Capital expansion and reduction with fixed and proportional costs under demand and irreversibility risk[1]

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1 Overview

This study examines a firm's capital expansion and reduction problem when the demand of output and the degree of irreversibility are stochastic. The firm controls the level of capital stock according to the demand of output, which is assumed to be governed by a geometric Brownian motion. When the firm expands the capital, it incurs capital purchase costs. In contrast, when the firm reduces the capital, the firm can sell the capital at a price lower than the purchase price. These prices are termed as the *proportional cost* in this study. The difference between these prices presents the irreversibility of capital investment. In this study, we consider the degree of irreversibility is governed by a Jacobi diffusion so that the degree moves between 0 and 1. Furthermore, we assume that changing the level of capital requires a fixed cost as well. The *fixed cost* represents the cost associated with investment decision-making, such as research costs. Thus, changing the level of capital requires the fixed and proportional costs. Therefore, the firm's problem is to decide when and how much to change the level of capital under output demand and irreversibility risk. To solve the problem, we formulate it as a stochastic impulse control problem.

2 Methods

2.1 Firm's Problem

The firm's operating profit $\hat{\pi}$ is specified as $\hat{\pi}(K_t, X_t) = K_t^{\alpha} X_t^{1-\alpha}$, where K_t is the capital stock, X_t is the output demand, and $\alpha \in (0,1)$. The firm controls the level of capital according to the output demand, which is governed by the geometric Brownian motion: $\mathrm{d}X_t^x = \mu_X X_t^x \mathrm{d}t + \sigma_X X_t^x \mathrm{d}W_t^X$, $X_{0^-}^x = x > 0$. Let $\zeta_i \in \mathbb{R}$ be the ith amount of change in capital at time τ_i, $i \geq 0$. The process of the capital stock is governed by the following differential equation:

$$\begin{cases} \mathrm{d}K_t^k = -\delta K_t^k \mathrm{d}t, & \tau_i \leq t < \tau_{i+1}, \\ K_{\tau_i}^k = K_{\tau_i^-}^k + \zeta_i > 0, \\ K_{0^-} = k > 0, \end{cases} \tag{2.1}$$

where $\delta \in (0,1)$ is a constant depreciation rate. At time τ_i, the firm can purchase capital at a constant unit price $p > 0$ or sell it at a price $(1 - \Lambda_{\tau_i})p > 0$, where Λ_{τ_i} represents the degree of irreversibility of capital investment. If Λ_t becomes 1, the investment cost is completely sunk, while if Λ_t becomes 0, the firm can sell the capital at the same price as the purchase price.

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Following this economic background, we assume that Λ_t is governed by the Jacobi diffusion: $\mathrm{d}\Lambda_t^\lambda = \eta(\mu_\Lambda - \Lambda_t^\lambda)\mathrm{d}t + \sigma_\Lambda\sqrt{\Lambda_t^\lambda(1-\Lambda_t^\lambda)}\mathrm{d}W_t^\Lambda$, $\Lambda_{0^-}^\lambda = \lambda \in (0,1)$. We assume that $\mathrm{d}W_t^X \mathrm{d}W_t^\Lambda = \rho \mathrm{d}t$, where $\rho \in (-1,1)$. In addition to the purchase/sell price, which is proportional cost, there is a cost associated with decision making when the firm changes the level of capital stock. It refers to as the fixed cost $\hat{c} > 0$. Thus, the capital expansion and reduction costs are given by

$$\hat{C}(\zeta_i, \Lambda_{\tau_i}) = \begin{cases} \hat{c} + p\zeta_i, & \zeta_i > 0, \\ \hat{c}, & \zeta_i = 0, \\ \hat{c} + (1-\Lambda_{\tau_i})p\zeta_i, & \zeta_i < 0. \end{cases} \tag{2.2}$$

The firm's expected discounted net profit $\hat{J}(k,x;\hat{v})$ is given by $\hat{J}(k,x,\lambda;\hat{v}) = \mathbb{E}[\int_0^\infty \mathrm{e}^{-rt}\hat{\pi}(K_t^{k,\hat{v}}, X_t^{x,\hat{v}})\mathrm{d}t - \sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}\hat{C}(\zeta_i, \Lambda_{\tau_i}^{\lambda,\hat{v}})\mathbf{1}_{\{\tau_i<\infty\}}]$, where $\hat{v} := \{(\tau_i,\zeta_i)\}_{i\geq 0}$ is the capital expansion and reduction policy.

Henceforth, we use the change variables as $Y_t := K_t/X_t$ for simplicity. Then, $\hat{\pi}$, ζ_i, $\hat{c}$, and $\hat{J}(k,x,\lambda;\hat{v})$ can be rewritten as follows: $\hat{\pi}(k,x) = k^\alpha x^{1-\alpha} = y^\alpha x = \pi(y)x$; $\zeta_i/x =: \xi_i$; $\hat{c}/x =: c$; $\hat{J}(k,x,\lambda) = x\hat{J}\left(\frac{k}{x},1,\lambda\right) = xJ(y,\lambda)$. Then, the firm's expected discounted net profit is rewritten as follows:

$$J(y,\lambda;v) = \mathbb{E}\left[\int_0^\infty \mathrm{e}^{-rt}\pi(Y_t^{y,v})\mathrm{d}t - \sum_{i=1}^\infty \mathrm{e}^{-r\tau_i}C(\xi_i, \Lambda_{\tau_i}^v)\mathbf{1}_{\{\tau_i<\infty\}}\right], \tag{2.3}$$

where $v := \{(\tau_i,\xi_i)\}_{i\geq 0}$. J is well defined and finite under certain conditions (Cadenillas and Zapatero, 1999).

Therefore, the firm's problem is to choose the capital expansion and reduction policy v over a set of admissible capital expansion and reduction policy $\mathcal{V}$ to maximize the expected discounted net profit:

$$V(y,\lambda) = \sup_{v\in\mathcal{V}} J(y,\lambda;v) = J(y,\lambda;v^*). \tag{2.4}$$

The firm's problem (2.4) is formulated as a stochastic impulse control problem.

2.2 Quasi-variational Inequalities and Viscosity Solutions

We introduce the quasi-variational inequalities (QVI) to solve the firm's problem (2.4):

$$\max\{\mathcal{L}V(y,\lambda) + \pi(y), \mathcal{M}V(y,\lambda) - V(y,\lambda)\} = 0, \tag{2.5}$$

where $\mathcal{L}$ is the degenerate elliptic differential operator: $\mathcal{L}V(y,\lambda) := -(\delta+\mu_X)yV_Y + [\eta(\mu_\Lambda - \lambda) + \rho\sigma_X\sigma_\Lambda\sqrt{\lambda(1-\lambda)}]V_\Lambda + \frac{1}{2}\sigma_X^2 y^2 V_{YY} - \rho\sigma_X\sigma_\Lambda\sqrt{\lambda(1-\lambda)}yV_{Y\Lambda} + \frac{1}{2}\sigma_\Lambda^2\lambda(1-\lambda)V_{\Lambda\Lambda} - (r-\mu_X)V$, and $\mathcal{M}$ is the capital expansion and reduction operator defined by $\mathcal{M}V(y,\lambda) = \sup_\xi\{V(y+\xi,\lambda) - C(\xi,\lambda); \xi \in \mathbb{R}, y+\xi \in \mathbb{R}_{++}\}$.

The definition of the QVI implies three regions: the continuation region $\mathcal{H}$, capital expansion region $\mathcal{E}$, and capital reduction region $\mathcal{R}$. For a certain λ, they are respectively defined as follows:

$$\mathcal{H} := \{y \in (0,\infty); V(y,\lambda) > \mathcal{M}V(y,\lambda) \text{ and } \mathcal{L}V(y,\lambda) + \pi(y) = 0\}; \tag{2.6}$$

$$\mathcal{E} := \{y \in (0,\infty); V(y,\lambda) = \mathcal{M}V(y,\lambda) \text{ and } \mathcal{L}V(y,\lambda) + \pi(y) < 0, \xi > 0\}; \tag{2.7}$$

$$\mathcal{R} := \{y \in (0,\infty); V(y,\lambda) = \mathcal{M}V(y,\lambda) \text{ and } \mathcal{L}V(y,\lambda) + \pi(y) < 0, \xi < 0\}. \tag{2.8}$$

The QVI drives the following capital expansion and reduction policy.

Definition 2.1 (QVI policy)**.** *Let ϕ be a solution to QVI (2.5). Then, the following capital expansion and reduction policy $\tilde{v} = \{\tilde{\tau}_i, \tilde{\xi}_i\}_{i\geq 0}$ would be the QVI policy:*

$$(\tilde{\tau}_0, \tilde{\xi}_0) = (0, 0); \tag{2.9}$$

$$\tilde{\tau}_i = \inf\{t \geq \tilde{\tau}_{i-1}; Y_t^{y,\tilde{v}} \notin \mathcal{H}\}; \tag{2.10}$$

$$\tilde{\xi}_i = \arg\max_{\xi} \left\{ \phi\left(Y_{\tilde{\tau}_i^-}^{y,\tilde{v}} + \xi_i, \lambda \right) - C(\xi_i, \lambda); \xi_i \in \mathbb{R}, Y_{\tilde{\tau}_i^-}^{y,\tilde{v}} + \xi_i \in \mathbb{R}_{++} \right\}. \tag{2.11}$$

The QVI policy means that once ϕ and $\mathcal{M}\phi$ coincide, the firm expands/reduces the capital by using a QVI policy. We verify that the QVI policy is the optimal capital expansion and reduction policy.

Theorem 2.1. *(I) For given λ, there is supposed to exist $0 < \underline{y}(\lambda) < \overline{y}(\lambda) < \infty$ such that ϕ is linear in $y \in (0, \underline{y}(\lambda)] \cup [\overline{y}(\lambda), \infty)$. Suppose that ϕ is in $C^{1,1}(\mathbb{R}_{++} \times (0,1))$ and $C^{2,2}(\mathbb{R}_{++} \times (0,1) \backslash \mathcal{N})$, where $\mathcal{N} \subset \mathbb{R}_{++}$. We assume that the family $\{\phi(Y_\tau^{y,v}, \Lambda_\tau^{\lambda,v})\}_{\tau<\infty}$ is uniformly integrable for all $(y, \lambda) \in \mathbb{R}_{++} \times (0,1)$ and $v \in \mathcal{V}$. If a solution ϕ of the QVI to the firm's problem (2.4) exists, then, for all $(y, \lambda) \in \mathbb{R}_{++} \times (0,1)$, we obtain*

$$\phi(y, \lambda) \geq V(y, \lambda). \tag{2.12}$$

(II) If the QVI policy corresponding to ϕ is admissible, $\tilde{v} \in \mathcal{V}$, then, for all $y \in \mathbb{R}_{++}$, ϕ is the value function

$$\phi(y, \lambda) = V(y, \lambda), \tag{2.13}$$

and $\tilde{v}$ is optimal.

Theorem 2.1 mentions if there exists an enough regular function satisfying the QVI, the function is the value function of the firm's problem (2.4). We shall show that the function satisfies the QVI in a weak sense. The value function is called a viscosity solution of the QVI in this case. We first define a viscosity solution of the QVI.

Definition 2.2 (Viscosity solution)**.** *The function $\phi \in C(\mathbb{R}_{++} \times (0,1))$ is referred as a* viscosity solution *of the QVI if the following hold.*

(i) ϕ is a viscosity subsolution *to the QVI if for every $\varphi \in C^{2,2}(\mathbb{R}_{++} \times (0,1))$ and every $(y_0, \lambda_0) \in \mathbb{R}_{++} \times (0,1)$ such that $\phi - \varphi$ has a local maximum at (y_0, λ_0) and $\phi(y_0, \lambda_0) = \varphi(y_0, \lambda_0)$, then we have*

$$\max\{\mathcal{L}\varphi(y_0, \lambda_0) + \pi(y_0)), \mathcal{M}\phi(y_0, \lambda_0) - \phi(y_0, \lambda_0)\} \geq 0. \tag{2.14}$$

(ii) ϕ is a viscosity supersolution *to the QVI if for every $\varphi \in C^{2,2}(\mathbb{R}_{++} \times (0,1))$ and every $(y_0, \lambda_0) \in \mathbb{R}_{++} \times (0,1)$ such that $\phi - \varphi$ has a local minimum at (y_0, λ_0) and $\phi(y_0, \lambda_0) = \varphi(y_0, \lambda_0)$, then we have*

$$\max\{\mathcal{L}\varphi(y_0, \lambda_0) + \pi(y_0)), \mathcal{M}\phi(y_0, \lambda_0) - \phi(y_0, \lambda_0)\} \leq 0. \tag{2.15}$$

We show that the value function of the firm's problem (2.4) is a viscosity solution, as proposed by Øksendal and Sulem (2002, 2019). See also Ishii (1993, 1995) which discussed the viscosity solutions to an impulse control problem.

Theorem 2.2. *The value function V of the firm's problem (2.4) is the viscosity solution of the QVI.*

2.3 Solution of the Quasi-variational Inequalities of the Firm's Problem

We assume that an optimal capital expansion and reduction policy $v^* \in \mathcal{V}$ is specified by four thresholds: $y_E(\lambda)$, $y_e(\lambda)$, $y_r(\lambda)$, and $y_R(\lambda)$ with $0 < y_E(\lambda) < y_e(\lambda) < y_r(\lambda) < y_R(\lambda) < \infty$ for a certain λ. For notational simplicity, the thresholds are respectively expressed as: $y_E^\lambda := y_E(\lambda)$, $y_e^\lambda := y_e(\lambda)$, $y_r^\lambda := y_r(\lambda)$, and $y_R^\lambda := y_R(\lambda)$. Once Y reaches the threshold y_E^λ (resp., y_R^λ), the firm purchases (resp., sells) the capital, and Y spontaneously increases (resp., decreases) to y_e^λ (resp., y_r^λ). Consequently, the level of Y changes by $y_e^\lambda - y_E^\lambda$ or $y_r^\lambda - y_R^\lambda$ at each time τ_i.

Based on the assumption above, we can define the optimal capital expansion and reduction policy $v^* = (\tau^*, \xi^*) \in \mathcal{V}$ such that

$$\tau_i^* := \inf\left\{t > \tau_{i-1}^*; Y_{t^-} \notin \left(y_E^\lambda, y_R^\lambda\right)\right\}; \tag{2.16}$$

$$\xi_i^* := Y_{\tau_i} - Y_{\tau_i^-} = \begin{cases} y_e^\lambda - y_E^\lambda, & Y_{\tau_i^-} = y_E^\lambda, \\ y_r^\lambda - y_R^\lambda, & Y_{\tau_i^-} = y_R^\lambda. \end{cases} \tag{2.17}$$

The thresholds y_E^λ, y_e^λ, y_r^λ, y_R^λ are numerically derived through the simultaneous equations:

$$\phi(y_E^\lambda, \lambda) = \phi(y_e^\lambda, \lambda) - (c + p(y_e^\lambda - y_E^\lambda)), \tag{2.18}$$

$$\phi(y_R^\lambda, \lambda) = \phi(y_r^\lambda, \lambda) - (c + (1-\lambda)p(y_r^\lambda - y_R^\lambda)), \tag{2.19}$$

$$\phi_Y(y_E^\lambda, \lambda) = p, \tag{2.20}$$

$$\phi_Y(y_R^\lambda, \lambda) = (1-\lambda)p, \tag{2.21}$$

$$\phi_Y(y_e^\lambda, \lambda) = p, \tag{2.22}$$

$$\phi_Y(y_r^\lambda, \lambda) = (1-\lambda)p, \tag{2.23}$$

where ϕ is the solution of the partial differential equation, $\mathcal{L}\phi(y, \lambda) + \pi(y) = 0$, for $y \in \mathcal{H}$.

2.4 Numerical Analysis

Results of numerical analysis will be presented at the conference.

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