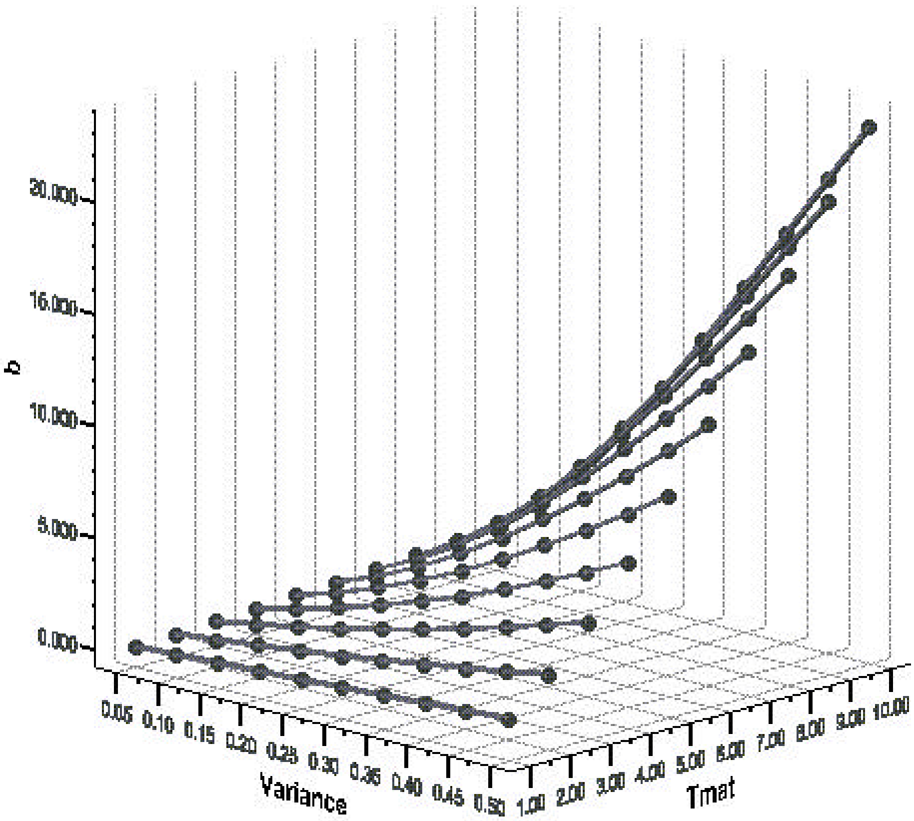

Figure 3. The values of the multiplier parameter b of logarithmic curves, *Sa+SblogN*, that were fitted to the maximum of N risky assets value curve. The option on the maximum was determined for N values ranging from 2 to 500. The logarithmic curve fitting was carried out 100 times, once for each variance. Increasing variance increases the multiplier and decreases the constant part that is independent of N. The input parameters are: risk-free rate R_f=*0.05*, initial asset value *S =10*, volatility and time to maturity of the underlying assets is used as a variable, and strike price *X=0*.