POLLUTION REDUCTION, ENVIRONMENTAL UNCERTAINTY, AND THE IRREVERSIBILITY EFFECT*

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ABSTRACT

This paper analyses the decision to invest to reduce the emissions of a stock pollutant under environmental uncertainty. It shows that this decision depends on the type and level of uncertainty. When uncertainty is small, there is no simple irreversibility effect because of the tension between environmental irreversibility (the stock of pollutant causes costly long-term social damages), and investment irreversibility (pollution abatement investments are sunk). When uncertainty is large enough, however, pollutant emissions should be curbed immediately. A continuous time formulation based on real options illustrates the link between flexibility and option value. These results have implications for global warming.

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I. INTRODUCTION

The "irreversibility effect" was originally introduced to the environmental literature by Arrow and Fisher (1974) and Henry (1974). Analyzing the development, with uncertain costs and benefits, of a natural area, these authors defined the irreversibility effect as the bias against conservation in a standard cost-benefit calculation that ignores the irreversible character of development. Building on Weisbrod's concept of option value (1964), Arrow and Fisher proposed to take into account the corresponding loss of flexibility by introducing an extra term in a cost-benefit analysis, which they called "quasi-option value." They showed that quasi-option value exists independently of risk-aversion.

Subsequent work has shown, however, that the "irreversibility effect" may not hold when some assumptions of the Arrow-Fisher model are relaxed (e.g. Epstein (1980) or Freixas and Laffont (1984)). This is the case, for example, when there is information to be gained by some level of development, which Fisher and Hanemann (1987) call "dependent learning" (Arrow and Fisher considered only full development or no development). Most of these results, however, were obtained with two or three periods discrete-time models, which do not allow for a full treatment of the dynamics of the problem.

The irreversibility effect has since been generalized. It is now usually perceived as the need to scale down or delay a project that is harmful to the environment or to adopt earlier an investment beneficial to the environment when an environmental irreversibility is involved. A decision is irreversible when it is very costly to reverse, for example because it limits the future choices of a decision-maker (Henry (1974)). A frequently used example of a development project with irreversible consequences is the construction of a dam in a scenic canyon, or the accumulation of long-lived greenhouse gases in the atmosphere.

At the end of their 1974 paper, Arrow and Fisher indicated how to extend their analysis to long-lived stock pollutants, such as DDT, and to cumulative "macro" environmental effects, such as the accumulation of greenhouse gases in the atmosphere. There is, however, one essential difference between the development of a natural area and the management of a stock pollutant. Indeed, whereas in the former type of problem the only irreversibility may be the loss of a natural area, two types of irreversibility are present in the decision to invest to reduce the emissions of a stock pollutant (Kolstad (1996)). The first irreversibility is environmental because society has to live for a long time with the flow of social damage caused by the stock of a slowly decaying pollutant. In the case of greenhouse gases, for example, it is estimated that the mean lifetime of CO_2 in the atmosphere is 500 years (IPCC (1992)). The second irreversibility, which could be termed investment irreversibility, comes from the fact that pollution control capital is often (at least partially) sunk. An example is the massive investments needed to switch from coal burning to natural gas burning power plants in order to reduce CO_2 emissions.

The optimal management of stock pollutants under uncertainty was recently analyzed again by Pindyck (in Dixit and Pindyck (1994)), Kolstad (1996), and Ulph and Ulph (1997). They used different approaches. Pindyck formulated the decision to invest to reduce the emissions of a stock pollutant as an optimal investment problem in continuous time. He considered uncertainty in the valuation of pollution damage and showed that an increase in the uncertainty of the future social costs of pollution leads to delay the decision to invest in emissions reduction. In this case, neglecting uncertainty would thus introduce a bias in favor of the environment. Kolstad, and Ulph and Ulph, on the other hand, used two-period, discrete time models, which do not allow for a full treatment of the dynamics of the problem.

They showed that the presence of an irreversibility effect is dependent on technical hypotheses on the social planner's utility function and on the resolution of uncertainty over time through learning. Kolstad seems to have used a strict definition of irreversibility, however, while Ulph and Ulph did not consider investment irreversibility. Despite the recent attention it has received, the joint impact of irreversibility and uncertainty on the management of stock pollutants does not seem to be fully understood.

This paper extends the literature on uncertainty and the irreversibility effect for the management of stock pollutants. We focus on uncertainty in the stock of pollutant (environmental uncertainty). To be in a framework favorable to the "irreversibility effect", we assume that information about the evolution of the stock of pollutant arrives in time independently of the decision to invest (independent learning). Like Pindyck, we use concepts from the theory of real options and a continuous-time formulation, which allows for a full treatment of the dynamics of the problem.[1] We show that there is no simple irreversibility effect for the management of stock pollutants because there is a tension between environmental and investment irreversibility. Moreover, the optimal decision to reduce emissions depends on both the level and the nature of uncertainty. Hence, when environmental uncertainty is "large enough," we find that the emissions of pollutant should be reduced immediately.

A by-product of this paper is a clarification of the concept of option value close in spirit, we believe, to Weisbrod's intuition. In the Arrow-Fisher framework, Conrad (1980) and Hanemann (1989) linked quasi-option value to the value of perfect information on the impact of development, conditional on preservation. As shown by Hanemann (1989), this interpretation cannot be generalized so, we rely on the theory of real options and we use the same formulation for the deterministic and stochastic cases. This gives us a deterministic option term, which we interpret as the value of the flexibility to modify an irreversible decision. Option value thus exists independently of the arrival of information over time.

This paper is organized as follows. In Section II, we introduce a simple continuous-time model, which features a single stock externality. We formulate our pollution control problem as an optimal stopping problem in which a risk-averse social planner has to choose when and how much to invest in a one-time reduction in pollutant emissions. In Section III, we solve the corresponding deterministic problem to get a benchmark for the impact of environmental uncertainty. In Section IV, we analyze two classes of stochastic models to explore the impact of the specification of uncertainty. One class gives finite expected social damage for all values of pollutant stock volatility, and the other one does not. A numerical illustration is provided in Section V. The last section summarizes our conclusions.

II. A MODEL OF POLLUTANT STOCK UNCERTAINTY

We consider a stylized model with one stock pollutant, which decays at rate $\alpha > 0$. We denote by X the stock of this pollutant and E_1 its rate of emission. To focus solely on the variability of X, we assume that E_1 is constant. Problems with more than one stochastic variable are notoriously difficult to solve analytically. Because of the randomness of physical and chemical processes that contribute to the decay of the pollutant, we suppose that X follows a diffusion process, which belongs to one of two classes of stochastic processes:

(1) $$dX = \alpha(\frac{E_1}{\alpha} - X)dt + \sqrt{vX^\rho}\,dz, \ \rho = 1,2$$

The quantity $vX^\rho \geq 0$ is the infinitesimal variance of the stock pollutant process and dz is an increment of a standard Wiener process (for an introduction to stochastic calculus for economists, see Dixit, 1993). The parameter v characterizes the volatility of the stock of pollutant. When $\rho = 1$ (Model 1), the infinitesimal variance of the process followed by X increases linearly with X, whereas when $\rho = 2$ (Model 2), it increases with the square of X, as for the geometric Brownian motion.[2] From Equation (1), we see that X remains non-negative and tends to revert to $\frac{E_1}{\alpha}$; thus, the decay rate, α, also characterizes the speed of reversion. We denote by X(0) the initial stock of pollutant.

We further assume that the flow of social costs resulting from pollution damage, noted C(X), is given by:

(2) $$C(X) = -\phi X^{\lambda}$$

where $\lambda \geq 1$. In the following, we normalize the valuation parameter ϕ to 1. This formulation is appropriate for stock externalities, i.e. situations where social damages result not from the emissions of a compound, but rather from its accumulation. The suspected impact on the climate of the accumulation of greenhouse gases provides a good example.

We assume that pollutant emissions can be decreased from E_1 to a constant E_2, at a cost K, which may depend on E_1 and on $E_1 - E_2$, but not on X. We suppose that K is sunk, which is often reasonable for pollution control measures (e.g., the installation of scrubbers by electric utilities). After emissions have been reduced, X follows the process:

(3) $$dX = \alpha\left(\frac{E_2}{\alpha} - X\right)dt + \sqrt{vX^{\rho}}\,dz,\ \ \rho = 1,2$$

We consider a risk-averse social planner with a temporally additive and separable utility function noted $U(\bullet)$ for the flow of social damage from pollution. To simplify our calculations, we assume further that the planner's utility is linear in K, and that $U(-y) = -y^{\delta}$, where $y \geq 0$ measures the flow of social damage from pollution, and where δ is such that $\lambda\delta$

$\equiv$ m is an integer strictly greater than 1. In this formulation, increasing constant risk aversion and increasing physical damage have the same qualitative impact, although this is clearly not the case in general. As for quasi-options, option value exists independently of risk aversion.

The objective of the social planner is to find the values of E_2 and T that maximize the present value function:

(4) $$J(T, E_2) = \mathcal{E}_0 \int_0^{\infty} -X^m e^{-rt} dt - e^{-rT} K(E_2, E_1 - E_2)$$

subject to Equation (1) for $0 \leq t \leq T$ and to Equation (3) for $t > T$, with X(0) given. E_2 ($0 \leq E_2 \leq E_1$) is the rate to which pollutant emission should be reduced; T is the socially optimal time at which to cut emissions to E_2; $\mathcal{E}_0$ is the expectation operator for information available at time t = 0; and r is the social discount rate. In this model, we thus have a continuum of possibilities to reduce pollutant emissions with independent learning. Indeed, information arrives over time in the form of a realization of the diffusion process followed by the stock of pollutant, independently of the decision to invest in the reduction of emissions.

This optimization problem can be solved in two steps. First, for an arbitrary value of E_2, such that $0 \leq E_2 < E_1$, we calculate the critical stock of pollutant, denoted x*, at which the rate of pollutant emission should be reduced from E_1 to E_2 for an arbitrary function $K(E_1, E_1\text{-}E_2)$. The values of X less than x* define the so-called "continuation region," or region 1, where the optimal decision is to wait. As soon as $X \geq x^*$, which defines the so-called "stopping region," or region 2, the rate of pollutant emissions should be reduced to E_2.

For the second step, it is first necessary to calculate $\mathcal{E}_0 T(x^*;E_2)$, the expected time at which the stock of pollutant reaches x* for the first time, given an initial stock of pollutant denoted X(0). We could then substitute these results back into the objective function and minimize it with respect to E_2. Since we are interested in how x* could vary with v, we focus only on the first step of the solution procedure and we take E_2 as given. The optimal value of E_2 would depend on the function $K(E_1, E_1-E_2)$. To simplify our notation, we omit E_2 as an argument of both the option term and expected social damages.

This is a standard optimal stopping problem, which bears similarities with an optimal investment problem. To solve it, we use stochastic dynamic programming and concepts from the theory of real options. Let $V_i(x;v)$ denote the value function in region "i". The Hamilton-Jacobi-Bellman (HJB) equation is:

$$(5) \qquad rV_i(x;v) = -x^m + (E_i - \alpha x)\frac{dV_i(x;v)}{dx} + \frac{vx^\rho}{2}\frac{d^2V_i(x;v)}{dx^2}, i = 1,2$$

The left side of Equation (5) can be interpreted as a return; the first term on the right side is the flow of social pollution costs; and the last terms represent the capital gains.

Equation (5) is a second-order linear differential equation. Its solution is the sum of a particular solution, noted $P_i(x;v)$, plus the general solution of the associated homogeneous equation, noted $\varphi(x;v)$. We select $P_i(x;v)$ so that it represents the expected social costs from emitting pollution at rate E_i forever, given x, the current stock of pollutant. $\varphi(x;v)$ is the value of the option to reduce emissions. Since it represents the value of the possibility of doing something, it is by definition non-negative. In this context, waiting decreases the present value of the cost of reducing pollutant emissions while cutting down on emissions earlier reduces the present value of pollution damages.

When we consider a one-time reduction in pollutant emissions, there is no option term after pollutant emissions have been reduced to E_2. Thus, the solutions of Equation (5) in regions 1 and 2 are respectively:

(6) $$V_1(x;v) = \varphi(x;v) + P_1(x;v),\ V_2(x;v) = P_2(x;v)$$

To find x*, we need two additional conditions (see Brekke and Oksendal, 1991.) First, at x*, the value of the option to reduce the rate of pollutant emissions plus the social cost of polluting forever at rate E_1 should equal the social cost of polluting forever at rate E_2 plus the cost of reducing emissions from E_1 to E_2. This is the continuity condition:

(7) $$\varphi(x^*;v) + P_1(x^*;v) = P_2(x^*;v) - K$$

The second condition, called "smooth-pasting," says that, when it is optimal to exercise the option to reduce pollutant emissions, the marginal change in the value of the option equals the marginal change in the difference of expected social pollution costs:

(8) $$\frac{d\varphi(x^*;v)}{dx} = \frac{dP_2(x^*;v)}{dx} - \frac{dP_1(x^*;v)}{dx}$$

By combining these two conditions, we obtain a "stopping rule" of the form:

(9) $$\frac{\varphi(x^*;v)}{\frac{d\varphi(x^*;v)}{dx}} = \frac{P_2(x^*;v) - P_1(x^*;v) - K}{\frac{dP_2(x^*;v)}{dx} - \frac{dP_1(x^*;v)}{dx}}$$

Equation (9) equates the instantaneous rate of return of the option to reduce pollutant emissions with that of the corresponding net reduction in the expected social costs of pollution. They are written in reverse form to avoid possible division by 0 (see the expressions of P_i below). The smallest non-negative root of this equation (denoted x*), if it exists, defines the critical stock of pollutant at which pollutant emissions should be reduced from E_1 to E_2. Moreover, we will see that if Equation (9) has only negative solutions pollutant emissions should be cut immediately.

III. SOLUTION OF THE DETERMINISTIC MODEL

In this section, we derive a deterministic benchmark for the stochastic models of Section IV. Using the same formulation for the deterministic and stochastic cases also allows us to illustrate the concept of option value in a real options framework. As shown below, we can define an option term under certainty. Following the investment literature on real options, we interpret this term as the value of the flexibility to modify an action with irreversible consequences (e.g., see Trigeorgis (1995)).

Until the end of this section, we thus assume that v = 0 so both stochastic models reduce to the same deterministic form. Equations (1), (3), and (5) simplify to first order linear differential equations. Integrating Equations (1) and (3), we obtain:

$$(10) \qquad X(t) = \begin{cases} \frac{E_1}{\alpha} + (X(0) - \frac{E_1}{\alpha})e^{-\alpha t}, 0 \le t \le T \\ \frac{E_2}{\alpha} + (X(T) - \frac{E_2}{\alpha})e^{-\alpha(t-T)}, t > T \end{cases}$$

Thus, when E_i is held constant, X converges monotically towards $\frac{E_i}{\alpha}$ and it never crosses this value. We can now calculate the present value of social pollution costs.

Lemma 1*: If the rate of pollutant emissions is fixed at E_i and the initial stock of pollutant is x, the present value of social pollution costs is:*[3]

$$(11) \qquad P_i(x;0) = -\sum_{k=0}^{m} x^k \frac{m!}{k!} \frac{E_i^{m-k}}{\prod_{j=k}^{m}(r + j\alpha)}, i = 1,2$$

Moreover, the option term is $\varphi(X;0) = \mathrm{Max}(\tilde{\varphi}(X), 0)$*, with:*

$$(12) \qquad \tilde{\varphi}(X) = A_0 \left|\alpha X - E_1\right|^{\frac{-r}{\alpha}}$$

A_0 is a constant to be determined jointly with the critical stock of pollutant at which it is optimal to reduce pollutant emission from E_1 to E_2.

Proof. We can find the present value of social pollution costs by calculating $-\int_0^{+\infty} X^m e^{-rt} dt$, with X(0) given, subject to $X(t) = (X(0) - \frac{E_i}{\alpha})e^{-\alpha t} + \frac{E_i}{\alpha}$. The option term is the solution of the homogeneous equation associated with Equation (5) with v = 0. ||

We can now derive the deterministic stopping rule.

Proposition 1. *The following equation has at most one non-negative root, denoted y*:*

(13) $$\frac{E_1 - \alpha x}{r} = \frac{P_2(x;0) - P_1(x;0) - K}{\frac{dP_2(x;0)}{dx} - \frac{dP_1(x;0)}{dx}}$$

If $0 \le y^* < \frac{E_1}{\alpha}$, y is the value of the stock of pollutant, denoted x_0^*, at which it is optimal to reduce pollutant emission from E_1 to E_2.*[4] *If Equation* (13) *has only negative roots, the emissions of pollutant should be reduced immediately. Finally, if $y^* \ge \frac{E_1}{\alpha}$, it is optimal to invest now in pollution reduction provided $P_2(x;0) - P_1(x;0) - K > 0$, and never otherwise.*

Proof. To derive Equation (13), introduce Equations (11) and (12) into Equation (9). Reorganizing the terms of Equation (13) leads to $f(x;\alpha, r, K) = 0$, where $f(x;\alpha, r, K) \equiv P_2(x;0) - P_1(x;0) + \frac{\alpha x - E_1}{r}\left(\frac{dP_2(x;0)}{dx} - \frac{dP_1(x;0)}{dx}\right) - K$. Introducing Equation (11) into the expression of f(x;α,r,K) and simplifying gives:

$f(x;\alpha,r,K)=\sum_{k=0}^{m-1} x^k \frac{m!}{k!}\frac{(E_1-E_2)E_2^{m-k-1}}{r\prod_{j=k+1}^{m}(r+j\alpha)}-K$. $f(x;\alpha,r,K)$ is thus continuous and strictly increasing in x, so Equation (13) has exactly one solution if $f(0;\alpha,r,K) \le 0$. Equation (13) is the deterministic counterpart of the stochastic stopping rule for this problem (Equation (9)).

On the other hand, if $f(0;\alpha,r,K) > 0$, Equation (13) has no solution. This is possible only because K is so small or α and r are so large that it is inexpensive (and optimal) to cut emissions right away compared to the expected reduction in social damages from pollution.

For the case where $x_0^* > \frac{E_1}{\alpha}$, let us show that the smooth-pasting condition forces the option term to be zero. Indeed, from Equation (11), the right hand-side of the smooth-pasting condition, given by $\frac{dP_2(x;0)}{dx}-\frac{dP_1(x;0)}{dx}$, is strictly positive for all $x \ge 0$. However, the left-hand side of the smooth pasting condition is $\frac{d\tilde{\varphi}(x)}{dx}=\frac{r\tilde{\varphi}(x)}{E_1-\alpha x}$, which is negative when $x>\frac{E_1}{\alpha}$, unless the constant A_0 in $\tilde{\varphi}(x)$ is negative. This forces the option term, $\varphi(x)$, to be zero. Investing to reduce pollutant emissions when pollution is decreasing is thus a "now or never" proposition. If $J(0,E_2)-J(+\infty,E_2)=P_2(x;0)-P_1(x;0)-K>0$, we should invest now to reduce pollution and never otherwise.[5] ||

Proposition 1 simply says that there are two cases where the possibility to delay an investment for cutting pollutant emissions has no value. In the first case, the stock of pollutant is increasing over time and reducing pollutant emissions is relatively inexpensive so it should be done immediately. In the second case, the stock of pollutant and associated social damages decrease over time. Since waiting decreases the benefits of reducing emissions without affecting its costs, it has no value. The decision to invest can thus be based on a conventional cost-benefit analysis. Fisher, Krutilla, and Cicchetti (1972) found a similar result in their analysis of the development of a scenic canyon on the Snake River. They showed that when the benefits of preservation are increasing over time relative to the net benefits from development, it is optimal to develop either now or never.

These results also show that $\frac{E_1}{\alpha}$ is a barrier that separates the range of values of X in two subsets: $X < \frac{E_1}{\alpha}$, where waiting is valuable if K is neither too cheap nor too expensive compared to a reduction in social damages from pollution, and $X \geq \frac{E_1}{\alpha}$, where there is no option value. We will see below that the singularity at $\frac{E_1}{\alpha}$ for the deterministic case has implications for the stochastic models.

IV. ANALYSIS OF THE STOCHASTIC MODELS

We now assume that v > 0. For both models, we start by calculating the expected social costs, denoted by P_i, and the option terms, denoted by $\tilde{\varphi}$. The superscripts "I" and "II" refer to Model 1 and 2 respectively (see Equation (1)).

Lemma 2. *For Models I and II, the expected social costs from continuing to pollute forever at rate E_i, given an initial stock of pollutant x, are respectively:*

(14) $$P_i^I(x; v) = -\sum_{k=0}^{m} x^k \frac{m!}{k!} \frac{\prod_{j=k}^{m-1}(E_i + jv/2)}{\prod_{j=k}^{m}(r + j\alpha)}, i = 1,2$$

(15) $$P_i^{II}(x;v) = -\sum_{k=0}^{m} x^k \frac{m!}{k!} \frac{E_i^{m\text{-}k}}{\prod_{j=k}^{m}(r + j\alpha - j(j-1)v/2)}, i = 1,2$$

For $P_i^{II}(x;v)$ *to be finite, we must have:* $0 \le v < 2\frac{r + m\alpha}{m(m-1)}$.

The corresponding option terms are given by $\varphi(X;0) = \mathrm{Max}(\tilde{\varphi}(X), 0)$*, with:*

(16) $$\tilde{\varphi}^I(x;v) = A_0 \Phi\left(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x\right)$$

(17) $$\tilde{\varphi}^{II}(x) = B_0 \left(\frac{E_1}{x}\right)^{\beta} \Psi\left(\beta, \zeta; \frac{2E_1}{vx}\right)$$

A_0 and B_0 are constants to be determined jointly with x. $\Phi(a,b;y)$ and $\Psi(a,b;y)$ are respectively the confluent hypergeometric functions of the first and second kinds with argument y and parameters a and b.*[6] *β and ξ are functions of r, v, and α defined by:*

(18) $$\beta = \frac{-\left(\frac{v}{2} + \alpha\right) + \sqrt{\left(\frac{v}{2} + \alpha\right)^2 + 2rv}}{v}, \xi = 2\beta + 2 + \frac{2\alpha}{v}$$

Proof. For both models, we derive expected social costs from the moment generating function of the stochastic process X, noted M(θ,t), and from the relationship:

(19) $$\frac{\partial^n M(0,t)}{\partial \theta^n} = (-1)^n \mathcal{E}(X_t^n).$$

Details of the calculations of M(θ,t) are presented in Appendix A1.

To find the option terms, we look for a solution to the homogeneous equation associated with Equation (5). This solution should be well defined at X = 0 and increasing in X because the larger is X, the larger are expected social damages, and so the more valuable is the possibility of reducing pollutant emissions. Details of the derivations are presented in Appendix A2. ||

Note that, for both models, the magnitude of expected social costs is increasing in v, but $P_i^{II}(x;v)$ is much more responsive to changes in v than $P_i^{I}(x;v)$. This feature seems reasonable for a number of stock pollutant problems. Indeed, for global climate change, an increase in the atmospheric concentration of greenhouse gases could cause increases not only in mean temperatures but also in temperature and rainfall volatilities, with potentially dramatic impacts on agriculture, for example.

We can now derive the stopping rules for both models. We find:

Lemma 3. *The critical stock of pollutant, denoted x*, is the smallest non-negative real which verifies the equations:*

$$\textit{(20)} \qquad \frac{\Phi\left(\frac{r}{\alpha}, \frac{2E_1}{v}; \frac{2\alpha}{v} x^*\right)}{\frac{r}{E_1}\Phi\left(\frac{r}{\alpha}+1, \frac{2E_1}{v}+1; \frac{2\alpha}{v} x^*\right)} = \frac{P_2^I(x^*;v) - P_1^I(x^*;v) - K}{\frac{dP_2^I(x^*;v)}{dx} - \frac{dP_1^I(x^*;v)}{dx}}$$

$$\textit{(21)} \qquad \frac{\frac{x^*}{\beta}\Psi\left(\beta, \xi; \frac{2E_1}{vx^*}\right)}{\frac{2E_1}{vx^*}\Psi\left(\beta+1, \xi+1; \frac{2E_1}{vx^*}\right) - \Psi\left(\beta, \xi; \frac{2E_1}{vx^*}\right)} = \frac{P_2^{II}(x^*;v) - P_1^{II}(x^*;v) - K}{\frac{dP_2^{II}(x^*;v)}{dx} - \frac{dP_1^{II}(x^*;v)}{dx}}$$

Proof. Introduce the results from *Lemma 2* into the continuity and smooth-pasting conditions and rearrange to get rid of the unknown constants A_0 and B_0. ||

One important difference with the deterministic case is that $\frac{E_1}{\alpha}$ is no longer a barrier for X (see Equation (1)). Thus, there can be a positive option value even if $X > \frac{E_1}{\alpha}$, although it may be small as we will see in the numerical illustration. Equations (20) and (21) define implicitly x* as a function of v. Unfortunately, it is not possible to find an explicit expression for x* nor to analyze how x* changes with v because of the complexity of the hypergeometric functions and their derivatives with respect to their second parameter. We thus examine how x* changes for "small" and "large" values of v. Considering first "small" values of v, we have:

Proposition 2. *The limit when v goes to zero of the stochastic critical value of the stock of pollutant, denoted* $\tilde{x}_0^*$, *equals the deterministic critical value, denoted* x_0^*. *Thus, if* $x_0^* < \frac{E_1}{\alpha}$, $\tilde{x}_0^*$ *is the non-negative root of Equation (13) if it exists, and zero otherwise. If, however,* $x_0^* > \frac{E_1}{\alpha}$, $\tilde{x}_0^*$ *is the non-negative root of:* $\left[P_2(x;0) - P_1(x;0) - K\right] = 0$.

In addition, for small values of v, x(v) may be larger or smaller than* $\tilde{x}_0^*$, *depending on the cost of cutting emissions relative to the benefits of reducing expected social damages from pollution.*

Proof. To prove this proposition, we derive first-order expansions in v of each side the stopping rules, which we denote by LHS(x*;v) and RHS(x*;v) for left hand-side and right hand-side respectively. Details of the derivations can be found in Appendix B.

Equating the constant terms in LHS(x*;v) and RHS(x*;v) proves the first part of this proposition. It is clear from the expressions of the expected social damages for both models that taking their limit when v goes to zero gives the expression of social damages in the deterministic case. Using results from Appendix B, it could also be shown that the limits of the stochastic option terms, when v goes to zero, gives the deterministic option term.

Equating the first order terms in v of LHS(x*;v) and RHS(x*;v) gives $\frac{dx^*}{dv}(0)$ for each model, but its sign is ambiguous (see Equation (A29)). Considering specific values for $\tilde{x}_0^*$ in the expression of $\frac{dx^*}{dv}(0)$, we find:

$$\text{(22)}\qquad \frac{dx^{*I}}{dv}(0) \approx \begin{cases} -\dfrac{m(m-1)}{4(r+\alpha)}\dfrac{E_1^{m-2}(E_1-E_2)}{(E_1^{m-1}-E_2^{m-1})} < 0, \text{ when } \tilde{x}_0^* \approx 0 \\ \dfrac{E_1}{2\alpha|\varepsilon|} > 0, \text{ when } \tilde{x}_0^* = \dfrac{E_1 \pm \varepsilon}{\alpha}, \text{ with } |\varepsilon| \text{ small} \\ \dfrac{r-(m-1)\alpha}{2\alpha[r+(m-2)\alpha]}, \text{ when } \tilde{x}_0^* >> \dfrac{E_1}{\alpha} \end{cases}$$

$$\text{(23)}\qquad \frac{dx^{*II}}{dv}(0) \approx \begin{cases} \dfrac{-E_2^{m-1}(E_1-E_2)\sum_{j=1}^{m}\dfrac{j(j-1)}{r+j\alpha}}{(r+\alpha)(E_1^{m-1}-E_2^{m-1}+(E_1-E_2)E_2^{m-2})} < 0, \text{ when } \tilde{x}_0^* \approx 0 \\ \dfrac{E_1^2}{2\alpha^2|\varepsilon|} > 0, \text{ when } \tilde{x}_0^* = \dfrac{E_1 \pm \varepsilon}{\alpha}, \text{ with } |\varepsilon| \text{ small} \\ \dfrac{\tilde{x}_0^*}{2}\dfrac{r^2+r\alpha-m(m-2)\alpha^2}{\alpha(r+m\alpha)(r+(m-1)\alpha)}, \text{ when } \tilde{x}_0^* >> \dfrac{E_1}{\alpha} \end{cases}$$

Both models give qualitatively similar results. When $\tilde{x}_0^* \approx 0$, we see from Equations (22) and (23) that $\frac{dx^*}{dv}(0) < 0$. This case occurs when K, r, and α combine to make reducing emissions attractive, so environmental irreversibility drives the solution. Then, as K increases or as r and α decrease, $\tilde{x}_0^*$ increases and investing to reduce emissions becomes less attractive. Investment irreversibility becomes more important compared to environmental irreversibility. When $\tilde{x}_0^* \approx \frac{E_1}{\alpha}$, we reach the singularity for the deterministic case, which is a barrier for the existence of the deterministic option term. This causes $\frac{dx^*}{dv}(0)$ to tend towards +∞ for both models, but faster for Model 2 than for Model 1. Away from this singularity, for $\tilde{x}_0^* >> \frac{E_1}{\alpha}$, $\frac{dx^*}{dv}(0) < 0$ if $\frac{r}{\alpha}$ is smaller than an increasing function of m, where m increases with the severity of social damages and with the social planner's risk aversion, and $\frac{dx^*}{dv}(0) \geq 0$ otherwise. Specifically, $\frac{dx^*}{dv}(0) < 0$ if $\frac{r}{\alpha} < m - 1$ for Model 1, and if $\frac{r}{\alpha} < \frac{-1 + \sqrt{4m^2 - 8m + 1}}{2}$ for Model 2. ‖

Thus, when the volatility of the stock of pollutant is small, we do not know a-priori how we bias our decision to invest to reduce pollutant emissions if we rely only on the deterministic model, which is the common approach in a standard cost-benefit analysis. Depending on the model parameters, either environmental or investment irreversibility could dominate, so there is no simple irreversibility effect.

Let us now investigate how x* changes when v is "larger." We have:

Proposition 3. *Unless reducing emissions is inexpensive compared to the resulting savings in expected social damages from pollution, there exists a unique permissible v such that x*(v) = 0.*

Proof. To prove this result, we set x* to zero in the stopping rules and look for a permissible v, i.e. $0 \leq v$ for Model 1 and $0 \leq v < 2\frac{r + m\alpha}{m(m-1)}$ for Model 2.

Let us start with Model 1. Setting x* to zero in Equation (20) and rearranging terms gives $f^I(v; \alpha, r, K) \equiv \frac{m!}{r + m\alpha} \frac{E_1 - E_2}{E_2} \prod_{j=0}^{m-1} \left(\frac{E_2 + 0.5jv}{r + j\alpha} \right) - K = 0$. The function $f^I(v; \alpha, r, K)$

is clearly continuous in its arguments, decreasing in K, r and α, increasing in v, and positive for large enough values of v. If $f^{I}(v;\alpha,r,K)\leq 0$, there exists a unique v such that x*(v) = 0. Otherwise, $f^{I}(v;\alpha,r,K)>0$ implies that K, r, or α are too low for $f^{I}(v;\alpha,r,K)=0$ to admit a positive solution in v. This means that reducing pollutant emissions is relatively inexpensive, so it should be done immediately.

For Model 2, we proceed the same way. We set x* to zero in Equation (21) and simplify, using $U(\alpha,\gamma,z)=z^{-\alpha}(1-\frac{\alpha(1+\alpha-\gamma)}{z}+o(\frac{1}{z}))$ when $z\to+\infty$, (see (9.12.3) p.270 in Lebedev) and the identity $-v\beta(1+\beta-\xi)=2r$. This leads to $f^{II}(v;\alpha,r,K)=0$, where

$f^{II}(v;\alpha,r,K)\equiv\frac{m!\,E_2^{m-1}(E_1-E_2)}{r\prod_{j=1}^{m}(r+j\alpha-0.5vj(j-1))}-K$. The rest of the proof is identical. ||

The intuition here is the same as in Proposition 1. If the cost of reducing pollutant emissions relative to the corresponding reduction in expected social damages is small enough under certainty, then we should invest right away to cut down pollutant emissions. Otherwise, there exists a permissible value of v such that it is optimal to invest right away to reduce pollutant emissions. The key here is to notice that, as the volatility of the stock of pollutants, v, increases, the expected social damages from pollution go to infinity for both models (see Equations (14) and (15)), while K remains unchanged.

V. A NUMERICAL ILLUSTRATION

A numerical illustration is presented in Tables I and II and in Figure I. Table I shows how the critical stock of pollutant at which pollutant emissions should be reduced, x*(v), varies with v when v ≈ 0+, for a range of parameter values. Although the primary purpose of this table is to illustrate the main result of Proposition 2 (x*(v) may increase or decrease as v increases from 0), it also helps us confirm our intuition on a number of points.

First, as expected, we observe that, the larger is α or r, the larger is x*(v): indeed, if the pollutant decays faster or if the social discount rate is higher, expected future damages decrease. Moreover, all else being equal, the faster expected social damages increase with the stock of pollutant (i.e. the higher is m) the earlier pollutant emissions should be reduced. Since x*(v) is very sensitive to m, we have to use much higher values of K for m = 3 than for m = 2 to obtain values of x* comprised between 0 and $\frac{E_1}{\alpha}$. Moreover, variations of x*(v) and φ*/K, the option value at the critical stock of pollutant normalized by the sunk cost of reducing pollutant emissions, are much steeper for Model 2 than for Model 1. Note that the option value can be as large as K or larger (e.g., see the case α = 0.03, r = 0.02). This suggests that ignoring the option value, as is often done in conventional cost-benefit analysis, can lead to very sub-optimal decisions. Finally, note that the option value goes to zero as v goes to zero when $x_0^* > \frac{E_1}{\alpha}$, because the option term is continuous in v.

Table II shows values of the volatility parameter v for which x*(v) goes to zero. Again, the larger is α and r, the larger is the corresponding value of v. Likewise, the larger is m, i.e. the higher is social damage associated with the level of stock pollutant, the smaller is v. It is also of interest to focus on the results for Model 2 in Table II, because very small values of v require investing right away in pollution reduction (e.g., consider r = 0.02). This illustrates the danger of using a certainty equivalent in stochastic models that are sensitive to a volatility parameter, even when volatility is "small" since "smallness" is context dependent.

Finally, Figure I illustrates the variations of $x^*(v)$ with v for Model 1 when $m = 2$. Qualitatively similar results were obtained for other values of m and for Model 2.

VI. CONCLUSIONS

We have analyzed the tension between environmental irreversibility and irreversibility in pollution control capital investment under environmental uncertainty using a real options approach in continuous-time. This formulation allows for a full treatment of the dynamics of the problem and illustrates the link between option value and flexibility. When uncertainty is low, we have found that we cannot a-priori know the bias introduced by neglecting uncertainty, an approach that would probably be followed in a conventional cost-benefit analysis. When uncertainty is "high enough," however, we have shown that it becomes optimal to invest right away to reduce pollutant emissions because expected social damages increase with the level of uncertainty. Thus, there is no simple "irreversibility effect" when more than one type of irreversibility is present. Moreover, to properly account for the impact of uncertainty, we have to consider not only the level and the nature, but also the specification of uncertainty. This is illustrated by a comparison between our two stochastic models, and by a comparison between our results and Pindyck's (1994).

Our model is obviously too simple to capture the salient features of the global warming problem. For example, it considers only fixed emission levels, and it ignores technological change as well as uncertainty in the valuation of damages from global warming. It does nonetheless indicate that global warming scientists should concentrate on modeling uncertainty accurately, and that high levels of environmental uncertainty probably warrant early action to reduce the build-up of greenhouse gases.

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APPENDIX A

In Appendix A1, we derive the moment generating functions of the processes considered and in Appendix A2, we show how to derive the option terms for both models.

A.1 Expected social damages.

We first summarize the method used, and then provide the main intermediate results. For more details, see Cox and Miller (1965). Let X(t) be a diffusion process which verifies:

(A1) $$dX = a(X,t)dt + b(X,t)dz$$

where dz is an increment of a standard Wiener process. The moment generating function of X(t), noted M(θ,t), is defined by:

(A2) $$M(\theta,t) = \mathcal{E}(e^{-\theta x}) = \int_{-\infty}^{+\infty} \phi(x_0, t_0; x, t) e^{-\theta x} dx$$

$\phi(x_0, t_0; x, t)$ is the probability density function for x at t, given $x(t_0) = x_0$. Then:

(A3) $$\frac{\partial M(\theta,t)}{\partial t} = \int_{-\infty}^{+\infty} \frac{\partial \phi(x_0, t_0; x, t)}{\partial t} e^{-\theta x} dx$$

To derive M(θ,t), we insert the left-hand side of the Kolmogorov forward equation:

(A4) $$\frac{1}{2}\frac{\partial^2}{\partial x^2}(b^2(x,t)\phi(x_0,t_0;x,t)) - \frac{\partial}{\partial x}(a(x,t)\phi(x_0,t_0;x,t)) = \frac{\partial}{\partial t}\phi(x_0,t_0;x,t)$$

into (A3), integrate by parts, and solve the resulting partial differential subject to the boundary conditions:

(A5) $$M(0,t) = 1,\ \frac{\partial M(0,0)}{\partial \theta} = -x_0,\ \frac{\partial^2 M(0,0)}{\partial \theta^2} = x_0^2$$

- Model 1: $dX = (E - \alpha X)dt + \sqrt{vX}dz$

For this process, the Kolmogorov forward equation is:

(A6) $$\frac{\partial\phi}{\partial t}=\frac{vx}{2}\frac{\partial^2\phi}{\partial x^2}+(v+\alpha x-E)\frac{\partial\phi}{\partial x}+\alpha\phi$$

After substituting (A6) into (A3), we integrate to obtain:

(A7) $$\frac{\partial M}{\partial t}=-\theta(\alpha+\frac{v}{2}\theta)\frac{\partial M}{\partial\theta}-E\theta M$$

Solving this partial differential equation subject to the boundary conditions (A5), we find:

(A8) $$M(\theta,t)=\left(1+\frac{v\theta}{2\alpha}\right)^{\frac{-2E}{v}}\left[1+C_1\frac{2\theta e^{-\alpha t}}{2\alpha+\theta v}+C_2\left(\frac{2\theta e^{-\alpha t}}{2\alpha+\theta v}\right)^2\right]$$

with $C_1=E-\alpha x_0$, $C_2=\frac{1}{2}\left(\left(E-\alpha x_0+\frac{v}{2}\right)^2-\frac{v}{2}\left(E+\frac{v}{2}\right)\right)$

- Model 2: $dX=(E-\alpha X)dt+\sqrt{vX^2}dz$

For this process, the Kolmogorov forward equation is:

(A9) $$\frac{\partial\phi}{\partial t}=\frac{vx^2}{2}\frac{\partial^2\phi}{\partial x^2}+(2vx+\alpha x-E)\frac{\partial\phi}{\partial x}+(v+\alpha)\phi$$

After substituting (A9) into (A3), we integrate to obtain:

(A10) $$\frac{\partial M}{\partial t}=\frac{v}{2}\theta^2\frac{\partial^2 M}{\partial\theta^2}-\theta\alpha\frac{\partial M}{\partial\theta}-E\theta M$$

The solution to this partial differential equation with boundary conditions (A5) is:

(A11) $$M(\theta,t)=\sum_{k=0}^{+\infty}\frac{\left(\frac{2E\theta}{v}\right)^k}{k!\left(-\frac{2\alpha}{v}\right)_k}+B_1e^{-\alpha t}\frac{v}{2E}\sum_{k=0}^{+\infty}\frac{\left(\frac{2E\theta}{v}\right)^{k+1}}{k!\left(2-\frac{2\alpha}{v}\right)_k}+B_2e^{(v-2\alpha)t}\left(\frac{v}{2E}\right)^2\sum_{k=0}^{+\infty}\frac{\left(\frac{2E\theta}{v}\right)^{k+2}}{k!\left(4-\frac{2\alpha}{v}\right)_k}$$

with $B_1=\frac{E}{\alpha}-x_0$, $B_2=\frac{1}{2}\left(x_0^2+\left(\frac{E}{\alpha}-x_0\right)\frac{2E}{\alpha-v}-\frac{2E^2}{\alpha(2\alpha-v)}\right)$

Let us now look at the restrictions on v for $P^{II}_{i,m}(x;v)$ to be finite. From Equation (15), it is infinite if $v = 2\dfrac{r+k\alpha}{k(k-1)}$, $2 \le k \le m$. Since these ratios are decreasing in k, v has to be smaller than $2\dfrac{r+m\alpha}{m(m-1)}$. Thus, the larger is m (the larger are pollution damages or the social planner's risk aversion), the smaller is the range of v for which the expected social damages from pollution are finite.

A.2 Option terms.

- Model 1. The change of variables $Y = \dfrac{2\alpha X}{v}$ and W(Y) = V(X), leads to:

(A12) $$Y\frac{d^2W}{dY^2} + (\frac{2E_1}{v} - Y)\frac{dW}{dY} - \frac{r}{\alpha}W = 0$$

This is Kummer's Equation (Lebedev (1972)). A general solution to this second order ordinary differential equation can be written:

(A13) $$W(Y) = A_0\Phi(\frac{r}{\alpha}, \frac{2E_1}{v}; Y) + B_0 Y^{1-\frac{2E_1}{v}}\Phi(1+\frac{r}{\alpha} - \frac{2E_1}{v}, 2 - \frac{2E_1}{v}; Y)$$

If B_0 were non-zero, the second term on the right-hand side of Equation (A13) would cause $\dfrac{\partial\tilde{\varphi}(0;v)}{\partial x}$ to be infinite. We thus set B_0 to 0 and obtain Equation (16).

- Model 2. The change of variables $Y = \dfrac{2E_1}{vX}$ and $\left(\dfrac{E_1}{X}\right)^{\beta} W\left(\dfrac{2E_1}{vX}\right) = V(X)$ leads again to Kummer's equation, but this time we write its solution as follows:

(A14) $$V(X;v) = A_0\left(\frac{E_1}{X}\right)^{\beta}\Phi(\beta, \xi; \frac{2E_1}{vX}) + B_0\left(\frac{E_1}{X}\right)^{\beta}\Psi(\beta, \xi; \frac{2E_1}{vX})$$

The option term should also be well defined at X = 0, and increasing in X. Since $X^{-\beta}\Phi(\beta,\xi,\frac{2E_1}{vX}) = \frac{\Gamma(\xi)}{\Gamma(\beta)} e^{\frac{2E_1}{vX} - \xi Ln(X)}[1+o(X)]$, the first term on the right-hand side of Equation (A14) grows unbounded when $X \to 0+$. From (9.12.1) in Lebedev (1972), $\Psi(a,b,z) \approx z^{-a}$ when $z \to +\infty$, so when $X \to 0+$ the second term on the right-hand side is equivalent to $B_0\left(\frac{2}{v}\right)^{-\beta}$. The value of the second term on the right-hand side can thus be defined by continuity at X = 0. Setting A_0 to zero yields Equation (17).

APPENDIX B

In B1 and B2 below, we respectively outline the derivation of approximate expressions for the left and right hand-sides of the stopping rules given by Equations (20) and (21) when v increases from 0+. We combine these results in B3 to obtain a first-order expansion of x*(v) when $v \approx 0+$.

B.1 First-order expansion in v of the left hand-side (LHS) of the stopping rule.

In the following, we show that:

$$\text{(A15)} \quad LHS(x^*;v) = \begin{cases} \dfrac{E_1 - \alpha\tilde{x}_0^*}{r} + \left(\dfrac{r+\alpha}{2r}\dfrac{\tilde{x}_0^{*\rho}}{(E_1-\alpha\tilde{x}_0^*)} - \dfrac{\alpha}{r}\dfrac{dx^*(0)}{dv}\right)v + o(v), \text{if } \tilde{x}_0^* < \dfrac{E_1}{\alpha} \\ \dfrac{\tilde{x}_0^{*\rho}\, v}{2(\alpha\tilde{x}_0^* - E_1)} + o(v), \text{if } \tilde{x}_0^* > \dfrac{E_1}{\alpha} \end{cases}$$

where ρ = 1 for Model 1, and ρ = 2 for Model 2.

- Model 1. When $v \to 0$, we have the formal convergence, for a non-negative integer k:

(A16) $$\Phi(k+\frac{r}{\alpha};k+\frac{2E_1}{v};\frac{2\alpha x^*}{v}) \to S(\tilde{x}_0^*) \equiv \sum_{n=0}^{+\infty} \frac{\left(k+\frac{r}{\alpha}\right)_n}{n!}\left(\frac{\alpha\tilde{x}_0^*}{E_1}\right)^n$$

From Lebedev (p. 275), the last series on the right side of Equation (A16) converges towards $\left(1-\frac{\alpha\tilde{x}_0^*}{E_1}\right)^{-\frac{r}{\alpha}-k}$ provided $\frac{\alpha\tilde{x}_0^*}{E_1}<1$ $\left(\text{i.e. } \tilde{x}_0^* < \frac{E_1}{\alpha}\right)$, where $\tilde{x}_0^* = \lim_{v\to 0} x^*(v)$. When this condition does not hold, $S(\tilde{x}_0^*) = +\infty$. We thus need to distinguish between two cases.

When $\tilde{x}_0^* < \frac{E_1}{\alpha}$, we use the recurrence relations on page 262 of Lebedev to derive:

(A17) $$\frac{\Phi(a,b,bz)}{\Phi(a+1,b+1,bz)} = 1 - z + \frac{(a+1)z}{(b+1)\frac{\Phi(a+1,b+1;bz)}{\Phi(a+2,b+2;bz)}}$$

We substitute $a=\frac{r}{\alpha}, b=\frac{2E_1}{v}, z=\frac{\alpha x}{E_1}$ in the above and simplify to find the first part of Equation (A15).

When $\tilde{x}_0^* > \frac{E_1}{\alpha}$, we extend (9.12.8) in Lebedev to derive, for k non-negative integer:

(A18) $$\Phi(k+\frac{r}{\alpha},k+\frac{2E_1}{v};\frac{2\alpha x^*}{v}) = \frac{\Gamma(k+\frac{2E_1}{v})}{\Gamma(k+\frac{r}{\alpha})} e^{\frac{2\alpha x^*}{v}} \left(\frac{2\alpha x^*}{v}\right)^{\frac{r}{\alpha}-\frac{2E_1}{v}} \left\{\left(1-\frac{E_1}{\alpha x^*}\right)^{\frac{r}{\alpha}+k-1} + o(1)\right\}$$

Using this result for k = 0 and 1 gives the second part of Equation (A15).

- Model 2. It is convenient to proceed slightly differently. First, we consider a sequence $(v_n)_{n\geq 1}$, defined by: $1+\beta_n - \xi_n = -n$, so that:

(A19) $$v_n = 2\frac{r+n\alpha}{n(n-1)}$$

We add the subscript "n" to indicate that we consider this sequence in the definition of various parameters. From (9.12.3) p. 270 in Lebedev, this sequence greatly simplifies the calculation of $\Psi(\beta_n, \xi_n, z_n)$, where $z_n = \frac{2E_1}{xv_n}$, since:

$$\text{(A20)} \quad \Psi(\beta_n, \xi_n, z_n) = z_n^{-\beta_n} \sum_{k=0}^{n} (-1)^n \frac{(\beta_n)_k (-n)_k}{k!} z_n^{-k}$$

We use this relationship for our calculations in the numerical illustration. Then, using recurrence relations (9.10.13) and (9.10.14) on p. 266 in Lebedev, we find:

$$\text{(A21)} \quad R_n = -\beta_n + \frac{\frac{\xi_n - \beta_n}{z_n} \tilde{R}_n}{1 - \frac{\tilde{R}_n}{z_n}}$$

where $R_n = \frac{\Psi(\beta_n, \xi_n, z_n)}{\Psi(1+\beta_n, 1+\xi_n, z_n)}$ and $\tilde{R}_n = \frac{\Psi(\beta_n, 1+\xi_n, z_n)}{\Psi(1+\beta_n, 2+\xi_n, z_n)}$. As $n \to +\infty$, $R_n \approx \tilde{R}_n$, so Equation (A21) allows us to calculate $R \equiv \lim_{n \to +\infty} R_n$. Since R_n is non-negative, either $0 \le R < +\infty$ or $R = +\infty$. First, if $0 \le R < +\infty$, taking limits in Equation (A21) gives:

$$\text{(A22)} \quad R = \frac{r}{\alpha} \frac{E_1}{\alpha x - E_1}$$

Since R is non-negative, we cannot obtain this result if $x < \frac{E_1}{\alpha}$. Next, if $R = +\infty$, we rewrite Equation (A21) as $\frac{\tilde{R}_n}{z_n} = 1 - \frac{\frac{\xi_n - \beta_n}{z_n} \frac{\tilde{R}_n}{R_n}}{1 + \frac{\beta_n}{R_n}}$ and take the limit as $n \to +\infty$. We find:

$$\text{(A23)} \quad \lim_{n \to +\infty} \frac{R_n}{z_n} = \lim_{n \to +\infty} \frac{\tilde{R}_n}{z_n} = \frac{E_1 - \alpha x}{E_1}$$

Since R is non-negative, the above result is impossible if $x > \frac{E_1}{\alpha}$. Hence, if $x < \frac{E_1}{\alpha}$ Equation (A23) holds, whereas if $x > \frac{E_1}{\alpha}$, Equation (A22) holds. These results give us a first approximation of the left hand-side of the stopping rule for this case. When $x < \frac{E_1}{\alpha}$, this is just a constant, however, and we want the term of order v. We thus use recurrence relationships (9.10.15) and (9.10.16) p. 266 in Lebedev to derive:

$$\text{(A24)}\quad \frac{\Psi(a,b,y)}{y\Psi(a+1,b+1,y)} = 1 - \frac{b}{y} + \frac{a+1}{\frac{\Psi(a+1,b+1;y)}{\Psi(a+2,b+2;y)}}$$

We iterate this relationship once, substitute β for a, ξ for b, and $\frac{2E_1}{vx}$ for y, and simplify. We introduce the result in the left-hand side of Equation (21) and use:

$$\text{(A25)}\quad \beta = \frac{r}{\alpha}\left(1 - \frac{\alpha + r}{2\alpha^2} v + o(v)\right)$$

to get Equation (A15) for the case $x < \frac{E_1}{\alpha}$.

B.2 First-order expansion in v of the right hand-side (RHS) of the stopping rules.

This simply requires a fair amount of tedious algebra. The important relationship for Models 1 and 2 are respectively:

$$\text{(A26)}\quad \prod_{j=k}^{m-1}(E + j\frac{v}{2}) = E^{m-k}(1 + \frac{v}{2E}\frac{(m-k)(m+k-1)}{2} + o(v))$$

$$\text{(A27)}\quad \prod_{j=k}^{m}(r + j\alpha - \frac{v}{2}j(j-1)) = \frac{\prod_{j=k}^{m}(r + j\alpha)}{1 + \frac{v}{2}\sum_{j=k}^{m}\frac{j(j-1)}{r + j\alpha}} + o(v)$$

We find:

$$\text{(A28)}\quad \text{RHS}(x^*;v)=\frac{\tilde{P}}{\tilde{P}_x}+\left[\frac{\tilde{P}_x\tilde{P}_x-\tilde{P}\tilde{P}_{xx}}{(\tilde{P}_x)^2}\frac{dx(0)}{dv}+\frac{\tilde{P}_v\tilde{P}_x-\tilde{P}\tilde{P}_{xv}}{(\tilde{P}_x)^2}\right]v+o(v)$$

where $\tilde{P}$ is an abbreviation for $P(\tilde{x}_0^*;0)\equiv P_2(\tilde{x}_0^*;0)-P_1(\tilde{x}_0^*;0)-K$. It represents the net benefit of reducing emissions at $\tilde{x}_0^*$ under certainty; and $\tilde{P}_x$, $\tilde{P}_v$, $\tilde{P}_{xx}$ and $\tilde{P}_{xv}$ are respectively the first derivatives with respect to x and v, and the second derivatives with respect to x twice and x and v, of P(x;v) evaluated at ($\tilde{x}_0^*$,0). P_i is given by Equation (14) for Model 1 and by Equation (15) for Model 2. The superscripts "I" and "II" are omitted to lighten our notation.

B.3 Expressions of $\tilde{x}_0^*$ and $\frac{dx^*}{dv}$ when $v \approx 0+$.

First, we derive the equation verified by $\tilde{x}_0^*$ by equating the terms of order 0 in Equations (A15) and (A28). Hence, if $\tilde{x}_0^* < \frac{E_1}{\alpha}$, it verifies $\frac{E_1-\alpha\tilde{x}_0^*}{r}=\frac{\tilde{P}}{\tilde{P}_x}$, which is Equation (13). On the other hand, if $\tilde{x}_0^* > \frac{E_1}{\alpha}$, then $\tilde{x}_0^*$ verifies $\tilde{P}=0$. P(x;0) is increasing in x so it has at most one root. This proves the first part of Proposition 2.

For $\frac{dx^*}{dv}$, we combine Equations (A15) and (A28) to find:

$$\text{(A29)}\quad \frac{dx^*(0)}{dv}=\begin{cases}\dfrac{\dfrac{r+\alpha}{2r}\dfrac{x_0^{*\rho}}{(E_1-\alpha x_0^*)}-\dfrac{\tilde{P}_v\tilde{P}_x-\tilde{P}\tilde{P}_{xv}}{(\tilde{P}_x)^2}}{\dfrac{\alpha}{r}+\dfrac{\tilde{P}_x\tilde{P}_x-\tilde{P}\tilde{P}_{xx}}{(\tilde{P}_x)^2}}, & \text{when } \tilde{x}_0^*<\dfrac{E_1}{\alpha}\\[2ex] \dfrac{\tilde{x}_0^{*\rho}}{2(\alpha\tilde{x}_0^*-E_1)}-\dfrac{\tilde{P}_v}{\tilde{P}_x}, & \text{when } \tilde{x}_0^*>\dfrac{E_1}{\alpha}\end{cases}$$

where ρ = 1 for Model 1 and ρ = 2 for Model 2. The notation is defined in B.2.

TABLE I: x*(v) and corresponding option value for "small" v.

		α=0.02		α=0.03		α=0.04	
	v	x*	φ*/K	x*	φ*/K	x*	φ*/K
m=2, K=7500, E_2=0.7; Model 1							
r=0.02	0.0000	-2.50	- -	6.00	82.0%	13.33	31.1%
	0.0025	- -	- -	5.98	82.1%	13.35	31.2%
	0.0050	- -	- -	5.97	82.2%	13.36	31.3%
r=0.04	0.0000	28.33	18.1%	45.00	0.0%	77.50	0.0%
	0.0025	28.39	18.2%	45.13	0.2%	77.52	0.0%
	0.0050	28.45	18.4%	45.25	0.4%	77.53	0.0%
m=2, K=7500, E_2=0.7; Model 2							
r=0.02	0.0000	-2.50	- -	6.00	82.0%	13.33	31.1%
	0.0025	- -	- -	5.43	86.6%	13.10	34.2%
	0.0050	- -	- -	4.86	91.3%	12.73	37.4%
r=0.04	0.0000	28.33	18.1%	45.00	0.0%	77.50	0.0%
	0.0025	28.77	22.6%	47.61	5.7%	78.30	2.8%
	0.0050	28.62	26.0%	48.44	9.5%	78.66	5.3%
m=3, K=175000, E_2=0.7; Model 1							
r=0.02	0.0000	-5.43	- -	7.35	188.4%	13.739	69.6%
	0.0025	- -	- -	7.31	189.0%	13.739	70.1%
	0.0050	- -	- -	7.27	189.6%	13.738	70.5%
r=0.04	0.0000	16.75	96.4%	23.33	28.3%	31.82	0.0%
	0.0025	16.74	96.8%	23.40	28.8%	31.90	0.4%
	0.0050	16.73	97.2%	23.45	29.3%	31.97	0.8%
m=3, K=175000, E_2=0.7; Model 2							
r=0.02	0.0000	-5.43	- -	7.35	188.4%	13.74	69.6%
	0.0025	- -	- -	6.12	212.4%	13.19	82.2%
	0.0050	- -	- -	4.80	238.9%	12.40	96.2%
r=0.04	0.0000	16.75	96.4%	23.33	28.3%	31.82	0.0%
	0.0025	15.69	111.9%	23.60	40.8%	32.89	10.4%
	0.0050	14.47	127.2%	23.10	51.9%	32.88	18.5%

TABLE II: Value of v such that x*=0

	α	Model 1		Model 2	
		m=2	m=3	m=2	m=3
r=0.02	0.02	- -	- -	- -	- -
	0.03	0.600	0.304	0.024	0.012
	0.04	1.600	0.791	0.053	0.027
r=0.03	0.02	1.225	0.350	0.033	0.011
	0.03	2.650	0.925	0.059	0.026
	0.04	4.375	1.569	0.083	0.039
r=0.04	0.02	3.400	0.914	0.057	0.021
	0.03	5.600	1.633	0.080	0.034
	0.04	8.200	2.424	0.102	0.046

Note: K = 7500 for m = 2, and K =175 000 for m = 3. For both values of m, E_2=0.7.

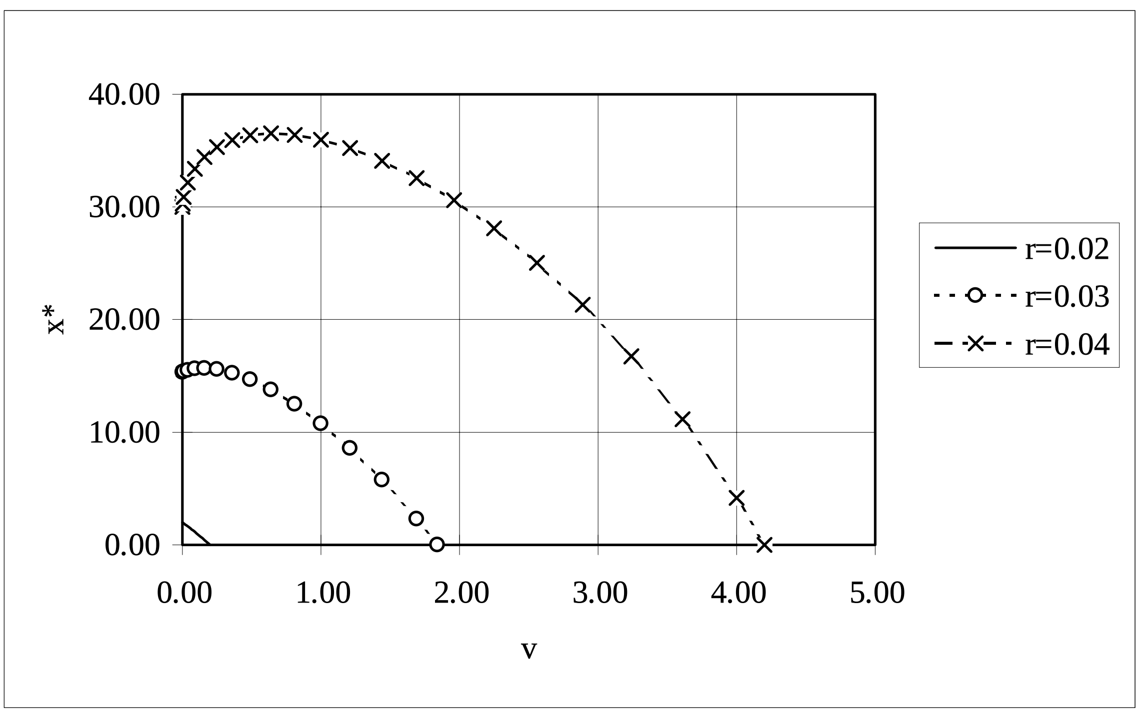

FIGURE I: x* vs. v for Model I with E_1=1, E_2=0.7, K=6000, and α=0.03

[1] See Dixit and Pindyck (1994) for an introduction to the theory of real options, and Trigeorgis (1995) for examples of applications.

[2] Model I is the square root process used by Cox, Ingersoll, and Ross (1985).

[3] We include 0 in the arguments of P_i for uniformity of notation with the stochastic models. See Section IV.

[4] When m = 2, we find that $x_0^* = \frac{r(2\alpha + r)K}{2(E_1 - E_2)} - \frac{E_2}{\alpha + r}$ when $x_0^* < \frac{E_1}{\alpha}$. Thus, x_0^* increases when r, α, or K increase, but the variations of x_0^* with E_1 or E_2 depend on K. In general, however, it is not possible to find closed-form expressions for x_0^*.

[5] We can also find this result by solving the maximization problem directly. Hence, for m = 2, the second derivative with respect to T of the objective function at x_0^* is $\frac{2(E_1 - E_2)}{2\alpha + r} e^{-(r+\alpha)T}(E_1 - \alpha X(0))$. This expression is non-negative only when $X(0) \leq \frac{E_1}{\alpha}$. From Equation (10), if $X(0) > \frac{E_1}{\alpha}$, X(t) remains above $\frac{E_1}{\alpha}$ for all t, which shows the result.

[6] They are defined by: $\Phi(a, b; z) = \sum_{k=0}^{+\infty} \frac{(a)_k}{(b)_k} \frac{z^k}{k!}$, where $(a)_0=1$, $(a)_k=a\cdot(a+1)\cdots(a+k-1)$, and $\Psi(a, b; z) = \frac{\Gamma(1-b)}{\Gamma(1+a-b)} \Phi(a, b; z) + \frac{\Gamma(b-1)}{\Gamma(a)} z^{1-b} \Phi(1 + a - b, 2 - b; z)$. $\Psi(a,b;z)$ is well defined for all values of a and b, including negative integer values of b, and it is bounded with respect to z (Lebedev (1972)). $\Gamma(x) \equiv \int_0^{+\infty} e^{-t} t^{x-1} dt, \text{ if } \operatorname{Re}(x) > 0$ is the gamma function.