A Recombining Binomial Tree for Valuing Real Options With Complex Structures

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1 Introduction

In this paper, we allow the (risk-neutral) drift and the volatility of the underlying asset vary as an arbitrary (continuous) function of the underlying asset value and time. We develop a lattice technique for valuing derivative assets (including real options) that have values dependent on the underlying asset. The prior literature shows how to do this for specific functional forms for the drift and volatility, such as for a lognormal diffusion. These papers do this by transforming the diffusion to a simple diffusion, such as an additive normal diffusion and generating a lattice with constant parameters.

We allow the lattice jumps to vary to accommodate the arbitrary changes in the volatility. We accomplish this by defining a lattice and probabilities on the lattice that achieve the correct volatility in the limit as the step size vanishes even though the drift may be incorrect in the limit. Then we consider a change of measure to correct the drift, in the spirit of a discrete-time version of Girsanov's theorem. The revised process converges to the desired underlying process.

2 Asset Price Processes and Convergence from Discrete- to Continuous-Time Processes

Consider a general stochastic differential equation:

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$$dS_t = \mu(S,t)\,dt + \sigma(S,t)\,dW_t \text{ where} \tag{1}$$

$$S_0 = \text{constant and } t \in [0,T] \tag{2}$$

We allow for two possible interpretations of this process for the underlying asset or state variable. First, it could represent the true stochastic process for a state variable, and we are merely interested in approximating this with a discrete process. Second, it could represent the "equivalent martingale process" for the asset that is used for pricing the asset and derivatives of the asset by discounting expected payoffs. This is sometimes called a risk-neutral process. In this case, the drift $\mu(S,t)$ can be equivalently represented as true drift minus a systematic risk premium, or as the return on the riskless asset r minus any convenience yield (netted for the cost of carry) δ. In this interpretation, the discrete process can be used to price and assess derivatives on the underlying asset. For example, under this interpretation, we can assess real options to acquire a commodity by undertaking a production process. The optimal time to build the production system can be analyzed by backward recursion on a tree of project values that is derived from the risk-neutral process for the underlying commodity.

We need suitable assumptions of smoothness and bounds on the local variation for the parameter functions μ and σ. More precisely, we will assume the following:

Condition 1 *μ is of class C^1 (admits continuous partial derivatives) with respect to S and it has locally bounded variation.*[1]

Condition 2 *σ is a strictly positive function of class C^2 with respect to S and C^1 with respect to time t and it has locally bounded variation.*

These conditions are sufficient to guarantee the existence and uniqueness of weak solutions to (1) and are general enough to be satisfied by the majority of price processes in financial models.

The goal is to obtain a sequence of recombining binomial (lattice) discrete processes that converge in distribution to the solution of (1). Then we can appeal to results of Nelson, Ramaswamy, Amin and Khanna [2], [1] to ensure the convergence of discrete option prices to the continuous values for European and, respectively American style payoffs. The general sufficient conditions for convergence (on the binomial parameters) given by Nelson and Ramaswamy in [2] will be used. First, we define the discrete processes:

[1] For any (ascending) collection of compact sets $\{C_i\}_{i\geq 0}$, (such that $C_i \subset C_{i+1}$ for every i, and $\cup_{i=0}^{\infty} C_i$ covers the spatial domain), and for every $i \geq 0$, there exist $c_i, d_i > 0$ so that $|\mu(S,t) - \mu(S',t)| \leq c_i\,|S - S'|$ and $|\mu(S,t) - \mu(S,t')| \leq d_i\,|t - t'|$ for any $S, S' \in C_i$ and $t, t' \in [0,T]$.

Notation 3 *For every positive integer n, we divide the time interval $[0,T]$ into n discrete time intervals of length $\Delta t = h_n := T/n$. A binomial lattice process will be uniquely determined by the values $S_{i,j}^n$ at the nodes (i,j) of a binomial lattice and by the discrete conditional probabilities $p_{i,j}^n$. That is, the process will transition from a node (i,j) to either $(i+1,j+1)$ or $(i+1,j)$ with probabilities $p_{i,j}^n$ and $1-$ $p_{i,j}^n$, respectively. The first indices of S and p represent the current number of discrete time intervals ($t = i_n$) and the second indices represent the number of up-jumps experienced by S prior the time t. For any i and j, we call a transition from (i,j) to $(i+1,j+1)$ an up-jump, and a transition from (i,j) to $(i+1,j)$ a down-jump.*

The following conditions on the discrete process will ensure that the up- and down-jumps in S decrease at a sufficiently fast rate and that the discrete first and second moments "adequately" approximate the instantaneous continuous ones. The conditions are formulated in the spirit of [2]:

Condition 4 *For any interval $[S_{\min}, S_{\max}]$ in the spatial domain,*

$$\lim_{n\to\infty} \sup_{\substack{S_{\min}\le S_{i,j}^n \le S_{\max} \\ 0\le i\le n-1}} \left|S_{i+1,j+1}^n - S_{i,j}^n\right| = 0$$

and

$$\lim_{n\to\infty} \sup_{\substack{S_{\min}\le S_{i,j}^n \le S_{\max} \\ 0\le i\le n-1}} \left|S_{i+1,j}^n - S_{i,j}^n\right| = 0$$

In other words, the up- and down-jumps $u_{i,j}^n := S_{i+1,j+1}^n - S_{i,j}^n$ and $d_{i,j}^n := S_{i+1,j}^n - S_{i,j}^n$, converge to zero uniformly on compact sets.

Definition 5 *The discrete drift at a node (i,j) is*

$$\mu_{i,j}^n = \left(p_{i,j}^n\left(S_{i+1,j+1}^n - S_{i,j}^n\right) + \left(1-p_{i,j}^n\right)\left(S_{i+1,j}^n - S_{i,j}^n\right)\right)/h_n.$$

The discrete (non-central) second moment at (i,j) is

$$\left(\sigma_{i,j}^n\right)^2 = \left(p_{i,j}^n\left(S_{i+1,j+1}^n - S_{i,j}^n\right)^2 + \left(1-p_{i,j}^n\right)\left(S_{i+1,j}^n - S_{i,j}^n\right)^2\right)/h_n.$$

The second moment of the distribution and the variance of the distribution differ by the square of the mean, which is $O(h_n^2)$, so that the difference between the two becomes negligible in the limit as $n\to\infty$.

Condition 6 *For any interval $[S_{\min}, S_{\max}]$,*

$$\lim_{n\to\infty} \sup_{\substack{S_{\min}\le S_{i,j}^n \le S_{\max} \\ 0\le i\le n-1}} \left|\mu\left(S_{i,j}^n, i_n\right) - \mu_{i,j}^n\right| = 0$$

In other words, the discrete drift μ^n approaches μ as $n\to\infty$.

Condition 7 *For any interval* $[S_{\min}, S_{\max}]$,

$$\lim_{n\to\infty} \sup_{\substack{S_{\min}\le S_{i,j}^n \le S_{\max} \\ 0\le i\le n-1}} \left|\sigma\left(S_{i,j}^n, i_n\right) - \sigma_{i,j}^n\right| = 0$$

The following theorem [2] gives the desired convergence property.

Theorem 8 *If conditions 1, 2, 4, 6 and 7 are satisfied, then the discrete process* S_i^n *converges weakly in distribution to the solution of (1).*

3 Construction of The Binomial Lattice

3.1 The Basic Concept

The idea is to produce a discrete process that matches the second moment of the continuous diffusion and has the first (discrete) moment altered with the purpose of making recombination possible.[2] Once this is achieved, an appropriate change in the discrete probabilities will force a match of the correct first moment without affecting the second moment in the limit as $h_n \to 0$. We will compute the discrete lattice parameters $S_{i,j}^n$ for the altered process and keep the same lattice for the corrected process. Denote the probability of an up-move at node (i,j) by $q_{i,j}^n$. The drifts $\mu_{i,j}^{*n}$ of the altered process are determined recursively backwards in time — this is back-folding for computing the underlying tree itself!

At the end of the procedure, the discrete computed initial value $S_{0,0}^n$ may not equal the true initial value S_0. We will then modify[3] the start-up values on the boundary $\{t = T\}$ and repeat the procedure until $S_{0,0}^n$ converges to S_0.

This is useful if we desire the value of an option on the underlying asset when the underlying asset price at time $t = 0$ is S_0. If we desire an array of option prices for various underlying asset prices at time 0, we might not need to be so precise at ensuring the tree goes through S_0 and instead start the tree a few periods prior to time 0 so that the tree has several prices at time 0.

3.2 Recursive Computation of The Lattice

For the first iteration of the procedure and every $0 \le j \le i \le n$, the computation of the lattice points $S_{i,j}^n$, the conditional probabilities and the drifts of the altered process $q_{i,j}^n$ and $\mu_{i,j}^{*n}$ are computed recursively. The correct probabilities $p_{i,j}^n$ are then calculated from the resulting lattice to approximate the desired drift of the actual process (which may be the true or risk-neutral drift). We proceed recursively as follows:

[2] Note that recombination is already implicitly assumed by the notation — indexing the nodes of the tree by the number of up-jumps implies that the values of the discrete process do not depend on the order in which these jumps ocurred.

[3] Several methods can be used here – we have experimented with a tangent (Newton type) method that hits the correct initial value in only one and two iterations for normal and, respectively lognormal difussions.

3.2.1 The Boundary:

As the recursive computation requires knowledge of the discrete parameters at the next time moment, we start the process by assigning the parameters at time n.

Let $j' = \left[\frac{n+1}{2}\right]$, where $[x]$ denotes the largest integer less than or equal to x and set

$$S^n_{n,j'} = S_0. \tag{3}$$

Next, compute the asset values $S^n_{n,j}$ at the last moment in time in such a way that the differences between consecutive values equal twice the local standard deviation, as given by the continuous process. This is chosen so that the discrete process has the correct second moment in the limit. Set:

$$S^n_{n,j+1} - S^n_{n,j} = 2 \cdot \sigma \left(\frac{1}{2} \left(S^n_{n,j} + S^n_{n,j+1} \right), n h_n \right) \cdot \sqrt{h_n} \text{ for } 0 \leq j \leq n-1. \tag{4}$$

The values of the probabilities and the drifts at the last time moment are not relevant.

3.2.2 The recursive step

Assume everything is computed for all $l > i$.

1. We first define the time-i transformed drifts $\mu^{*n}_{i,j}$ for the number of up-jumps $j \leq i$.

   First, let $j' := \arg\max_j \{S^n_{i+1,j} | S^n_{i+1,j} \leq S_0\}$

   Recursively, define the transformed drifts at time i as follows:

   $$\mu^{*n}_{i,j'} = 0$$

   For $j \geq j'$:

   $$\mu^{*n}_{i,j+1} = \mu^{*n}_{i,j} + \frac{1}{2\sqrt{h_n}} \left(\begin{array}{c} \sigma\left(S^n_{i+2,j+1}, (i+2) h_n\right) \\ +\sigma\left(S^n_{i+2,j+2}, (i+2) h_n\right) \\ -2\sigma\left(S^n_{i+1,j+1}, (i+1) h_n\right) \end{array} \right) \tag{5}$$

   For $j \leq j'$:

   $$\mu^{*n}_{i,j-1} = \mu^{*n}_{i,j} - \frac{1}{2\sqrt{h_n}} \left(\begin{array}{c} \sigma\left(S^n_{i+2,j}, (i+2) h_n\right) \\ +\sigma\left(S^n_{i+2,j+1}, (i+2) h_n\right) \\ -2\sigma\left(S^n_{i+1,j}, (i+1) h_n\right) \end{array} \right) \tag{6}$$

   When $i = n-1$, however, $S^n_{i+2,\cdot}$ are not computed so we will use instead:

2. Compute the node values for the asset at time i:

$$S_{i,j}^n = \frac{1}{2}\left(S_{i+1,j+1}^n + S_{i+1,j}^n\right) - 2h_n \cdot \mu_{i,j}^{*n} \tag{7}$$

3. Compute the node values for the probabilities at time i:

$$q_{i,j}^n = \frac{h_n \cdot \mu_{i,j}^{*n} + S_{i,j}^n - S_{i+1,j}^n}{S_{i+1,j+1}^n - S_{i+1,j}^n} \tag{8}$$

and

$$p_{i,j}^n = \frac{h_n \cdot \left(\mu\left(S_{i,j}^n, ih_n\right) - \mu_{i,j}^{*n}\right) + S_{i,j}^n - S_{i+1,j}^n}{S_{i+1,j+1}^n - S_{i+1,j}^n} \tag{9}$$

Iterate the entire procedure by replacing the starting value S_0 in equation 3 by a new starting value. The goal is to end up with $S_{0,0}^n$ within a specified error away from the correct initial value S_0. We denote by $S_0(k)$ the starting value in equation 3 at the beginning of the kth iteration. Of course, on the first iteration:

$$S_0(1) = S_0.$$

Then,

$$S_0(k) = S_0(k-1) \cdot \frac{S_0}{S_{0,0}(k-1)}$$

Here, $S_{0,0}(k-1)$ denotes the value $S_{0,0}^n$ obtained at the root of the lattice after the $(k-1)$th iteration. Stop the iteration when

$$|S_0 - S_{0,0}(k-1)| \le \varepsilon_n$$

where $\varepsilon_n > 0$ is a pre-determined error margin.

3.3 Proof of Convergence

We prove that conditions 4, 6 and 7 are satisfied and therefore the discrete process converges to the continuous diffusion as $h_n \to 0$.

The steps of the proof:

- Step1

  Show inductively on i that the following relationship holds for every i, j and n:

$$S_{i,j+1}^n - S_{i,j}^n = 2 \cdot \sigma\left(S_{i+1,j+1}^n, (i+1)h_n\right) \cdot \sqrt{h_n}. \tag{10}$$

For $i = n$, equation 10 is satisfied because of the choice of the boundary values for σ and equation 4. Assume now that equation 10 holds for every $i' > i$. To show it holds for i, we first obtain the terms in the left side from equation 7:

$$S^n_{i,j+1} = \frac{1}{2}\left(S^n_{i+1,j+2} + S^n_{i+1,j+1}\right) - 2h_n \cdot \mu^{*n}_{i,j+1}$$

$$S^n_{i,j} = \frac{1}{2}\left(S^n_{i+1,j+1} + S^n_{i+1,j}\right) - 2h_n \cdot \mu^{*n}_{i,j}$$

The left side of equation 10 becomes

$$S^n_{i,j+1} - S^n_{i,j} = \frac{1}{2}\left(S^n_{i+1,j+2} - S^n_{i+1,j+1}\right) + \frac{1}{2}\left(S^n_{i+1,j+1} - S^n_{i+1,j}\right) \qquad (11)$$

$$- 2h_n\left(\mu^{*n}_{i,j+1} - \mu^{*n}_{i,j}\right) \qquad (12)$$

The expressions $\left(S^n_{i+1,j+2} - S^n_{i+1,j+1}\right)$ and $\left(S^n_{i+1,j+1} - S^n_{i+1,j}\right)$ satisfy the induction hypothesis equation 10 and can be replaced in equation 11. Also, $\mu^{*n}_{i,j+1}$ and $\mu^{*n}_{i,j}$ can be replaced with the expressions given by equations 5 or 6 (depending on j). Simple algebra turns equation 10 into an identity and the proof of the inductive step is completed.

- Step 2

Assume without loss of generality that $j \geq j^{'}$ and note that equation 5 can be written as follows:
$\mu^{*n}_{i,j+1} - \mu^{*n}_{i,j} =$

$$\frac{1}{2\sqrt{h_n}}\left(\begin{array}{c} 2\sigma\left(S^n_{i+1,j+1},(i+2)\,h_n\right) - 2\sigma\left(S^n_{i+1,j+1},(i+1)\,h_n\right) \\ +2\sigma\left(\frac{S^n_{i+2,j+1}+S^n_{i+2,j+2}}{2},(i+2)\,h_n\right) - 2\sigma\left(S^n_{i+1,j+1},(i+2)\,h_n\right) \\ +\sigma\left(S^n_{i+2,j+1},(i+2)\,h_n\right) + \sigma\left(S^n_{i+2,j+2},(i+2)\,h_n\right) \\ -2\sigma\left(\frac{S^n_{i+2,j+!}+S^n_{i+2,j+2}}{2},(i+2)\,h_n\right) \end{array}\right) \qquad (13)$$

Using the notation $o\left(E_n\right)$ for an expression that is negligible with respect to E_n when $n \to \infty$, we have a limiting function $v = \mu^*$ where:

$$\mu^{*n}_{i,j+1} - \mu^{*n}_{i,j} = \partial_S\mu \cdot 2\sigma\sqrt{h_n} + o\left(\sqrt{h_n}\right),$$

$$2\sigma\left(S^n_{i+1,j+1},(i+2)\,h_n\right) - 2\sigma\left(S^n_{i+1,j+1},(i+1)\,h_n\right) = 2\cdot\partial_t\sigma\cdot h_n + o\left(h_n\right),$$

$$2\sigma\left(\frac{S^n_{i+2,j+1}+S^n_{i+2,j+2}}{2},(i+2)\,h_n\right) - 2\sigma\left(S^n_{i+1,j+1},(i+2)\,h_n\right)$$
$$= 2\cdot\partial_S\sigma\cdot 2h_n\mu^* + o\left(h_n\right)$$

and

$$\begin{aligned}
&\sigma\left(S^n_{i+2,j+1},(i+2)\,h_n\right)+\sigma\left(S^n_{i+2,j+2},(i+2)\,h_n\right)\\
&-2\sigma\left(\frac{S^n_{i+2,j+1}+S^n_{i+2,j+2}}{2},(i+2)\,h_n\right)\\
&=\partial_{SS}\sigma\cdot\left(\sigma\sqrt{h_n}\right)^2+o\left(h_n\right)=\\
&=\partial_{SS}\sigma\cdot\sigma^2 h_n+o\left(h_n\right)
\end{aligned}$$

Multiplying through equation 13 by $\frac{1}{2\sqrt{h_n}}$, followed by a re-arrangement of the terms, leads to

$$\sigma\cdot\partial_S\mu^*=\frac{1}{2}\mu^*\cdot\partial_t\sigma+\mu^*\cdot\partial_S\sigma+\frac{1}{4}\sigma^2\cdot\partial_{SS}\sigma+o\left(1\right)$$

where $o\left(1\right)$ is a remainder that goes to zero when $n\to\infty$. Therefore, the computed discrete modified drift μ^* converges to the solution v of the differential equation 14.

Let

$$v\left(S,t\right)=\frac{1}{2}\sigma\left(S,t\right)\cdot\int_{S_0}^{S}\left(\frac{\partial_t\sigma\left(s,t\right)}{\sigma^2\left(s,t\right)}+\frac{1}{2}\partial_{SS}\sigma\left(s,t\right)\right)ds$$

be the solution to the differential equation

$$\sigma\cdot\partial_S v-v\cdot\partial_S\sigma-\frac{1}{2}\partial_t\sigma-\frac{1}{4}\cdot\sigma^2\cdot\partial_{SS}\sigma=0 \tag{14}$$

with boundary condition

$$v\left(S_0,t\right)=0\text{ for }t\in\left[0,T\right].$$

Then, the following equation (similar to the one given by Condition 6) holds:

$$\lim_{n\to\infty}\sup_{\substack{S_{\min}\le S^n_{i,j}\le S_{\max}\\ 0\le i\le n-1}}\left|v\left(S^n_{i,j},ih_n\right)-\mu^{*n}_{i,j}\right|=0 \tag{15}$$

It can be derived from the fact that the recursive definition of $\mu^{*n}_{i,j}$ is a valid finite-difference discretization of v in the above differential equation. Also, note that v satisfies Condition 1.

- Step 3

  Show that the discrete volatilities implicitly defined by

$$\left(\sigma^n_{i,j}\right)^2=\left(q^n_{i,j}\left(S^n_{i+1,j+1}-S^n_{i,j}\right)^2+\left(1-q^n_{i,j}\right)\left(S^n_{i+1,j}-S^n_{i,j}\right)^2\right)/h_n$$

  where

$$q^n_{i,j}=\frac{h_n\cdot\mu^{*n}_{i,j}+S^n_{i,j}-S^n_{i+1,j}}{S^n_{i+1,j+1}-S^n_{i+1,j}}$$

satisfy Condition 7. This is “straightforward” algebra as long as

$$0 \leq q_{i,j}^{n} \leq 1.$$

- Step 4

  Using the theorem in the previous section, conclude that the sequence of discrete processes given by $\left(S_{i,j}^{n}, q_{i,j}^{n}\right)$ is converging to the solution of

  $$dS_t = v\,(S,t)\,dt + \sigma\,(S,t)\,dW_t.$$

- Step 5

  The discrete processes with corrected probabilities $p_{i,j}^{n}$ converges to the original diffusion as the new discrete second moments

  $$(\sigma'^{n}_{i,j})^2 = \left(p_{i,j}^{n}\left(S_{i+1,j+1}^{n} - S_{i,j}^{n}\right)^2 + \left(1 - p_{i,j}^{n}\right)\left(S_{i+1,j}^{n} - S_{i,j}^{n}\right)^2\right)/h_n$$

  still satisfy Condition 7. This statement can be regarded as a discrete consequence of Girsanov's Theorem, namely that changing the probability measure to match a different drift does not change the variance of the process. In the discrete setting, it is enough to show that $\left(\sigma_{i,j}^{n}\right)^2 - (\sigma'^{n}_{i,j})^2 = o\,(1)$ uniformly on compacts of the form $[S_{\min}, S_{\max}] \times [0,T]$.

4 Numerical Examples

A first example of an application will be to price a European and an American call option written on an underlying S which (in risk-neutralized form) follows a generalized diffusion:

$$dS_t = \alpha\,(L - S)\,dt + a\exp\,(bt)\,dW_t$$

where:

- α is the strength of mean-reversion
- L is a (constant) long term mean
- a and b are parameters that give an exponential growth formulation for the (absolute) volatility

We assume that possession of the underlying incurs a net dividend yield δ at each moment in time and it is expressed as an annualized (constant) proportion of the underlying.

The following values for the input parameters will be used as the starting point:

- Riskless rate $r = 5\%$
- Value of S at time zero $S_0 = 100\$$
- Time to expiry $T = 1$ year
- Strike price $K = 100\$$
- $\alpha = 0.5$
- $L = 100\$$
- $\delta = 10\%$
- $a = 100\$$
- $b = 0.1$
- Number of discrete time steps $n = 200$

The computed values of the European and American call options with the above contract and model specifications are

$$E = 26.758\$$$

and, respectively

$$A = 34.695\$.$$

The following table displays the option values when one parameter at a time is changed as indicated in the left coloumn:

	European	American
Base case	26.758	34.695
$T = 0.5$	22.200	25.291
$T = 2$	30.170	48.368
$K = 90$	31.518	40.325
$K = 110$	22.712	29.686
$\alpha = 0$	32.910	35.864
$\alpha = 1$	22.612	34.239
$L = 80$	23.643	32.181
$L = 120$	30.131	37.410
$a = 50$	11.821	16.090
$a = 150$	41.781	53.366
$b = 0$	25.495	33.046
$b = 0.2$	28.183	36.502
$\delta = -5\%$	31.325	37.652
$\delta = -15\%$	36.592	41.105

References

[1] Amin, K. I., and A. Khanna, 1994, “Convergence of American Option Values from Discrete to Continuous-time Financial Models,” *Mathematical Finance*, 4, 289-304.

[2] D. B. Nelson, and K. Ramaswamy, 1990, “Simple Binomial Processes as Diffusion Approximations in Financial Models,” *The Review of Financial Studies*, Vol 3, number 3, 393-430