Equilibrium Investment Strategies and Output Price Behavior:
A Real-Options Approach

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Abstract

This paper studies the effect of competitive interactions on investment decisions and on the dynamics of the price of a non-storable commodity. We develop a model of incremental investment with time to build and operating flexibility. We examine the implications of a construction lag and the option to vary output for capacity choice and for the dynamics of capacity expansion and capacity utilization. We find that an increase in uncertainty may encourage firms to hold more capacity. Furthermore, we show that it may be optimal to invest in additional capacity even during periods in which part of the completed capacity is not being utilized. These results contrast with previous studies on irreversible investment and capacity choice that assume that investment is instantaneous and/or that installed capacity is always fully utilized. Our results on capacity choice and capacity utilization are qualitatively the same under oligopoly and perfect competition. However, the industry effect on the properties of the endogenous output price process can be very different. In particular, during the periods in which the completed capacity is fully utilized the price can increase to a very high level relative to its average during periods of excess capacity. The relative size of these spikes in the price series is increasing in the number of firms in the market. In addition, the dynamics of capacity utilization induces heteroskedasticity in the output price process, and the price volatility may be increasing in the number of competitors in the industry. Our results on competition and output price behavior appear to conform to actual behavior in wholesale electricity markets.
1. Introduction

The purpose of this paper is to study the effect of competitive interactions among firms in a given industry on investment decisions and on the price of a non-storable commodity. We analyze strategic behavior in a real-options framework that includes realistic features of investment projects, such as time to build, operating flexibility and capacity choice.

Many irreversible investment decisions have the following characteristics: (i) investment takes time to build, (ii) firms can choose how much to invest and can add more capacity later as demand increases, (iii) once the investment is completed, firms have the flexibility to vary output in response to changes in demand, costs, and other variables, and (iv) investment strategies take place in a industry context and, thus, firms must consider the strategies of their competitors. For example, consider investment in electricity generation. The construction of new power plants can take 6 to 10 years to complete. The capacity of a power generating facility is determined by, among other considerations, future demand. An electric utility can expand its generating capacity in response to an increase in expected future demand. Producers can vary the amount of energy generated in response to changes in demand or costs. Other operating options that are common in the electricity supply industry include the choice of the fuel to burn. Finally, an electricity producer has to formulate its investment strategies considering the strategies of other electric utilities operating in its area of service. In this paper we develop a model of investment under uncertainty that includes time to build, capacity choice, flexibility to vary output and considers the effect of competition.

Time to build is an important characteristic of many investment projects. Besides electricity generation, there are many examples of investment with significant construction lags. Grenadier (1995a) reports lags of about two to three years for office building construction. Majd and Pindyck (1987) note that the construction of an underground mine, or the development of a large petrochemical plant can take at least five or six years to complete. Pindyck (1991) observes investment lags of five to ten years in pharmaceuticals and aerospace. Real option models with time to build include Majd and Pindyck (1987), Grenadier (1995a), and Bar-Ilan and Strange (1996). These
models study the investment decisions of an individual firm and their focus is on a single project of a given fixed size, like the decision to build a new factory or a new office building.

Capacity choice and operational flexibility are important characteristics of many real investments. For example, the owner of real estate can select the scale or density at which to develop his property. This choice is often subject to legal limitations, such as zoning regulations. Williams (1991) features a model of real estate development as an option that analyses both the optimal timing and the scale of the development. Many land development projects are examples of “lumpy” investments, where the size of the project is chosen when the investment occurs. Investment can also proceed incrementally, where the initial size of the project can be expanded later. For example, a firm must decide how large a factory to build and when to expand it. Pindyck (1988) develops a model of irreversible and incremental investment in which the firm has the option to not utilize the incremental unit when demand is low. He and Pindyck (1992) extend Pindyck’s model to allow for technology choice. An example of technology choice arises when a firm can use one or two variable inputs in production. An electric utility, for instance, can choose between a coal-fired plant, an oil-fired plant, or a plant that can burn either fuel. The problem is to decide which type and how much capital to install. These models of capacity choice assume that investment is instantaneous and the firm is a monopoly.

In many real investment contexts the payoffs from a firm’s investment are affected by the investment strategies of its competitors. Models of irreversible investment under uncertainty and competitive equilibrium include Dixit (1991, 1993), Leahy (1993), and Grenadier (1995b). All these papers assume that the investment is instantaneous. Grenadier (1999a) derives a competitive equilibrium with time to build, and Grenadier (1996) studies an oligopolistic industry equilibrium with construction lags. These last two papers by Grenadier assume that the firms’ capacity is always fully used. The closest work on equilibrium investment strategies using option pricing methods to this paper are in Baldursson (1998) and Grenadier (1999b). Baldursson studies an oligopoly were firms facing a stochastic linear demand curve use capacity as strategic variable. The equilibrium framework of Baldursson is very similar to our model.

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1 See also Chapter 8 in Dixit and Pindyck (1994).
However, Baldursson assumes that investment is instantaneous and that installed capacity is always fully utilized. The examples analyzed in Baldursson (1998) indicate that the qualitatively the price process will be the same in oligopoly and perfect competition. Grenadier (1999b) develops an approach to solving for the investment equilibrium that is applicable to more general specification of demand and considers time to build. However, like in Baldursson (1998), Grenadier (1999b) does not assume flexibility in the use of the installed capacity. Furthermore, with respect to the models of competitive equilibrium cited above, the model of Grenadier (1999b) does not provide new economic insights on output price behavior.

Our model extends the classic real-options model of irreversible investment and capacity choice by introducing time to build and competition. A firm must decide how much capacity to build initially and when to expand it later. The firm has the option to not use any incremental unit of capacity if demand falls. Our analysis makes several interesting contributions. First, we show that with time to build more uncertainty may encourage the firm to hold more capacity. Furthermore, the firms’ optimal capacity choice may be higher under uncertainty than under certainty. These results contrast with models of incremental investment that assume no construction lags and find a negative effect of uncertainty on capacity choice.

The optimal capacity choice under uncertainty involves a trade-off. On the one hand, the irreversibility of investment induces firms to hold less capacity as a buffer against unanticipated drops in demand. On the other hand, the potential profits lost by not having enough capacity to satisfy an unanticipated increase in demand creates an incentive to hold more capacity. However, when new capacity can be used immediately, the income from any additional capacity depends on current demand, which is independent of uncertainty. Therefore, without construction lags, more uncertainty reduces the level of capacity that firms are willing to hold.

With time to build there is a lag between the time at which new capacity is installed and the time at which it can be used in production. More uncertainty increases the likelihood of extreme future demand states. Since firms have the option to not utilize

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2 See, for example, Pindyck (1988) or He and Pindyck (1992).

3 See for example Pindyck (1988), or Chapter 11 in Dixit and Pindyck (1994).
any additional capacity when demand falls, the profit from incremental capacity is always positive. This means that the opportunity cost of insufficient capacity rises with uncertainty. Thus, the net effect of uncertainty on the level of optimal capacity is ambiguous. An increase in uncertainty may encourage firms to hold more capacity.

We also find that the level of optimal capacity may increase with a longer investment lag. This result has implications for the dynamics of capacity utilization. Specifically, the frequency of times in which all the completed capacity is used may be decreasing in the length of the investment lag. This result is consistent with the chronic excess capacity that has been observed in commercial real estate and electricity generation, where construction lags are long. We discuss the dynamics of capacity in Section 5.

Our second main result refers to the dynamics of capacity choice and capacity utilization. We show that firms may find it optimal to invest in additional capacity even during periods in which part of their current operational capacity is not being used. Thus, we provide a new insight for an observed phenomenon in electricity generation, in which the decision to start the construction of a new power plant is often made even in the presence of excess capacity. The intuition is that firms expand their capacity in anticipation of a stronger future demand even if current demand is not enough to operate at full capacity. We elaborate further on this result in Section 5.

The third important finding in this paper is that more competition does not change the above results on capacity choice and capacity expansion. This latter result contrasts with the analysis in Pindyck (1993) that shows that competition has a depressing effect on irreversible investment even when the opportunity cost of insufficient capacity is increasing in uncertainty. We elaborate further on this point in Section 7.

Finally, although the dynamics of capacity are qualitatively the same under oligopoly and perfect competition, the properties of the equilibrium output price can be remarkably different. Specifically, we show that during the periods in which the industry’s operational capacity is fully used the output price can increase to a very high level relative to the price when completed capacity is not fully used. The relative size of these spikes in the price series is increasing in the number of firms in the market. Furthermore, more competition can reduce the percent variation in prices when the
industry capacity is not fully used, but when the industry is operating at full capacity the variation in relative prices increases with the number of competitors. The reason is that, when capacity is enough to satisfy current demand, the output price approaches the marginal production cost (which is the price under perfect competition) as the number of firms in the industry increases. Thus, more competition reduces the effect of changes in demand on prices when the industry capacity is not fully used. However, when demand is high and capacity is insufficient, competition can not buffer the price changes.

The presence of spikes and heteroskedasticity are empirically stylized facts about commodity price series (See Deaton and Laroque (1992)). A well-established line of research explains these properties through the dynamics of storage. (For a comprehensive treatment see Williams and Wright (1991)). In our model these properties follow from the interaction of time to build, capacity choice, flexibility and competition. Thus, our analysis complements the storage-based models. Furthermore, our model provides a useful framework for studying the prices of commodities for which storage is relatively expensive, such as electricity for example. We discuss the output price dynamics in Section 6.

This paper is organized as follows. Section 2 analyzes the problem of incremental investment and capacity choice with time to build for a single firm. Section 3 presents our model of capacity choice with time to build. The properties of the optimal capacity choice are examined in Section 4, and the dynamics of capacity are analyzed in Section 5. Section 6 examines the effect of competition on investment and output prices, and Section 7 compares our findings to recent research on competitive investment under uncertainty. Section 8 concludes.

2. Incremental Investment and Capacity Choice with Time to Build

In this section we study the capacity choice problem of a monopolist. We initially abstract from competition to focus on how time to build affects the firm’s investment problem. The extension to oligopoly and perfect competition is presented in Section 6.
Consider a firm facing an inverse demand curve $P = D(q, Y)$, with $D_q(q, Y) < 0$ and $D_Y(q, Y) > 0$, where $q$ is the level of output and $Y$ is a demand shock. This demand shock evolves as geometric Brownian motion

$$dY(t) = \mu Y(t)dt + \sigma Y(t)dZ(t) \tag{1}$$

where $\mu$ is the instantaneous conditional expected proportional change in $Y$ per unit time, $\sigma$ is the instantaneous conditional standard deviation per unit time, and $Z$ is a standard Wiener process. Both $\mu$ and $\sigma$ are constant.

The firm operates a simple production technology. Each unit of operational capacity can produce one unit of output per unit time. Let $C(q)$ be the cost function, with $C(0) = 0$. At any time $t$ the firm chooses output to maximize its current profit. It follows from the assumed production technology that the output chosen can not exceed the current level of operational capacity, which is denoted by $O(t)$. Thus,

$$\pi(O(t), Y(t)) = \max_{0 \leq q \leq O(t)} D(q, Y(t))q - C(q)$$

is the instantaneous profit that the firm earns when the operating capacity level is $O(t)$ and the demand parameter is $Y(t)$.

The firm can add units of capacity one at a time at a sunk cost of $k$ per unit. However, there is a lag between the time at which a new unit is purchased and the time at which this unit can be used in production. Let us denote this investment lag or time to build by $h$, where $h \geq 0$. That is a unit of capacity purchased at time $t$ will be ready to use in production at time $t + h$.

The introduction of investment lags in a model of incremental investment can substantially complicate the investment decision. The reason is that the information needed to solve the investment problem includes the initiation times of all the units that are currently under construction. Thus, the dimension of the information set can potentially be infinity. To handle this apparent complexity we follow the procedure outlined in Grenadier (1999a). With time to build, at any point in time $t$ there may be both operational units, whose total number is $O(t)$, and units that are currently under construction. Let $N(t)$ denote the number of units that are under construction. These are the units that began construction during the interval $(t - h, t]$ and, consequently, will be
completed during the interval \((t, t+h]\). Let \(K(t)\) denote “committed capacity” where \(K(t) = O(t) + N(t)\). Investment is irreversible, technically this means that the process for the committed capacity, \(\{K(t)\}\), is non-decreasing. Any unit that is completed today must have been part of committed capacity \(h\) years ago. Therefore, for any \(t\), we have \(O(t) = K(t-h)\).

The firm’s problem is to choose the path of capacity expansion that maximizes the present value of its future cash flows. We assume that the firm is risk neutral. This means that future cash flows are discounted at the risk-free rate of interest, \(r\), which is assumed constant. Alternatively we can assume that there exits an asset whose return is perfectly correlated with \(Y\). Thus markets are dynamically complete in the sense that contingent claims written on \(Y\) can be priced by taking the expectation of the discounted cash flows under the risk neutral probability measure.

The cash flow at time \(t\) is

\[
\pi(O(t), Y(t)) - kdK(t) = \pi(K(t-h), Y(t)) - kdK(t)
\]

Thus, the firm solves the following optimal control problem

\[
W(\Omega_0) = \max_{\{K(t); t \geq 0\}} \mathbb{E} \left\{ \int_0^\infty e^{-rt} [\pi(K(t-h), Y(t))\pi - kdK(t)] \mid \Omega(0) = \Omega_0 \right\}
\]

(2)

where \(\Omega(t)\) is the information set of the firm at time \(t\). This set includes the current state of demand, \(Y(t)\), and the current operational capacity, \(O(t)\), because these two variables determine the current level of profit. The firm must also include in its information set the amount of capacity that is currently under construction, \(N(t)\), and the times at which these units will be completed and become operational. These four state variables define the current information set, \(\Omega(t)\).

The potential difficulty in solving the problem of capacity choice with lags arises from the inclusion of the set of entry times of the units currently under construction in \(\Omega(t)\). If there is no time to build then the new capacity can be used immediately and, thus, the current information set can be summarized by the current value of the demand

\[4\] At any time \(t\) the set \(E(t) = \{ \tau \in [t-h, t) : dK(\tau) > 0 \}\) specifies all the times during the previous \(h\) years at which new construction was initiated.
parameter, $Y(t)$, and the current operational capacity, $O(t)$. By allowing for construction lags the state may become one of infinite dimensionality. However, as Grenadier (1999a) points out, this apparent complexity can be greatly simplified if we note that, because of the construction lag, the choice of the optimal path of $K(t)$ does not influence the profit flow over the interval $(0,h)$. Furthermore, since all the units currently under construction will be completed by time $h$, the profit flow during the period $[0, h)$ depends only on the entry times over the period $[-h,0)$ while the profit flow after time $h$ depends only on the current value of committed capacity, $K_0$. Therefore, the firm’s problem can be rewritten as

$$W(\Omega_0) = E\left\{ \int_0^h e^{-rt} \pi(K(t-h), Y(t)) dt \mid \Omega(0) = \Omega_0 \right\} +$$

$$\max_{\{K(s) > 0\}} E\left\{ \int_h^\infty e^{-rt} \pi(K(t-h), Y(t)) dt - \int_0^\infty e^{-rt} K(t) \mid K(0) = K_0, Y(0) = Y_0 \right\}$$

Equation (3) demonstrates that the solution to the original optimization problem $W(\Omega_0)$ in (2) is equivalent to the solution of a simpler optimization problem, in which the only state variables are the current level of committed capacity and the value of the demand parameter, plus a term over which no optimization is required.

To solve this new optimization problem define

$$V(K, Y, h) = \max_{\{K(s) > 0\}} E\left\{ \int_h^\infty e^{-rt} \pi(K(t-h), Y(t)) dt - \int_0^\infty e^{-rt} K(t) \mid K(0) = K, Y(0) = Y \right\}$$

$V(K,Y,h)$ can be though as the value of a firm that has no operational capacity and no capacity under construction before $t = 0$ and starts the construction of $K$ units at $t = 0$. The net value of this firm would be $V(K,Y,h) - kK$. Since the construction costs are assumed proportional, $V(K,Y,h)$ can be derived as the solution to an optimal instantaneous control problem.

Instead of solving this problem directly, we extend the alternative way that Pindyck (1988), and He and Pindyck (1992) propose for solving this type of capacity choice problem by including an investment lag. As mentioned above $V(K,Y,h)$ is
equivalent to the value of a firm that begins with no capacity and decides to build a starting capacity $K$. We can write this value as the sum of two parts

$$V(K,Y,h) = G(K,Y,h) + F(K,Y)$$

$G(K,Y,h)$ is the value of the firm’s initial capacity, that is the present value of the expected flow of profits that $K$ units will generate, given that the current value of the demand parameter is $Y$. Thus,

$$G(K,Y,h) = E\left\{ \int_{h}^{\infty} e^{-\pi(K,Y(t))} dt \mid K(0) = K, Y(0) = Y \right\}$$  \hspace{1cm} (5)$$

$F(K,Y)$ is the value of the firm’s options to expand its capacity in the future. Thus, $F(K,Y)$ is the present value of any additional profits that might result if the firm adds more capacity in the future less the present value of the cost of that capacity. Notice that $F(K,Y)$ is positive because the firm is not committed to any investment plan.

The optimal capacity choice can now be interpreted as the first-order condition for a value-maximizing firm. That is, if the firm’s initial capacity is zero, then the firm’s optimal capacity level, $K^*$, maximizes the firm’s net value, $G(K,Y,h) + F(K,Y) - kK$. This maximization gives the following optimality condition

$$\frac{\partial G(K^*,Y,h)}{\partial K} = k - \frac{\partial F(K^*,Y)}{\partial K}$$  \hspace{1cm} (6)$$

Thus, the firm should invest until the value of a marginal unit of capacity, $\partial G(K,Y,h)/\partial K$, is equal to its total cost, the purchase cost $k$, plus the opportunity cost $\partial F(K,Y)/\partial K$. This latter cost arises because investment is irreversible. This means that the firm can not reduce its capacity if demand changes adversely. Hence, by purchasing an additional unit of capacity the firm extinguishes the option to wait for new demand information that might affect the desirability of the expenditure.

Therefore, to solve the firm’s capacity problem we need to find the value of a marginal unit of capacity and the value of the option to invest in this unit. Define $\Delta G(K,Y,h) = \partial G(K,Y,h)/\partial K$ and $\Delta F(K,Y) = - \partial F(K,Y)/\partial K$. Differentiating (5) with respect to $K$,
\[ \Delta G(K, Y, h) = \mathbb{E} \left[ \int_0^\infty e^{-rt} \Delta \pi(K, Y(t)) dt \mid K(0) = K, Y(0) = Y \right] \]  

(7)

where \( \Delta \pi(K, Y) = \partial \pi(K, Y)/\partial K \). The value of the marginal unit of capacity can be evaluated by direct integration in (7).

The marginal value of the growth options, \( \Delta F(K, Y) \), is the value of an option to purchase an additional unit of capacity when the current committed capacity level is \( K \). The exercise price of this option is equal to the cost of construction. Therefore, for any level of committed capacity \( K \), \( \Delta F(K, Y) \) is a perpetual American call option whose value depends on \( Y \). Hence, there will be a threshold value at which it will be optimal to exercise this option. Specifically, for any \( K \) there will exists a threshold, \( Y(K) \), such that the option to build an additional unit of capacity will be exercised the first time that \( Y \) equals or exceeds \( Y(K) \).

The value of the option to invest in an incremental unit of capacity can be derived using the methods of contingent claim pricing. Consider the instantaneous return on \( \Delta F(K, Y) \) over the region \( Y < Y(K) \). By Ito’s formula, the instantaneous change in \( \Delta F \) is

\[
d\Delta F = \left( \frac{\sigma^2}{2} Y^2 \Delta F_{yy} + \mu Y \Delta F_y \right) dt + \sigma \Delta F_y dZ(t)
\]

Since the economy is risk neutral, the equilibrium expected instantaneous return on \( \Delta F \) is equal to \( r \Delta F dt \). Therefore the value of the option to invest satisfies the following differential equation

\[
\frac{\sigma^2}{2} Y^2 \Delta F_{yy} + \mu Y \Delta F_y - r \Delta F = 0
\]

(8)

The optimal exercise policy, \( Y(K) \), is determined by the following boundary conditions.

\[
\Delta F(K, Y(K)) = \Delta G(K, Y(K), h) - k
\]

\[
\Delta F_y (K, Y(K)) = \Delta G_y (K, Y(K), h)
\]

(9)

These conditions are commonly termed as “value-matching” and “smooth pasting” respectively. The first condition states that upon exercise the payoff of the option to invest in a marginal unit of capacity is the value of this unit minus the cost of
construction. The second condition ensures that \( Y(K) \) is the threshold that maximizes the value of the option to invest. As the level of capacity \( K \) varies we obtain \( Y(K) \) for each \( K \).

Once we find the marginal values of a unit of committed capacity and the growth options, the value \( V(K,Y,h) \) is obtained by

\[
V(K,Y,h) = \int_0^K \Delta G(x,Y,h)dx + \int_K^\infty \Delta F(x,Y)dx
\]

where we use the fact that \( \lim_{x \to 0} G(x,y,h) = \lim_{x \to \infty} F(x,y) = 0 \) for any \( y \neq 0 \).

3. A Model of Capacity Choice with Time-to-Build

In this section we apply the methodology developed in the previous section to the case of a firm that faces a linear inverse demand function and has a constant marginal production cost. Specifically

\[
D(q,Y) = Y - \gamma q, \quad \text{and} \quad C(q) = c_1q + 0.5c_2q^2 \tag{10}
\]

Our model extends the classic model of irreversible investment and capacity choice of Pindyck (1988) by introducing time to build. We assume that the firm starts with no operational capacity and no capacity under construction, so at \( t = 0 \) it must decide how much capacity to build. In the previous section we concluded that the solution to the capacity choice problem involves two steps. First, the value of an extra unit of capacity must be determined. Second, the value of the option to invest in this unit must be determined together with the decision rule for exercising this option. This decision rule is the solution to the optimal capacity problem. It maximizes the net value of the firm.

3.1 The Value of a Marginal Unit of Capacity

The value of a marginal unit of committed capacity is the present value of the expected flow of profits from this unit. For given a current committed capacity \( K \), and demand \( Y, \Delta G(K,Y,h) \) denotes the value of a marginal unit of capacity.
It follows from the demand and cost functions of equation (10) and the assumed production technology that at any future time \( t \mu h \) the profit of a firm with current committed capacity \( K \) is

\[
\pi(K, Y(t)) = \max_0 \; q \; K \left[ (Y(t) - \gamma q)q - c_1q + 0.5c_2q^2 \right] \tag{11}
\]

This implies that the profit from an additional unit of committed capacity at any \( t \mu h \) is

\[
\Delta \pi(K, Y(t)) = \max \left[ Y(t) - (2\gamma + c_2)K - c_1, 0 \right] \tag{12}
\]

Thus, after it is completed, each incremental unit of capacity will be used only when the additional profit it generates is positive.

To find \( \Delta G(K, Y, h) \) we first determine the value of the marginal unit after it is completed and can be used in production. The value of an incremental unit of operational capacity when the current level of completed capacity is \( O \) is denoted by \( \Delta H(O, Y) \), and it can be written as

\[
\Delta H(O, Y) = E \left[ \int_0^\infty e^{-rt} \Delta \pi(O, Y(t)) dt \mid Y(0) = Y \right] \tag{13}
\]

Instead of evaluating this double integral directly, it is easier to value an incremental unit of operational capacity as a simple contingent claim that depends on \( Y \). The procedure for obtaining \( \Delta H(O, Y) \) is standard. The details can be found in chapter 6 of Dixit and Pindyck (1994). The solution gives the following expression for the value of an additional unit of operational capacity

\[
\Delta H(O, Y) = \begin{cases} A(Y)^\alpha + \frac{Y}{r - \mu} \frac{(2\gamma + c_2)O + c_1}{r} & \text{for } Y \geq (2\gamma + c_2)O + c_1 \\
B(Y)^\beta & \text{for } Y \leq (2\gamma + c_2)O + c_1 \end{cases} \tag{14}
\]

where \( \alpha \) and \( \beta \) are respectively the negative and positive roots of the characteristic equation

\[
\frac{\sigma^2}{2} + \xi(\xi - 1) + \mu \xi - r = 0 \tag{15}
\]

and the constants \( A \) and \( B \) are given by
The expression for $\Delta H(O,Y)$ is interpreted as follows. If the marginal unit is always used its value is $Y/(r - \mu) - [(2\gamma + c_2)K - c_1]/r$. Thus, when $Y > (2\gamma + c_2)K + c_1$, $A(O)Y^\alpha$ is the value of the option to not use the unit should $Y$ fall. Similarly, when $Y < (2\gamma + c_2)K + c_1$, the unit is not used and $B(O)Y^\beta$ is the value of the option to use it should $Y$ increase.

Once we have the value of an incremental unit of operational capacity, the value of an unit of capacity under construction $\Delta G(K,Y,\theta)$, where $0 < \theta$ [ $h$ is the remaining time until the unit becomes operational, is obtained as follows

$$\Delta G(K,Y,\theta) = E\left[\int_0^\infty e^{-r\tau}\Delta \pi(K,Y(t))dt \mid Y(0) = Y\right]$$

$$= e^{-\theta Y} E\left[\int_0^\infty e^{-r\tau}\Delta \pi(K,Y(t+\theta))dt \mid Y(0) = Y\right]$$

$$= e^{-\theta Y} E[\Delta H(K,Y(\theta))] \mid Y(0) = Y$$

In general at any time $t$

$$\Delta G(K(t),Y(t),\theta) = e^{-\theta Y} E_t[\Delta H(K(t),Y(t+\theta))]$$

where $E_t[.] = E[.\mid Y(t)]$.

Appendix A derives the solution to this conditional expectation

$$\Delta G(K,Y,\theta) = (1 - \Phi(v - \alpha \sigma \sqrt{\theta}))A(K)Y^\alpha + (1 - \Phi(v - \sigma \sqrt{\theta}) Ge^{-\tau}) \frac{Y^e(r - \mu)h}{r - \mu}$$

$$- (1 - \Phi(v)) \frac{(2\gamma + c_2)K + c_1}{r} e^{-\theta Y} + \Phi(v - \beta \sigma \sqrt{\theta})B(K)Y^\beta$$

where $\Phi(.)$ is the standard normal cumulative distribution function and $v$ is defined as
\[ v = v(K, Y, \theta) = \frac{\log[(2\gamma + c_2)K + c_1] - \log Y - (\mu - \sigma^2/2)\theta}{\sigma \sqrt{\theta}} \]  

(20)

3.2 The Decision to Invest in the Marginal Unit

Having valued the marginal unit of committed capacity, we can now value the option to invest in this unit \( \Delta F(K, Y) \), and determine the optimal investment threshold \( Y(K) \). In the previous section we demonstrated that \( \Delta F(K, Y) \) satisfies the following partial differential equation

\[ \frac{\sigma^2}{2} Y^2 \Delta F_{yy} + \mu Y \Delta F_y - r \Delta F = 0 \]  

(21)

which is subject to the following boundary conditions:

\[ \Delta F(K,0) = 0 \]

\[ \Delta F(K,Y(K)) = \Delta G(K,Y(K),h) - k \]  

(22)

\[ \Delta F_y(K,Y(K)) = \Delta G_y(K,Y(K),h) \]

The first boundary condition arises because \( Y = 0 \) is an absorbing barrier for the process described in (1), and therefore the option to invest has no value at that point. This implies the following functional form for the option to invest in a marginal unit of capacity

\[ \Delta F(K,Y) = D(K)Y^\beta \quad \text{for} \quad Y < Y(K). \]  

(23)

where \( \beta \) is the positive root of (15) and \( D(K) \) is a “constant” to be determined. The last two boundary conditions form the system of equations that must be solved to get the values of \( Y(K) \) and \( D(K) \). They are the value-matching and the smooth-pasting condition respectively, and they imply that \( Y(K) \) is the value of \( Y \) that maximizes the value of the option to invest.

Using the functional forms for \( \Delta F(K,Y(K)) \) and \( \Delta G(K,Y(K),h) \) this system becomes
\[ D(K)Y(K)^\beta = (1 - \Phi(v_K - \alpha \sigma \sqrt{h}))A(K)Y(K)^\alpha + (1 - \Phi(v_K - \sigma \sqrt{h})) \frac{Y(K)e^{-(r-\mu)h}}{r-\mu} \]

\[ - (1 - \Phi(v_K)) \left[ \frac{(2\gamma + c_2)K + c_1}{r} \right]e^{-rth} + \Phi(v_K - \beta \sigma \sqrt{h})B(K)Y(K)^\beta - k \]

(24)

\[ \beta D(K)Y(K)^{\beta-1} = (1 - \Phi(v_K - \alpha \sigma \sqrt{h}))\alpha A(K)Y(K)^{\alpha-1} + A(K)Y(K)^\alpha \frac{\phi(v_K - \alpha \sigma \sqrt{h})}{Y(K)\sigma \sqrt{h}} \]

\[ + (1 - \Phi(v_K - \sigma \sqrt{h})) \frac{e^{-(r-\mu)h}}{r-\mu} \phi(v_K - \sigma \sqrt{h}) e^{-(r-\mu)h} - \phi(v_K)\left[ (2\gamma + c_2)K + c_1 \right]e^{-rth} \]

\[ + \Phi(v_K - \beta \sigma \sqrt{h})\beta B(K)Y(K)^{\beta-1} - \frac{\phi(v_K - \beta \sigma \sqrt{h})}{Y(K)\sigma \sqrt{h}} B(K)Y(K)^\beta \]

(25)

where \( v_K = v(K,Y(K),\theta) \), and \( \phi(\cdot) \) is the standard normal density function. Eliminating \( D(K) \) we are left with an equation for the investment threshold. A reduced expression for this equation is given by

\[ \beta \left[ \Delta G(K, Y(K), h) - k \right] - Y(K)\Delta G_Y(K, Y(K), h) = 0 \]

(26)

This last equation cannot be solved analytically to get an expression for \( Y(K) \), however it is easily solved numerically. In the next section we illustrate the solution to the investment problem.

### 3.3 The Firm’s Optimal Capacity

The function \( Y(K) \) is the firm’s optimal investment rule, if \( Y \) and \( K \) are such that \( Y > Y(K) \), the firm should add capacity, increasing \( K \) until \( Y = Y(K) \). Equivalently, given the current level of the demand shock \( Y \), we can determine the firm’s optimal capacity by rewriting equation (26) in terms of \( K(Y) \)

\[ \beta \left[ \Delta G(K(Y), Y, h) - k \right] - Y\Delta G_Y(K(Y), Y, h) = 0 \]

(27)

In the next section we analyze the properties of the solution to (27).
4. Characteristics of the Optimal Capacity Choice

In the previous Section we derived the solution to the firm’s optimal capacity choice. Assuming that the firm has no initial capacity, the solution to equation (27) gives the optimal level of capacity to build for a current level of demand $Y$. Because there is no closed form expression for $K(Y)$, it is necessary to solve equation (27) numerically in order to understand the properties of the firm’s capacity choice. We examine the effects of demand volatility, $\sigma$, and of the length of the construction lag, $h$, on $K(Y)$. The numerical examples presented below use the following parameters $r = 0.05$, $\mu = 0$, $\gamma = 0.5$, $c = 1$, $k = 5$.

To analyze the effect of uncertainty on capacity choice when it takes time to build, we first examine the solution to the benchmark case in which there is no investment lag. Figure 1 shows $K(Y)$ when investment is instantaneous ($h = 0$) for different levels of demand volatility. Since the output price is strictly increasing in $Y$, so is $K(Y)$. For any level of demand, optimal capacity is smaller when future demand is more uncertain. As shown in Figure 1 the curve $K(Y)$ for $\sigma = 0.4$ is below that for $\sigma = 0.2$, which is below $K(Y)$ for $\sigma = 0$. The firm’s capacity choice is decreasing in demand volatility when new capacity can be used immediately. We explain the intuition behind this result below.

The effect of uncertainty on the level of optimal capacity is remarkably different when it takes time to build. Figure 2 shows $K(Y)$ for $\sigma = 0$, 0.2 and 0.4. Uncertainty has a negative effect on the level of optimal capacity for low demand values. However, as the level of demand increases, more uncertainty may encourage the firm to build more capacity. As shown in Figure 2 as $Y$ increases the curve $K(Y)$ for $\sigma = 0.4$ will be eventually above $K(Y)$ for $\sigma = 0.2$. Furthermore the capacity curves for $\sigma = 0.2$ and 0.4 will eventually be above the curve for $\sigma = 0$. This implies, for example, that for $Y$ large enough a firm facing an annual level of demand volatility of 0.4 will invest more than an otherwise identical firm facing no demand uncertainty. Thus, the firm may invest more under uncertainty than under certainty.
Figure 3 shows the optimal capacity for different levels of the demand parameter. For $Y = 5$, $K(Y)$ is decreasing in $\sigma$. For $Y = 7$ and $Y = 9$, $K(Y)$ initially decreases to a local minimum and then increases to a local maximum as $\sigma$ increases. As the level of the demand parameter increases, the local minimum occurs at a lower $\sigma$, while the local maximum occurs at a higher $\sigma$. Thus, the set of volatility values for which capacity is increasing in demand uncertainty is increasing in the value of the demand parameter. It is important to note that these results do not rely on the assumption of a trendless demand process ($\mu = 0$); the same results obtain for a positive trend ($\mu > 0$).

Uncertainty affects the firm’s optimal capacity choice in two opposing ways. First, the irreversibility of investment creates a cost of excess capacity. Specifically, if future demand falls the firm will find itself holding more capacity than it would have chosen had the fall in demand been anticipated. More uncertainty increases the likelihood of lower future demand and, thus, the cost of excess capacity. The value of an additional unit of capacity includes the option to not utilize the unit when demand falls. This option offsets the effect of uncertainty on the cost of excess capacity as follows. For any given level of demand volatility a firm that can shut down part of its operational capacity when demand is low will hold more capacity than an otherwise identical firm that has to use all its completed capacity. However, to determine the net effect of an increase in demand volatility on the firm’s capacity choice, we have to compare the effects of uncertainty on both the cost of excess capacity and the opportunity cost of insufficient capacity.

The firm’s output in every period is constrained by the number of operational units. The potential profits lost from not having enough capacity create an opportunity cost that is decreasing in the level of capacity. When new capacity can be used immediately the opportunity cost of insufficient capacity depends on current demand, which is independent of uncertainty. Therefore, with no investment lags, more uncertainty reduces the level of optimal capacity.

With time to build there is a lag between the time at which new unit of capacity is purchased and the time at which it can be used in production. More uncertainty increases the likelihood of extreme levels of future demand. But the firm need not use the additional unit of capacity when demand falls. Therefore, the opportunity cost of

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5 Examples available from the author.
insufficient capacity raises with uncertainty. Thus, the net effect of uncertainty on the level of optimal capacity is ambiguous. More uncertainty raises the level of optimal capacity if the increased opportunity cost of insufficient capacity outweighs the increased cost of excess capacity.

The ability to vary output by not using the marginal unit of capacity at times when demand is low is essential to generating the above result. Without this option the firm has to operate the unit at a loss during periods of extremely low demand that may result from a high level of uncertainty. In Appendix B we analyze the capacity choice of an otherwise identical firm that does not have this operating option. This implies that completed capacity is always fully used. For this case we derive the firm’s optimal capacity choice in closed form and show that the level of committed capacity is decreasing in demand volatility.

Figure 4 shows the effects of changes in the length of the investment lag on the optimal level of capacity, \( K(Y) \), at different levels of current demand for the base case parameters, with \( \sigma = 0.1 \). For the values of the demand parameter shown, optimal capacity is initially increasing in the length of the investment lag, and then it decreases, as time to build is longer. As Figure 4 shows, the value of the investment lag for which capacity reaches the maximum is increasing in the demand parameter.

The intuition for the comparative static result of time to build on capacity choice is as follows. As the construction time increases the present value of the flow of profits from an additional unit of capacity decreases. In addition the likelihood of extreme levels of demand at the time of completion is increasing in the construction lag. This implies that both the cost of excess capacity and the opportunity cost of insufficient capacity increase with the length of the investment lag. Thus, the net effect of time to build on capacity is ambiguous.

5. The Dynamics of Capacity Expansion and Capacity Utilization

In this section we characterize the dynamics of committed capacity, operational capacity and output.
When the current level of demand is \( Y \), the solution to equation (27) gives the optimal level of committed capacity, \( K(Y) \), for a firm that has no capacity. Alternatively, since investment is irreversible, at any time \( t \), \( K(Y(t)) \) is the level of “desired capacity”. Thus, the irreversibility constraint implies that \( K(t) \geq K(Y(t)) \) for all \( t \), and \( K(t) = K(Y(t)) \) if \( dK(t) > 0 \). Therefore, for any time \( t \) the level of committed capacity \( K(t) \) may depend on the whole past history of the demand process \{\( Y(s), 0 \leq s \leq t \)\}. Additionally, since output price is strictly increasing in \( Y \), so is the desired committed capacity \( K(Y) \). Finally, the process for the committed capacity \{\( K(t) \)\} is non-decreasing because we assume that the installed capacity does not depreciate.

With these properties we can characterize the dynamics of committed capacity. The firm begins with no capacity at \( t = 0 \), it observes \( Y(0) \), and begins the construction of \( K(Y(0)) \) units of capacity. Then, if \( Y \) increases from its initial value the firm will expand its initial level of committed capacity. After demand has reached a temporary maximum, the level of committed capacity remains fixed until demand reaches its historic high again. Thus, capacity expansion occurs when demand is rising and only when it is rising above previous highs. Figure 5 shows a sample path for the demand level, \( Y(t) \), and the corresponding behavior of the committed capacity, \( K(t) \). The following variables and parameters values are used, \( Y(0) = 5, r = 0.05, \mu = 0.03, \gamma = 0.5, c_1 = 1, c_2 = 0.5, k = 5, h = 3 \).

Once we have obtained the path of committed capacity, the path of operational capacity follows from the fact that \( O(t) = K(t - h) \). Given \( O(t) \) the output choice is

\[
Q(t) = \min \left[ \frac{1}{2\gamma + c_2} (Y(t) - c_1), O(t) \right]
\]

This follows from (11) with \( O(t) \) replacing \( K \).

The lower graph in Figure 5 shows the paths of committed capacity, \( K(t) \), completed capacity, \( O(t) \), and output, \( Q(t) \), that result from the demand path shown in the upper graph. The times at which the paths \( O(t) \) and \( Q(t) \) coincide are the times when operational capacity is fully utilized.

Figure 6 shows the paths of capacity expansion, \( \Delta K(t) \), and the fraction of completed capacity that is used in production, \( Q(t)/O(t) \), that result from the demand path
realization of Figure 5. The completed capacity is fully used when \( Q(t)/O(t) = 1 \). The flat parts on the path \( Q(t)/O(t) \) correspond to the periods of time in which the operational capacity is fully used. Notice that there are periods of capacity expansion in which part of the operational capacity is not used in production. This behavior occurs whenever \( Y(K(t)) > Y(t) \), and \( Y(t) < (2\gamma + c_2)O(t) + c_1 \). In Figure 6 this occurs during the intervals \( T_1 \), \( T_2 \), and \( T_3 \). During \( T_1 \) the firm begins the construction of new capacity when 8% of its current operational capacity is not used. At the beginning of the interval \( T_2 \) the firm expands its capacity when 10% of its completed capacity is idle. During \( T_3 \) the firm adds more capacity when 16% of its operational capacity is not used. It is worth noting that this result does not rely on the assumed positive trend in the demand process. A similar investment behavior can be obtained for a trendless demand process \((\mu = 0)\). \( ^6 \)

The result that the firm may find it optimal to invest in additional capacity when its completed capacity is not fully used contrasts with the dynamics of capacity utilization found in Pindyck (1988). The reason is that Pindyck assumes that new capacity can be used immediately. Hence, it is not optimal to add more capacity only to keep it idle for some time. Thus, all capacity will be utilized during expansions.

With time to build the decision to invest in additional capacity depends on the expected demand after the construction is completed. Hence, even if current demand is not enough to operate at full capacity, a rise above historic levels induces the firm to add more capacity in anticipation of a stronger future demand.

Therefore, by allowing for flexibility in the use of the operational capacity and a construction lag, the simple model of capacity choice studied in this paper offers a rational explanation for the observed behavior of several markets in which new units are built when there is current excess capacity. The electricity supply industry is one example of such markets. Bar-Ilan and Strange (1996) notice that new power plants continue being built even in the presence of current excess capacity. However, the model studied in Bar-Ilan and Strange (1996) can suggest only a part of the explanation for this phenomenon because their analysis focuses on the effect of construction lags on the timing of a investment in a single project of a given fixed size.

\( ^6 \) Details available from the author.
Grenadier (1995a) presents a model that attempts to explain the recurrence of cycles of overbuilding in real estate markets, in which large quantities of newly completed buildings enter the market during periods of already high vacancy rates. This overbuilding phenomenon is different from the behavior discussed in the last paragraph. More precisely, in the context of our model the overbuilding phenomenon would correspond to new capacity being completed at a time when demand is insufficient to use it. The analysis in Grenadier (1995a) focuses on the probability of overbuilding, which he defines as the probability (as of the beginning of the construction) that a new building will be completed at a time in which demand is insufficient to rent a specified fraction of its units. Furthermore, Grenadier (1995a) studies the decision to invest in a single building of fixed capacity and does not address the dynamics of capacity utilization.

The length of the construction lag plays a crucial role in the dynamics of capacity utilization. The top graph in Figure 7 compares the path capacity when investment is instantaneous with the path of capacity for a 3-year construction lag that result from the realization of demand of Figure 5. Under the assumed parameters the firm holds less capacity when new units can be used immediately. As a result there will be periods of low demand when capacity is fully utilized. A large drop in demand is required for capacity utilization to fall below 100 percent. Thus, as can be seen in the middle graph in Figure 7, when investment is instantaneous capacity will be fully used most of the time. In contrast, for a lag of 3 years there are fewer periods in which the completed capacity is fully used, as can be seen in the bottom graph of Figure 7. This behavior is consistent with the chronic excess capacity that has been observed in commercial real estate and electricity generation where construction lags are long.

6. Capacity Choice with Time to Build in Oligopoly

In this section we extend the model of Section 3 to the case of an oligopolistic industry. To get the solution to each firm’s capacity choice problem in oligopoly we follow the procedure presented in Section 2. The relevant state variables are the current demand and committed capacity level. The main difference from the monopoly case is
that, in an oligopoly, the state space for each firm includes not only its capacity level but also the capacity levels of the other firms in the market. Committed capacity is the strategic variable and each firm must condition its capacity choice on the strategies of its competitors.

For each firm $i$, the value of an incremental unit of committed capacity is denoted by $\Delta G_i(K_i, K_{-i}, Y, h)$, where $K_i$ is the committed capacity of firm $i$ and $K_{-i} = (K_1, \ldots, K_{i-1}, K_{i+1}, \ldots, K_n)$ denotes the strategies of firm $i$’s competitors. The option to invest in an incremental unit of committed capacity is denoted by $\Delta F_i(K_i, K_{-i}, Y)$. Given the current demand, $Y$, and level of committed capacity of its competitors, $K_{-i}$, firm $i$’s reaction function, $K_i(Y, K_{-i})$ is its optimal choice of committed capacity. This function is the solution to the system of equations

\[
\frac{\partial \Delta F_i(K_i(Y, K_{-i}), K_{-i}, Y)}{\partial Y} = \frac{\partial \Delta G_i(K_i(Y, K_{-i}), K_{-i}, Y, h)}{\partial Y} - k
\]

An $n$-tuple of strategies $(K_1^*, \ldots, K_n^*)$ is a Nash industry equilibrium if

\[
K_i^* = K_i(Y, K_{-i}^*) , \ i = 1, \ldots, n
\]

To determine the instantaneous profit flow of each firm we note that at any time $t$ the firms play a static Cournot game. Each firm chooses its output to maximize its current profit. This choice depends on current demand and it is constrained by the firm’s current operational capacity. In addition, each firm must condition its output choice on the output choices of the other firms, which are also constrained by their operational capacity levels. The operational capacity of firm $i$ is denoted by $O_i$, and $O_{-i} = (O_1, \ldots, O_{i-1}, O_{i+1}, \ldots, O_n)$ summarizes the operational capacity of firm $i$’s competitors. To simplify the analysis we assume that the industry is composed of $n$ identical firms. That is, all firms have the same marginal cost function and investment cost. We also assume that all the firms start with the same initial capacity and, thus, they all have the same size at any time. Thus, our analysis focuses on a symmetric Nash equilibrium.
Let $K(t)$ and $O(t)$ denote the total industry committed and operational capacity respectively. Since all firms are identical it follows that $K_i(t) = K(t)/n$, and $O_i(t) = O(t)/n$. Hence, $\Delta G_i(K_i, K_{-i}, Y) = \Delta G_i(K, Y)$ and $\Delta F_i(K_i, K_{-i}, Y) = \Delta F_i(K, Y)$.

To obtain specific results about the effect of strategic behavior on investment and output prices we assume the demand and costs function given in (10). When the industry committed capacity is $K$ the additional profit from an incremental unit of capacity at any future time $t \leq h$ is

$$\Delta \pi_i(K, Y(t)) = \max \left[ Y(t) - \frac{(n+1)\gamma + c_2}{n} K - c_1, 0 \right], \quad i = 1, \ldots, n$$

(29)

This is similar to the marginal profit of equation (12), the only difference is that the coefficient multiplying the capacity is $[(n + 1)\gamma + c_2]/n$ instead of $2\gamma + c_2$. Thus, to get the solution to the capacity choice problem in oligopoly we make this substitution in equation (27). The capacity choice problem under perfect competition is obtained by taking the limit as $n$ increases to infinity. Therefore, the properties of the optimal capacity choice and the timing of the investment in an incremental unit of capacity that we have previously derived in a monopoly context are the same under oligopoly and perfect competition. However the properties of the price process can be very different. We illustrate this below.

Figure 8 shows the dynamics of capacity expansion for different numbers of firms in the industry that result from the realization of demand of Figure 5. The total capacity of the industry increases as the number of firm increases. However, the times at which new capacity is added are the same regardless of the number of firms in the market. Furthermore, the length of periods at which capacity is fully utilized, and the percentage of capacity utilization, do not vary with the number of firms in the industry. To prove this result it suffices to show that the ratio of total output to total operational capacity is the same for any number of firms in the industry. Let $K^n$, $O^n$ and $Q^n$ be the committed

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7 At any time $t$ the firm $i$’s profit given the industry operating capacity $O(t)$ is

$$\Delta \pi_i(O(t), Y(t)) = \max_{0 \leq q_i \leq O(t)/n} \left[ Y(t) - \sum_{j=1}^{n} q_j q_i - c_1 q_i - 0.5 c_2 q_i^2 \right]$$

Then it follows from the assumption of symmetric equilibrium that the profit from an additional unit of operational capacity is
capacity, operational capacity, and output of an \( n \)-firm industry respectively. Comparing equations (12) and (29) it is easily verified by substitution that for any number of firms \( n \),

\[
K^n = \frac{n(2\gamma + c_2)}{(n+1)\gamma + c_2} K^1
\]  

(30)

and, consequently,

\[
O^n = \frac{n(2\gamma + c_2)}{(n+1)\gamma + c_2} O^1
\]  

(31)

In addition the output for an \( n \)-firm market is

\[
Q^n = \min \left[ \frac{n}{(n+1)\gamma + c_2} (Y - c), O^n \right]
\]  

(32)

Equations (31) and (32) imply that \( Q^n/O^n = Q^1/O^1 \) for any \( n \).

In contrast, the properties of the endogenous price process change significantly as the number of firms in the industry increases. Figure 9 shows the price paths for a monopoly, a 3-firm oligopoly and a 10-firm oligopoly that result from the realization of the demand process of Figure 5. One standard consequence of a larger number of firms in the market is the reduction of the output price. As can be seen from Figure 9 the price path for 1 firm is completely above the path for 3 firms, which is above the path for 10 firms.

The most remarkable property of the resulting price paths is that during the periods in which the industry’s operational capacity is fully utilized the output price can increase to a very high level relative the price during periods when capacity is not completely used. The relative size of these spikes in the price series gets larger with more firms in the industry. Furthermore, the resulting price series is heteroskedastic with higher (lower) volatility when prices are high (low). This heteroskedasticity is a direct consequence of the dynamics of capacity utilization. More precisely, there are two possibilities or regimes of price volatility depending on whether or not the current operational capacity is fully used. The difference between the price volatility in these two regimes is increasing in the number of firms in the industry.

\[
\Delta \pi_t(O(t), Y(t)) = \max[(Y(t) - [(n + 1) \gamma + c_2]/n)O(t) - c_1, 0]
\]
To prove this last statement let $P_n$ be the output price for an $n$-firm industry. It follows from (32) and (10) that when the operational capacity is enough to satisfy the current demand, the output price is

$$P_n = \frac{(\gamma + c_2)Y + n\gamma c_1}{(n+1)\gamma + c_2}$$

By applying Ito’s formula, the instantaneous conditional standard deviation of the percent change in $P_n$ is $\sigma(\gamma + c_2)Y/[(\gamma + c_2)Y + n\gamma c_1]$, which is decreasing in $n$ for $c_1 > 0$, and constant for $c_1 = 0$. Therefore, during the periods in which part of the operational capacity is not used, the variance of the output price is non-increasing in the number of firms in the industry. When production equals completed capacity the output price is

$$P_n = Y - \gamma O^n$$

Where $O^n$ is the operational capacity of an $n$-firm industry. Thus, the effect of demand changes on the output price can only be reduced when newly completed capacity arrives, $(dO^n > 0)$, but during the periods in which completed capacity remains constant $(dO^n = 0)$ the instantaneous standard deviation of $P_n$ is $\sigma$ which is independent of $n$. Therefore, since $P_n$ is decreasing in $n$, the relative variation in the output price is increasing in the number of firms in the industry.

In sum, more competition reduces the level of output prices and can buffer the effect of demand uncertainty in the percent variation of prices. However, competition is not as effective in reducing the price volatility when demand is very high and capacity is insufficient.

Figure 10 displays the paths of percent change in price, $dP(t)/P(t)$, corresponding to the price paths of Figure 9. Each graph in Figure 10 also includes the path of capacity utilization, $Q(t)/O(t)$. The flat parts in the paths of capacity utilization correspond to the periods in which the industry’s capacity is fully used. When capacity is not fully used the relative variations in price decrease as the number of firms increases. In contrast, when the industry is operating at full capacity the relative variations in price are larger as the number of firms increases.
The existence of rare but violent explosions in prices is a stylized fact about commodity prices. Commodity price series are characterized by the presence of occasional spikes. In our model these spikes occur when the completed capacity is fully used. Thus, the frequency of these spikes depends on the dynamics of capacity utilization, which are affected by the length of the investment lag. As was discussed in Section 5, the frequency of the periods in which capacity is fully used may be decreasing in the length of the construction lag. Hence, the sudden increases in output price are rare for longer lags.

The properties of the commodity price series have been extensively analyzed within the context of models of competitive storage (e.g. Deaton and Laroque (1992, 1996), and Williams and Wright (1991)). In essence these models attribute the observed asymmetry in price changes - increases in price, while rarer, are larger than the typical price decreases – to the fact that storage itself is asymmetric. Low prices are associated with a large stockpile able to buffer price changes, while very high prices are associated with the stockpile empty and unable to buffer price changes.

Our model does not consider storage because we assume that consumption equals industry output in every period. Therefore, our analysis would apply to commodities for which storage is relatively expensive. One of such commodities is electricity. Thus, our model complements the models of competitive storage cited above.

It is worth noting that sudden price increases occur in competitive electricity markets. In particular, during the week of June 22-26, 1998 the wholesale electricity markets in the Midwestern United States experienced a dramatic price increase. A study by the Federal Energy Regulatory Commission (FERC) highlights insufficient generating and transmission capacity among the main causes of this price spike. In addition, the same study concludes that although a number of companies have announced plans for some new generation plants in the Midwest, the operational conditions that led to the June 1998 price spike are likely to recur. The reason cited is that the minimum lead time for installing new capacity is at least one year (for natural gas-fired combustion...
turbines). For many larger plants, the lead time will be much longer. Thus, our model implications appear to conform to actual behavior in wholesale electricity markets.

Our result that output price volatility is increasing in the number of firms in the industry is consistent with the cross-country evidence on electricity prices of Wolak (1997). He studies the behavior of spot electricity prices in England and Wales, Norway, the state of Victoria in Australia and New Zealand. Wolak finds that electricity supply industries with a larger component of private participation in the generation market tend to have more volatile prices. It is particularly interesting how the recent behavior of the Victoria power market conforms to our prediction on the relation between output price behavior and the number of firms in the industry. Restructuring and privatization of the electricity supply industry in the state of Victoria in 1994 took place at the power station level. Each power station was formed into a separate entity to be sold. Wolak reports that entry by new generators and changes in firm ownership as more new generating companies have been privatized has led to an increase in volatility of the Victoria Power Exchange prices. Wolak suggests that part of this price volatility may be explained by attempts to exercise market power. Our model shows that the observed behavior of electricity prices in competitive markets can be explained by the interaction of time to build, operating flexibility and capacity choice.

7. Relation to Previous Research on Competitive Investment Decisions

In this section we compare our findings with other recent papers that study irreversible investment and industry equilibrium. Baldursson (1998) studies an oligopoly where firms face a linear demand curve similar to the one in our model. Firms use capacity as their strategic variable. Baldursson assumes that investment is instantaneous and that the installed capacity is always fully utilized so production equals capacity at each instant. It follows from these assumptions that more uncertainty induces the firms to hold less capacity and delays the decision to expand their installed capacity. Furthermore,
the examples presented in Balduisson (1998) indicate that the qualitative nature of the price process will be the same in oligopoly and in competitive equilibrium. In contrast, by allowing for a construction lag and flexibility in the use of the installed capacity, our model of industry equilibrium generates more interesting implications about the industry effect on investment and output price.

The option that allows the firms to let part of their completed capacity go unused when demand is low is crucial in obtaining our industry equilibrium results. Without any operational flexibility the net effect of uncertainty on industry investment is always negative, even with construction lags. The model studied in Grenadier (1999a) illustrates this point. Grenadier derives a dynamic competitive equilibrium for a market subject to time to build. Grenadier assumes that all the industry capacity is always fully utilized and derives the firms’ investment rule in closed-form. The resulting investment threshold is increasing in demand uncertainty. Furthermore the price properties derived in Grenadier (1999a) for competitive equilibrium with time to build are identical for oligopoly (see Grenadier (1999b)).

Finally, we mention above that increased competition does not affect the comparative statics of the optimal capacity choice that were presented in Section 4 in a monopoly context. In particular, we find that more uncertainty may encourage firms to hold more capacity. We explained that the ability to vary output by not using the marginal unit of capacity at times when demand is low is essential to generating this result. Our findings, that the encouraging effects of uncertainty on investment are robust to increased competition, contrast to those of Pindyck (1993) on competitive investment under uncertainty. Pindyck shows that competition has a depressing effect on investment, even when the marginal cost of capital is increasing in uncertainty. In the example presented by Pindyck the marginal product of capital is convex in price which is affected by a stochastic demand shift factor. By Jensen’s inequality, increased uncertainty about future demand increases the expected marginal product of capital and hence increases the incentive to invest. However, when demand increases existing firms will expand or new firms will enter until the market clears. The increased supply limits any price increase under good demand outcomes. Furthermore, since investment is irreversible, it is impossible to prevent the price from falling under bad demand outcomes by reducing
capital. Therefore, the equilibrium response of firms offsets the increases in the value of a unit of capital that results from the convexity of the marginal profit function. Pindyck presents a two-period example in which investment is left unchanged by a mean-preserving spread in the demand shift variable. When the number of periods is more than two, the initial investment is lower when future demand is more uncertain.

Contrary to the findings of Pindyck (1993), competition does not affect the net effect of uncertainty on irreversible investment in our model. The marginal product of capital is convex in both models, the difference in results arises because the production technologies are different. Pindyck assumes a Cobb-Douglas production function\(^8\). This technology allows the substitutability of capital with other factors or inputs, such as labor for example. Thus, at any time the installed capacity is always fully used, and the firm changes its output level by varying other production factors to maximize its instantaneous profit. This ensures that the marginal product of capital is convex in the stochastic demand factor. In contrast capital is the only factor in production in our model, and the marginal product is convex because the firm can vary output by not using the marginal unit of capital when demand falls. More uncertainty increases the likelihood of extreme demand outcomes at the time of completion. New entry offsets any increase in the marginal profits under good demand outcomes, but the option to not utilize the marginal unit when completed prevents the marginal profits from falling below zero under bad demand outcomes. Thus, it follows from the interaction of irreversibility and flexibility in capacity use that increased competition does not change our results on the effect of uncertainty on investment.

8. Conclusions

This paper presents a model of investment under uncertainty that includes time to build, capacity choice, flexibility in the use of the installed capacity and considers the

\(^{10}\) Pindyck’s purpose is to show that the encouraging effect of uncertainty on investment found by Abel (1983) and Caballero (1991) is not robust to increased competition. These two papers assume a Cobb-Douglas production function.
effect of competition on the investment decision. Our analysis extends existing real options models by incorporating all these features in a simple model of investment under uncertainty. Our model provides new insights into the effects of uncertainty on capacity choice and on the dynamics of capacity expansion and capacity utilization that are not available in previous models that do not include one or more of the four features above.

Our main results are the following. First, more uncertainty may encourage firms to hold more capacity. Similarly the optimal capacity choice may be increasing in the length of the construction lag. This implies that a longer lag may reduce the frequency of times in which the operational capacity is fully used. Second, firms may find it optimal to invest in additional capacity even during times in which their current capacity is not fully used. These optimal responses by firms are consistent with the actual behavior of capacity utilization and capacity expansion in real estate and electricity generation. Finally, there are two regimes for the changes in the output price depending on whether or not the operational capacity is fully used and more competition increases the differences between these regimes.
Appendix A: The Value of an Incremental Unit under Construction

This appendix derives the value of an unit of capacity under construction, $\Delta G(K,Y,\theta)$. We follow the procedure outlined in the appendix of Bar-Ilan and Strange (1996).

The expectation in (18) is

$$E[\Delta H(K(t), Y(t+\theta))] =$$

$$\int_{0}^{\infty} \left[ A(K(t))(Y(t+\theta))^\alpha + \frac{Y(t+\theta)}{r-\mu} - \frac{w}{r} \right] f(Y(t+\theta))dY(t+\theta) \quad (A1)$$

$$+ \int_{0}^{\infty} \left[ B(K(t))(Y(t+\theta))^\beta \right] f(Y(t+\theta))dY(t+\theta)$$

where $w = (2\gamma + c_2)K(t) + c_1$ and $f(Y(t+\theta))$ denotes the probability density function (p.d.f.) of the demand at time $t + \theta$ when the construction is completed, given the current demand $Y(t)$.

Since $Y$ follows the geometric Brownian motion process specified in (1), the distribution of future demand $Y(t+\theta)$ given the current demand $Y(t)$, is lognormal

$$\log Y(t+\theta) = N \left( \log Y(t) + \left( \mu - \frac{\sigma^2}{2} \right) \theta, \sigma^2 \theta \right) \quad (A2)$$

The solution for the expected value (A1) uses the following two properties of the lognormal distribution. (See footnote 11 in Bar-Ilan and Strange (1996) for references).

(i) If $z = \log x$ is distributed $N(g,s^2)$, then the $u$-th moment of $x$ around 0 is

$$E(x^u) = e^{ug+us^2/2} \quad (A3)$$

(ii) When $z$ is truncated bellow al $\log x_0$ then the $u$-th moment of $x$ around 0 is

$$E(x^u) = \frac{1 - \Phi(v-u\sigma)}{1 - \Phi(v)} e^{ug+us^2/2} \quad (A4)$$

where $\Phi(.)$ denotes the cumulative distribution function of the standard normal distribution and $v$ is defined as
\[ v = \frac{1}{s} \left( \log x_0 - g \right) \]  

\[(A5)\]

The derivation of the value of the incremental unit of capacity when the time remaining to its completion is \( \theta \) uses the above properties of the lognormal distribution with

\[ g = \log Y(t) + \left( \mu - \frac{\sigma^2}{2} \right) \theta , \quad s = \sigma \sqrt{\theta} \quad \text{and} \quad x_0 = w \]

Equation (A4) implies

\[
\int_{w}^{\infty} (Y(t + \theta))^u f(Y(t + \theta))dY(t + \theta) = (1 - \Phi(v - u \sigma \sqrt{\theta}))E[(Y(t + \theta))^u] \]

\[(A6)\]

\[
\int_{0}^{w} (Y(t + \theta))^u f(Y(t + \theta))dY(t + \theta) = \Phi(v - u \sigma \sqrt{\theta})E[(Y(t + \theta))^u] \]

\[(A7)\]

Therefore equations (14), (A1), (A6) and (A7) imply

\[
\Delta G(K(t), Y(t), \theta) = e^{-r\theta} \left\{ A(1 - \Phi(v - \alpha \sigma \sqrt{\theta}))E[Y^\alpha(t + \theta)] \\
+ (1 - \Phi(v - \sigma \sqrt{\theta})) \frac{E[Y(t + \theta)]}{r - \mu} \\
- (1 - \Phi(v)) \frac{w}{r} + B \Phi(v - \beta \sigma \sqrt{\theta})E[Y^\beta(t + \theta)] \right\} 
\]

\[(A8)\]

equation (A3) implies

\[
E[Y^\alpha(t + \theta)] = Y^\alpha(t)e^{(\alpha(\mu - \sigma^2/2) + \alpha^2\sigma^2/2)\theta} \]

\[
E[Y^\beta(t + \theta)] = Y^\beta(t)e^{(\beta(\mu - \sigma^2/2) + \beta^2\sigma^2/2)\theta} \]

\[(A9)\]

\[
E[Y(t + \theta)] = Y(t)e^{\mu\theta} \]

since \( \alpha \) and \( \beta \) are the roots of the characteristic equation (15) we have

\[
E[Y^\alpha(t + \theta)] = Y^\alpha(t)e^{r\theta} \]

\[
E[Y^\beta(t + \theta)] = Y^\beta(t)e^{r\theta} \]

\[(A10)\]

Combining (A8) and (A10) and we get the expression for the value of the incremental unit under construction.
\[
\begin{align*}
\Delta G(K, Y, \theta) &= (1 - \Phi(v - \alpha \sigma \sqrt{\theta}))AY^\alpha + (1 - \Phi(v - \sigma \sqrt{\theta})) \frac{Ye^{-(r - \mu)\theta}}{r - \mu} \\
&- (1 - \Phi(v)) \left[ (2\gamma + c_2)K + c_1 \right] e^{-r\theta} + \Phi(v - \beta \sigma \sqrt{\theta})BY^\beta
\end{align*}
\] (A11)

where

\[
\nu = \nu(K, Y, \theta) = \frac{\log[(2\gamma + c_2)K + c_1] - \log Y - (\mu - \sigma^2/2)\theta}{\sigma \sqrt{\theta}}
\] (A12)

**Appendix B: Analytical Results**

The investment rule for additional capacity and the optimal capacity choice are the solutions to equations (26) and (27) respectively. These equations do not admit a closed form solution for all parameter values. However, there are two benchmark cases for which we can characterize analytically the solution to each of these equations, and their properties.

The first case of interest is when demand is certain, \(\sigma = 0\), and \(\mu \neq 0\). The benefit of delaying is growing if \(\mu > 0\), while the cost of investing is constant, therefore, \(\Delta F(K, Y) > 0\) for all \(K\). The value of the option to invest if exercised at \(T\) is

\[
\Delta F(K, Y) = \left[ \frac{Ye^{\mu(T+h)}}{r - \mu} - \frac{(2\gamma + c_2)K + c_1}{r} e^{-rh} - k \right] e^{-rT}
\] (B1)

The optimal exercise time \(T^*\) solves

\[
\frac{\partial \Delta F(K, Y)}{\partial T} = -Ye^{-(r - \mu)(T^* + h)} + [(2\gamma + c_2)K + c_1] e^{-r(T^* + h)} + rhe^{-rT^*} = 0
\] (B2)

To find the value of \(Y\) for which it is optimal to invest immediately we set \(T^* = 0\), this yields

\[
Y(K) = [(2\gamma + c_2)K + c_1 + rko^h] e^{-\mu h}
\] (B3)

This implies the following path of committed capacity
\[ K(t) = \max \left\{ \frac{Y(t)e^{\mu h} - c_1 - rke^{\gamma h}}{2\gamma + c_2}, 0 \right\} \quad (B4) \]

Since \( Y(t) = Y(t-h)e^{\mu h} \) and \( O(t) = K(t-h) \), the path of operational capacity is

\[ O(t) = \max \left\{ \frac{Y(t) - c_1 - rke^{\gamma h}}{2\gamma + c_2}, 0 \right\} \quad (B5) \]

Solving for \( Y \) in (B5) we have \( Y(t) > (2\gamma + c_2)O(t) + c_1 \). In particular \( Y(K) > (2\gamma + c_2)O(t) + c_1 \) for all \( K \) and \( O \). Thus, expansion always occurs when the completed capacity is fully used. With no demand uncertainty, future demand can be perfectly forecasted so that each marginal unit of capacity is always used when completed.

The second special case is when the firm does not have the option to reduce output if the level of demand falls. Since installed capacity is always fully utilized, the value of an incremental unit of capacity when \( 0 < \theta \) \( h \) is the remaining construction time is

\[ \Delta G(K,Y,\theta) = \frac{Ye^{-(r-\mu)\theta}}{r-\mu} - \frac{[(2\gamma + c_2)K + c_1]e^{-r\theta}}{r} \quad (B6) \]

The value of the investment threshold is

\[ Y(K) = \frac{\beta}{\beta - 1} \frac{r-\mu}{r} \left( (2\gamma + c_2)K + c_1 + rke^{\gamma h} \right) e^{-\mu h} \quad (B7) \]

In this case a lag between the addition of an incremental unit of capacity and its completion does not change the effect of uncertainty on investment when the firm has to use all its operating capacity. More demand uncertainty raises \( \beta(\beta - 1) \), and thus always raises \( Y(K) \). Notice also that \( Y(K) \) is equal to the certainty trigger times a factor \( \beta(r-\mu)/(\beta - 1)r \), which is greater than one for all parameters values. Therefore when the firm does not have the option to reduce output the investment triggers is always greater than the certainty trigger.
By solving for $K$ in (B7) we get the firm optimal capacity given the current value of the demand shock

$$K(Y) = \max \left\{ \frac{r}{2\gamma + c_2} \left[ Ye^{\alpha_s} \left( 1 - \frac{1}{\beta} \right) - ke^{\gamma/s} - \frac{c_1}{r} \right], 0 \right\} \text{ (B8)}$$

More demand uncertainty reduces $\beta$. This implies that the firm’s optimal capacity is decreasing in demand uncertainty. Given a demand path $Y(t)$ equation (B8) gives the firm’s desired committed capacity, $K(Y(t))$. Because investment is irreversible, the path of committed capacity, $K(t)$, is such that $K(t) \leq K(Y(t))$ for all $t$. Furthermore, since we do not allow for depreciation of the installed capacity, investment occurs only when demand reaches a new high. This implies that the firm’s path of optimal committed capacity is

$$K(t) = \max \left\{ \frac{r}{2\gamma + c_2} \left[ e^{\alpha_s} \left( 1 - \frac{1}{\beta} \right) \sup \{Y(s), 0 \leq s \leq t\} - ke^{\gamma/s} - \frac{c_1}{r} \right], K(0) \right\} \text{ (B9)}$$
References


Figure 1. Optimal Capacity as a function of $Y$ with instantaneous investment.

Note: This graph shows the capacity choice $K(Y)$ as a function of the demand parameter $Y$ for different levels of demand volatility $\sigma$. The assumed parameter values are $\mu = 0$, $\gamma = 0.5$, $\epsilon_1 = 1$, $\epsilon_2 = 0.5$, $r = 0.05$, $k = 5$, $h = 0$. 
Figure 2. Optimal Capacity as a function of $Y$ with time to build.

$K(Y)$

$\sigma = 0.4$

$\sigma = 0.2$

$\sigma = 0$

$Y$

Note: This graph shows the capacity choice $K(Y)$ as a function of the demand parameter $Y$ for different levels of demand volatility $\sigma$. The assumed parameter values are $\mu = 0$, $\gamma = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $r = 0.05$, $k = 5$, $h = 3$.  

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Figure 3. Optimal capacity as a function of demand volatility

Note: This graph shows the capacity choice $K(Y)$ as a function of demand volatility $\sigma$ for different levels of the demand parameter $Y$. The assumed parameter values are $\mu = 0$, $\gamma = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $r = 0.05$, $k = 5$, $h = 3$. 
Figure 4. Optimal Capacity as a function of the investment lag

$K(Y)$

$Y = 9$

$Y = 7$

$Y = 5$

$\sigma$  

$0$  $5$  $10$  $15$  $20$  

Note: This graph shows the capacity choice $K(Y)$ as a function of the investment lag $h$ for different levels of the demand parameter $Y$. The assumed parameter values are $\mu = 0$, $\sigma = 0.1$, $\gamma = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $r = 0.05$, $k = 5$, $h = 3$. 
Figure 5. The Dynamics of Committed Capacity, Operational Capacity and Output

Note: These graphs show a sample path of demand and the corresponding paths of committed capacity, operational capacity and output. The assumed parameter values are $\mu = 0.03$, $\sigma = 0.15$, $\gamma = 0.5$, $c_1 = 1$, $c_2 = 0.5$, $r = 0.05$, $k = 5$, $h = 3$, $Y(0) = 5$. 
Figure 6. The Dynamics of Capacity Expansion and Capacity Utilization.

Note: This graph shows the paths of capacity expansion and capacity utilization that result from the demand path of Figure 1. The change in committed capacity during \([t, t + \Delta t]\) is \(\Delta K(t)\). The operational capacity and output at time \(t\) are \(O(t)\) and \(Q(t)\) respectively, and \(Q(t)/O(t)\) is the fraction of completed capacity that is used in production at time \(t\). The flat parts on the path, \(Q(t)/O(t)\), correspond to the periods in which all the operational capacity is fully utilized. When completed capacity is fully used \(Q(t)/O(t) = 1\). During the time intervals \(T_1\), \(T_2\), and \(T_3\), corresponding to \([5.19, 5.22]\), \([6.31, 6.70]\) and \([11.02, 11.06]\) respectively the firm starts the construction of new capacity when the current operational capacity is not fully utilized. The assumed parameter values are \(\mu = 0.03\), \(\sigma = 0.15\), \(\gamma = 0.5\), \(r = 0.05\), \(k = 5\), \(h = 3\), \(c_1 = 1\), \(c_2 = 0.5\), \(Y(0) = 5\).
Figure 7. The Dynamics of Capacity and Capacity Utilization with and without a Construction Lag

Note: These graphs compare the dynamics of capacity expansion and capacity utilization with and without construction lags. The top graph shows the paths of committed capacity, $K(t)$, for $h = 0$ and for $h = 3$. The middle and bottom graphs show the fraction of completed capacity that is used in production, $Q(t)/O(t)$, for $h = 0$ and $h = 3$ respectively. The flat parts on the path, $Q(t)/O(t)$, correspond to the periods in which all the operational capacity is fully utilized. When completed capacity is fully used $Q(t)/O(t) = 1$. The assumed parameter values are $\mu = 0.03$, $\sigma = 0.15$, $\gamma = 0.5$, $r = 0.05$, $k = 5$, $h = 3$, $c_1 = 1$, $c_2 = 0.5$, $Y(0) = 5$. 
Figure 8. The dynamics of capacity expansion for different numbers of firms in the Industry

Note: This graph shows the paths of capacity expansion for a monopoly, a duopoly, a 3 firm oligopoly and perfect competition that result from the same realization of demand. The assumed parameter values are $\mu = 0.03$, $\sigma = 0.15$, $\gamma = 0.5$, $r = 0.05$, $k = 5$, $h = 3$, $c_1 = 1$, $c_2 = 0.5$, $Y(0) = 5$. 
Note: This graph shows the path of output price paths for a monopoly, a 3 firm oligopoly, and a 10 firm oligopoly, that result from the same realization of demand. The assumed parameter values are $\mu = 0.03$, $\sigma = 0.15$, $\gamma = 0.5$, $r = 0.05$, $k = 3$, $c_1 = 1$, $c_2 = 0.5$, $\gamma(0) = 5$. 
Figure 10. Capacity Utilization and the Relative Changes in Output Prices

Note: Each graph in this figure shows the path of capacity utilization, $Q(t)/O(t)$, and the change in the relative output price, $dP(t)/P(t)$, that result from the same realization of demand. The paths on the top graph correspond to a monopoly, those on the middle graph are for a 3-firm oligopoly, and the paths on the bottom are for a 10-firm oligopoly. The units on the vertical axis correspond to $dP(t)/P(t)$. The flat parts on the path, $Q(t)/O(t)$, correspond to the periods in which all the operational capacity is fully utilized. The assumed parameter values are $\mu = 0.03$, $\sigma = 0.15$, $\gamma = 0.5$, $r = 0.05$, $k = 5$, $h = 3$, $c_1 = 1$, $c_2 = 0.5$, $Y(0) = 5$. 

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