Optimal Randomization over Thresholds in an Abandonment Problem with a Spectrally Negative Lévy Process

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Abstract

This paper considers an abandonment problem in which the underlying uncertainty is modelled as a spectrally negative Lévy jump diffusion. We show that the solution to the corresponding optimal stopping problem may not be a threshold policy. We derive the conditions under the solution is a threshold policy and show these may fail when the jumps are large. In the latter case we show that the optimal abandonment strategy consists of randomizing over thresholds. As a result, abandonment may be optimal sooner than the classical solution implies.

We illustrate these results by applying them to a model of investment in carbon-capture and storage by an electricity producer faced with a stochastically evolving carbon price. In the presence of random jumps in the carbon price, e.g., due to supply restrictions of carbon permits, in addition to (Brownian) trading noise, the firm may be pushed into randomization over investment times and, consequently, invest in carbon abatement technology sooner. This suggests that policy uncertainty may speed up decarbonization.

Keywords: Abandonment option; Spectrally negative process; Threshold policy; Optimal stopping; CCS investment.

JEL Classification Numbers: C62; D81; G33.

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1 Introduction

2 Problem Set Up

Consider a risk neutral decision maker (DM) who has the option to abandon the production of some product if, for some reason, its demand deteriorates significantly. Future demand for the product is uncertain and the uncertainty is modelled on a probability space $(\Omega, \mathcal{F}, P)$ equipped with the standard filtration $(\mathcal{F}_t)_{t\geq 0}$ of a Lévy process $(L_t)_{t\geq 0}$.

The evolution of the revenues, $(X_t)_{t\geq 0}$, is assumed to follow a spectrally negative geometric jump process defined on the state space $\mathcal{X} = (0, \infty)$ and represented by the stochastic differential equation

$$\frac{dX_t}{X_{t-}} = \mu dt + \sigma dW_t - \int_{\mathbb{R}} z\widetilde{N}(dt, dz);\ X_0 = x,$$

where $(W_t)_{t\geq 0}$ is a standard Wiener process and $\widetilde{N}(dt, dz)$ a compensated Poisson random measure with Lévy measure $\lambda m(dz)$. The evolution of information is modeled by the natural filtration $\mathbf{F} = (\mathcal{F}_t)_{t\geq 0}$. The set of $\mathbf{F}$-stopping times is denoted by $\mathcal{T}$. We assume that

$$\lim_{t\to\infty} E_x\left[\int_0^t e^{-rs}|X_s|ds\right] = 0.$$

If the decision maker abandons production at time $\tau \in \mathcal{T}$, then the NPV is given by

$$F(x) = E_x\left[\int_0^\tau e^{-rt}(X_t - c)dt - e^{-r\tau}K\right] \tag{1}$$

where $r > \mu$ is the discount rate, $c > 0$ the per period production cost and $K > 0$ the cost of abandonment. We assume that if he abandons, he does not have the option to restart production.

We need to determine a value function and an optimal stopping time $\tau^* < \infty$ such that

$$\begin{aligned} F^*(x) :&= \sup_{\tau\in\mathcal{T}} E_x\left[\int_0^\tau e^{-rt}(X_t - c)dt - e^{-r\tau}K\right] \\ &= L(x) - \frac{c}{r} - \sup_{\tau\in\mathcal{T}} E_x\left[e^{-r\tau}\left(K - \frac{c}{r} + L(X_\tau)\right)\right], \end{aligned} \tag{2}$$

where $L(x) := E_x\left(\int_0^\infty e^{-rt}X_t dt\right)$ and, by assumption, $c > rK$ so that the solution to the optimal stopping problem is well defined.

The Bellman equation to solve is

$$0 = \max\{-K - F(x), \mathcal{L}_x F + (x - c) - rF(x)\}. \tag{3}$$

Applying the standard optimal stopping approach to solve for such a problem as outlined in, for example, Øksendal and Sulem [2] (Chapter 2), we obtain the following solution to (2):

$$x^* = \frac{\beta_2}{\beta_2 - 1}(r-\mu)\left(\frac{c}{r} - K\right) \tag{4}$$

and

$$F^*(x) \begin{cases} \frac{1}{1-\beta_2}\left(\frac{c}{r} - K\right)\left(\frac{x}{x^*}\right)^{\beta_2} + \frac{x}{r-\mu} - \frac{c}{r} & \text{for } x > x^* \\ -K & \text{for } x \leq x^* \end{cases}, \tag{5}$$

such that $\beta_2 < 0$ is the smaller root of the equation

$$\mathcal{Q}(\beta) = \frac{1}{2}\sigma^2\beta(\beta-1) + (\mu + \lambda E[z])\,\beta - (r+\lambda) + \lambda E[(1-z)^\beta] = 0 \tag{6}$$

and $x^* = X(\tau^*)$ where $\tau^* := \inf\{t > 0 | x \leq x^*\}$.

However, in the case of an abandonment problem with a spectrally negative process, a concern is that the process may jump below x^* at some time $\tilde{\tau} < \tau^*$. If this occurs, then $\tau^* \neq \inf\{x > 0 | x \leq x^*\}$ and, as such, the DM should abandon earlier than the solution suggests; i.e., at $\tilde{\tau}$. In this case, the solution to the optimal stopping problem is not a threshold policy.

Proposition 1. *The solution to the optimal stopping problem (2) is a threshold policy if, and only if,*

$$(1-\beta_2)\sigma^2 - 2(x^*)^2 \int_{\mathbb{R}} \left(\frac{1}{\beta_2}(1-z)^{\beta_2} + z - \frac{1}{\beta_2}\right) m(dz) > 0, \tag{7}$$

i.e. if, and only if, the expected jump size is small enough.

Proof. See Appendix A. ■

For the following parameter values, $K = 95$ and $c = 5$, in addition to those in Alvarez and Rakkolainen [1], $r = 4.5\%$, $\mu = 3\%$, $\lambda = 0.1$, $\sigma = 0.1$ and $a = 1$, we plot the condition (7) against $E[z] = a/(a+b)$ in Fig. 1 below. It is clear from the plot that the condition holds (i.e., it is positive) for smaller jumps, but does not hold for larger jumps, which supports Proposition 1.

This result implies that one must exercise caution in applying the classical approach to determine the optimal time to abandon production when the underlying process is governed by a spectrally negative Lévy jump diffusion because, if the jumps are large, abandonment may be optimal earlier than the classical solution suggests.

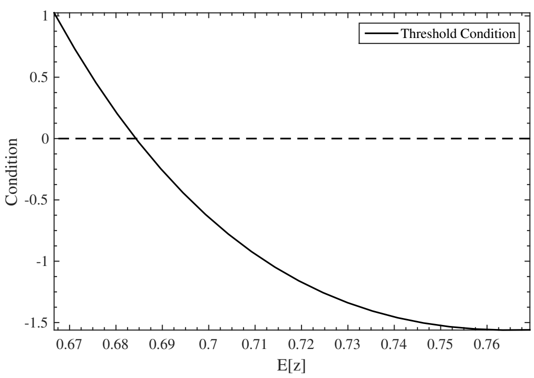

Figure 1: The effect of the expected jump size on the threshold condition.

3 Randomised Stopping Times

The significance of Proposition 1 is that if (7) is violated, there is the risk that a jump will occur for which the value to the DM instantly "plunges" into the stopping region before x^* is hit. Therefore, we assert that the DM should be alert to this risk and consider abandoning when the jumps are large once some boundary on x is reached. As such, we solve the stopping problem subject to a set of randomized threshold strategies in which the DM abandons at some hazard rate $h(t) := \frac{dG(t)}{1-G(t)}$, where $G(t)$ is the distribution function in the $t \in [0, \tau^*]$ interval; i.e., for $x \geq x^*$.

As well as the stopping rate, we also need to characterise a boundary, denoted hereafter as x_B, below which the DM will adopt the randomised threshold strategy. We know that when (7) is satisfied, the solution to the optimal stopping problem is a pure threshold policy so that $x_B = x^*$. Thus, when the condition is violated, it will be the case that $x_B > x^*$.

Moreover, we know from the proof of Proposition 1 (cf. Appendix A) that it is the negativity of the generator $\mathcal{L}_x F(x)$ that can lead to a solution to the optimal stopping problem that is not a pure threshold policy. As such, we characterise the boundary as follows:

$$x_B := \inf\{x > x^* : \mathcal{L}_x F(x) < 0)\}. \tag{8}$$

Proposition 2. *The DM's strategy at $t \in [0, \tau^*]$, where τ^* is such that $F(X_{\tau^*}) = -K$, is to abandon at a rate*

$$h(t) = \frac{dG(t)}{1-G(t)} = \mathbb{1}_{x \leq x_B} \left[\frac{\hat{\delta} \left(\frac{x}{x^*}\right)^{\beta_2} \left[(r - \mu x) L(x^*) - c + rK\right]}{\hat{\delta} \left(1 - \left(\frac{x}{x^*}\right)^{\beta_2}\right) \left(K - \frac{c}{r}\right) + x - \left(\frac{x}{x^*}\right)^{\beta_2} x^*} \right] dt \tag{9}$$

where

$$\hat{\delta} := r - \mu - \lambda E[z] - \lambda \left(1 - E\left[\frac{1}{1-z}\right]\right)$$

and

$$L(x) := \int_0^\infty e^{-rt} E[X_t] dt = \frac{x}{\hat{\delta}}.$$

The resulting expected payoff from randomly stopping at t is given by

$$\widehat{V}_t(x) = F^*(x) = \sup_{\tau \in \mathcal{T}} E_x \left[\int_0^\tau e^{-rt}(X_t - c)dt - e^{-r\tau} K \right].$$

Proof. See Appendix B. ■

An important finding, stated in the proposition, is that the value function from randomly stopping at $t \in [0, \tau^*]$, according to the hazard rate defined, coincides with the threshold policy value function $F^*(x)$ given by (5). Therefore, when the classical optimal stopping approach to solving an abandonment problem is not appropriate, i.e., when the underlying process follows a spectrally negative process with large jumps, our approach of defining an interval and a hazard rate at which the DM abandons production yields the same payoff in equilibrium as that which is attained from solving using classical techniques.

However, if the jump sizes and intensity are small, $\mathcal{L}_x F(x)$ will be positive so that $x_B = 0$, by its definition given by Eq. (8). As such, according to (9), $h(x) = 0$, which also coincides with the threshold policy approach.

Finally, according to the following corollary, stopping will be certain if and only if $x \leq x^*$. Indeed, this too coincides with the optimal strategy implied by threshold approach.

Corollary 1. *The cumulative probability of abandonment before x^* is reached is not equal to one.*

Proof. See Appendix B.1. ■

This result is depicted in Fig. 2 where $G(t) = 1$ for $x = x^*$, but for all $x > x^*$, it is less than one. It is driven by the fact that x can hover round, or indeed cross, the x_B boundary many times before x^* is hit; in other words, if $x \in (x^*, x_B]$, the outlook may improve at any stage, causing the process to drift out of the region and back above x_B. The further away is x from x_B, the less likely it is to cross the boundary and the greater is the cumulative probability. However, since there is always a chance of improvement while in this region, abandonment will never be certain until x^* is reached.

The overall significance of Proposition 2 and the subsequent inferences is stated in the following proposition. This is the main result of the paper.

Proposition 3. *It is sufficient to resort to the randomised threshold rule and avoid using classical techniques in solving an optimal stopping abandonment problem, such as that given by Eq. (2), when the underlying stochastic process is spectrally negative.*

Given this, it is important to understand how the underlying process impacts on the shape and size of this interval $[x^*, x_B)$, as well as on the hazard rate defined by Eq. (9).

Fig. 3 below helps us to infer how the state space will appear for different jump sizes and intensities by examining their effect on the boundary x_B. The upper plots show that x_B increases in both the expected jump size $E[z]$ and the jump intensity λ. The driving force underpinning these effects is the impact of the parameters on β_2. We show in Appendix A that β_2 increases in $E[z]$ and, indeed, we can infer in a similar way that β_2 also increases in λ. As such, x^* decreases in both. Furthermore, by standard calculus techniques, we can establish that the discount factor $(x/x^*)^{\beta_2}$ increases in $E[z]$ and λ when the jumps are large and/or intense. However, as shown in Proposition 1, it is not appropriate to use the classical approach to solve the optimal stopping problem

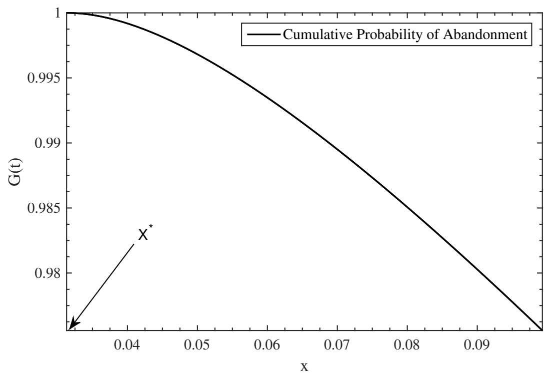

Figure 2: The cumulative probability of abandonment before x^* is reached.

when the jumps are large. Therefore, we can assert that $(x/x^*)^{\beta_2}$ increases in $E[z]$ and λ unconditionally for the analysis hereafter. This implies that the larger is $E[z]$ and/or λ, the shorter will be the time until x^* is reached for any $x > x^*$, which is intuitive. Therefore, to compensate for the risk of x jumping below the threshold too quickly, the DM will start to abandon sooner at a rate $dG(t)/(1 - G(t)) > 0$; i.e., x_B is higher.

The lower plots show that the difference between x_B and x^* is increasing in $E[z]$ and in λ since x^* decreases and x_B increases in both. However, $x_B - x^*$ increases at an increasing rate in $E[z]$ implying they keep diverging, but the rate of increase in the difference slows for larger values of λ implying the divergence between x_B and x^* flattens out for high values. This is not so apparent from the plots, but for $E[z_1]$, $E[z_2]$ and $E[z_3]$, the respective $x_{B_i} - x_i^*$ are 0.04, 0.5 and 1.01. However, for λ_i, the respective differences are 0.46, 2.13 and 2.46.

From this, we acquire an understanding of how the state space will appear as $E[z]$ and λ change, and provide a sketch of it in terms of both parameters in Fig. 4 below for $x_B > x^*$. For smaller jump sizes and intensities, where condition (7) is satisfied (i.e., the standard case), the state space will be divided into two regions separated by x^* and the randomisation region will be eliminated entirely.

Proposition 4. *The rate of abandonment before x^* is reached decreases in the expected jump size and intensity.*

The intuition here is that the DM will abandon with with positive probability earlier for larger values of $E[z]$ and/or λ owing to the higher discount factor, as discussed above. However, earlier abandonment implies abandoning at higher values of x and since $\lim_{x\to\infty} h(t) = 0$, where $h(t) := dG(t)/(1 - G(t))$, we can assert that the rate of abandonment is low for larger values of $E[z]$ and/or λ.

This proposition is supported from examining Fig. 5 below, in which we can infer that $h(t)$ does indeed decrease in $E[z]$ (left hand plot) and in λ (right hand plot) by considering its effect on x.

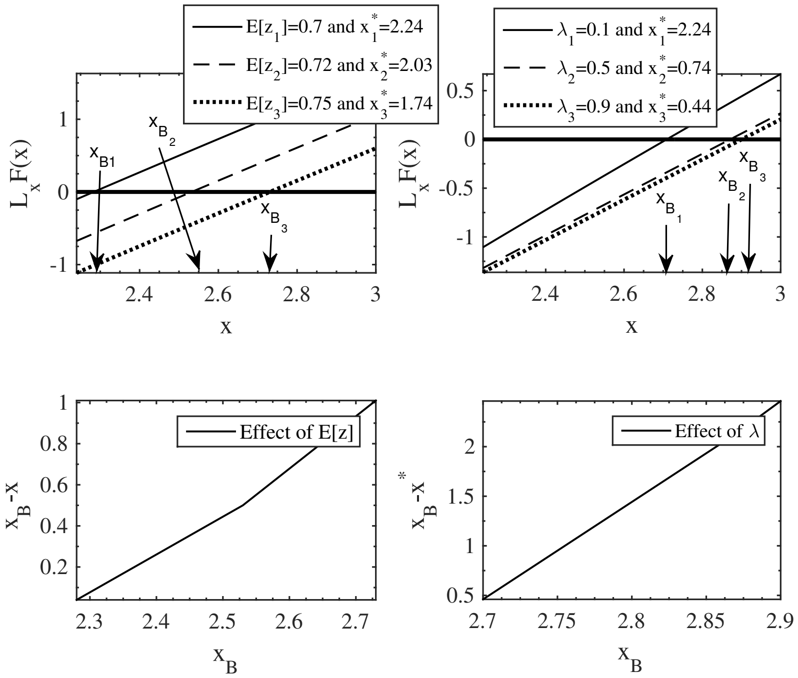

Figure 3: The boundary x_B for different jump sizes and intensities.

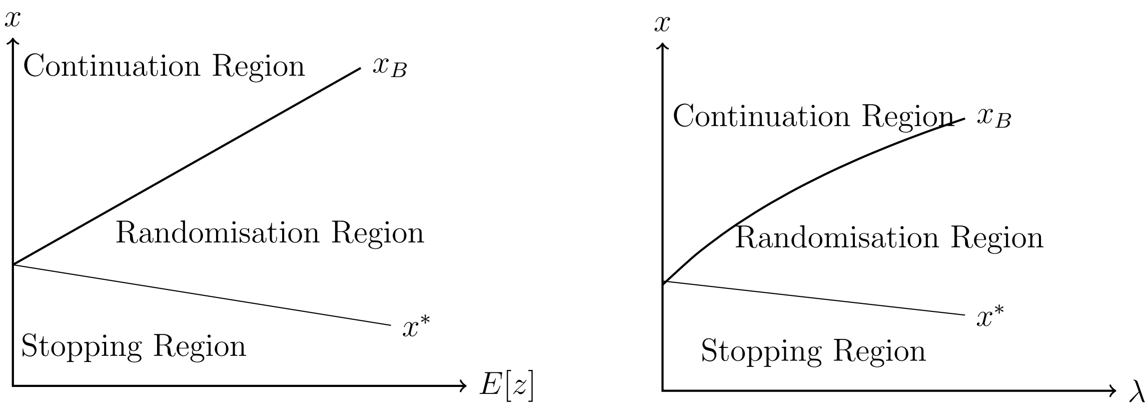

Figure 4: The state space as a function of the jump size and the jump intensity for $x_B > x^*$.

4 Conclusion

Appendix

A Proof of Proposition 1

1. Say stopping is optimal. Then, according to Eq. (3), $\mathcal{L}_x F + (x - c) - rF(x) < 0$ and $F(x) = -K$. From Thijssen [3], the generator is given by

$$\mathcal{L}_x F = \frac{1}{2}\sigma^2 x^2 F_{xx} + \mu x F_x + \int_{\mathbb{R}} \left[F(x - xz) - F(x) + zxF_x(x)\right] m(dz) \tag{A.1}$$

so that

$$x - c < -rK$$

must hold in the stopping region for $x \leq x^*$.

2. If continuation is optimal, then according to (3), $F(x) > -K$ must be satisfied for $x^* < x$ with $F(x)$ the solution to $\mathcal{L}_x F + (x - c) - rF(x) = 0$ subject to the condition that the value of the option to abandon is low for high values of x.

To determine if the solution to Eq. (2) is a threshold policy, consider x close to x^*, but $x > x^*$. If the DM is acting optimally, his value function is given by

$$F(x) = (x - c)dt + (1 - rdt)\left[E^x\left(F(x) + dF\right)\right].$$

By the usual manipulation, this reduces to

$$rF(x) = (x - c) + \mathcal{L}_x F(x)$$

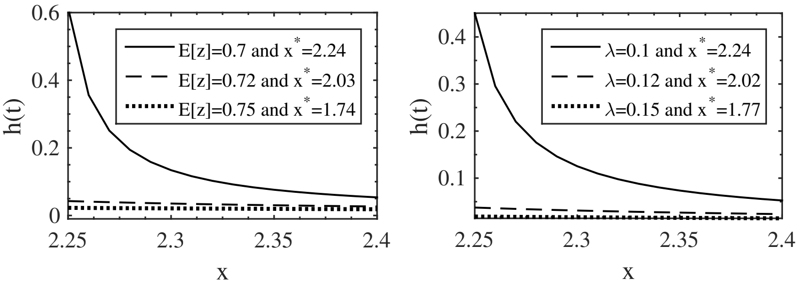

Figure 5: The effect of the jump size and the intensity on the stopping rate.

and, since $F > -K$ in the continuation region, we have that

$$-rK < (x - c) + \mathcal{L}_x F(x). \tag{A.2}$$

But, in the stopping region, the condition is that $x - c < -rK$, so that $(x - c) \to -rK$ as $(x - c)$ decreases in the continuation region. Therefore, to ensure the DM is acting optimally in the continuation region so that (A.2) is satisfied, we must have that

$$\mathcal{L}_x F(x) > 0$$

for all $x > x^*$ if the solution is a threshold policy.

To check the conditions under which the solution is a threshold policy, we check the conditions under which $\mathcal{L}_x F(x) > 0$ for $x > x^*$ where x^* is given by Eq. (4) and $F(x)$ by Eq. (5). Note, however, that if the solution is a threshold policy, the condition will hold for x very close to x^*. Thus, for ease of analysis, it is sufficient to check the condition for $x \approx x^*$ and, as such, for $F_x(x) \approx 0$. Thus, the solution will be a threshold policy iff

$$\begin{aligned} &\frac{1}{2}\sigma^2 (x^*)^2 F_{xx}(x^*) + \int_{\mathbb{R}} \left[F((1-z)x^*) - F(x^*)\right] m(dz) > 0 \\ \iff &(1-\beta_2)\sigma^2 > 2(x^*)^2 \int_{\mathbb{R}} \left(\frac{1}{\beta_2}(1-z)^{\beta_2} + z - \frac{1}{\beta_2}\right) m(dz). \end{aligned} \tag{A.3}$$

Let $z = 0$. It is easily verified from (6) that β_2 will still be negative. Therefore, the condition (A.3) is satisfied for $z = 0$; i.e., for small jumps.

We can also verify that β_2 increases in z:

$$\frac{\partial \beta_2}{\partial z} = -\frac{\partial \mathcal{Q}(\beta_2)/\partial z}{\partial \mathcal{Q}(\beta_2)/\partial \beta_2} > 0 \iff \frac{\partial \mathcal{Q}(\beta_2)}{\partial z} > 0$$

since $\mathcal{Q}(\beta)$ is an upward sloping parabola with β_2 being the smaller root. Now since $z \leq 1$ and $\beta_2 < 0$, we can indeed verify that $\frac{\partial \mathcal{Q}(\beta_2)}{\partial z} > 0$.

This implies that for larger jumps, the left hand side of the inequality is low. It is not possible to determine if the right had side of the inequality unambiguously increases in z, but if we let $z = 1$, we do indeed see that the RHS is positive. Since it is zero for $z = 0$, it is plausible to infer, therefore, that the RHS increases in z.

Given this conclusion, if jumps are large, it is possible that (A.3) may not hold and, if this is so, the solution to the optimal stopping problem is not a threshold policy.

B Proof of Proposition 2

Let $\widehat{V}_t(x)$ denote the expected value to the DM from randomising over stopping times in the $t \in [0, \tau^*]$ interval. He will either abandon at or before some t is reached with probability $G(t)$ and get $-K$ or he will not abandon by t; i.e., he will wait until τ^*, where his expected present value of abandoning at τ^* is given by $F^*(x)$, defined by Eq. (2). Therefore, $\widehat{V}_t(x)$ is given by

$$\widehat{V}_t(x) = E\left[-\int_0^t K dG(s) + (1 - G(t))F^*(x)\right]. \tag{B.1}$$

Now, by the general theory of optimal stopping, $F^*(x)$ is the smallest continuous supermartingale dominating the payoff process in the $[0, \tau^*]$ interval and $G(t)$ is continuous for all $t \in [0, \tau^*]$. As such, $\widehat{V}_t(x)$ satisfies

$$\begin{aligned} E[d\widehat{V}_t(x)] =& E\left[-\left(K + F^*(x)\right) dG(t) + (1 - G(t))dF^*(x)\right] \\ =& \mathcal{L}\widehat{V}_t(x). \end{aligned} \tag{B.2}$$

Now, to ensure the DM is indifferent over payoffs, $\widehat{V}_t(x)$ should be a martingale. It will be a martingale iff $E[d\widehat{V}_t(x)] = 0$; i.e., iff

$$\frac{dG(t)}{1 - G(t)} = \frac{E[dF^*(x)]}{K + F^*(x)}.$$

Applying Ito's Lemma to $F^*(x)$ gives

$$\begin{aligned} E[dF^*(x)] =& \left[-re^{-r\tau^*}\left[K - \frac{c}{r} + L(x^*)\right] + \mu x F^*_x(x) + \int_{\mathbb{R}} \left[F^*(x - xz) - F^*(x) + zxF^*_x(x)\right] m(dz)\right] dt \\ =& \left(\frac{x}{x^*}\right)^{\beta_2} \left[\left(r - \mu x\right) L(x^*) - c + rK\right] dt \end{aligned} \tag{B.3}$$

where $E(e^{-r\tau^*}) = \left(\frac{x}{x^*}\right)^{\beta_2}$ and $L(x) := \int_0^\infty e^{-rt} E[X_t] dt$, as previously defined. However, from Alvarez and Rakkolainen [1],

$$E(X_t) = x \exp\left(\mu + \lambda E[z] - \lambda\left(1 - E\left[\frac{1}{1 - z}\right]\right)\right)$$

and $r > \mu + \lambda\bar{m} + \lambda\left(1 - E\left[\frac{1}{1-z}\right]\right)$. Hence $L(x) = \frac{x}{\hat{\delta}}$ with $\hat{\delta} := r - \mu - \lambda E[z] - \lambda\left(1 - E\left[\frac{1}{1-z}\right]\right)$.

Therefore, $\widehat{V}_t(x)$ is a martingale for the hazard rate $dG(t)/(1 - G(t))$ given by Eq. (9).

Now, by integrating across Eq. (B.2), we obtain

$$\begin{aligned} E\left[\widehat{V}_t(x)\right] &= \widehat{V}_0(x) + E\left[\int_0^t \mathcal{L}\widehat{V}_s(x)ds\right] \\ &= F^*(x) \\ &= \sup_{\tau\in\mathcal{T}} E\left[\int_0^\tau e^{-rt}(X_t - c)dt - e^{-r\tau}K\right], \end{aligned} \tag{B.4}$$

since $\widehat{V}_0(x) = F^*(x)$ by Eq. (B.1) and $\mathcal{L}\widehat{V}_t(x)$ is zero for the hazard rate given by (9).

B.1 Proof of Corollary 1

Since $G(\tau^*) = 1$,

$$E[\widehat{V}_{\tau^*}(x)] = E\left[-\int_0^{\tau^*} KdG(s)\right] = F^*(x).$$

Thus, $\int_0^{\tau^*} dG(s) = 1$ if and only if $F^*(x) = -K$. This is true iff $t \geq \tau^*$. Hence, the cumulative probability that the DM will abandon before τ^* is not one.

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