Investment timing and optimal capacity choice with price floors and cellings*

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January 2020

Abstract

What is the optimal capacity and investment timing when there is a price floor and ceiling? We adapt previous solutions for the value of an active firm with such a price collar, and provide a novel solution for an investment opportunity with simultaneous determination of optimal capacity and the price threshold that justifies immediate investment. We show that higher price caps reduce the thresholds and also the optimal capacity (surprisingly). However, higher price floors also reduce the thresholds, but eventually raise the optimal capacity (not surprising). Increased price volatility raises both the price threshold and the optimal capacity. Thus, a government wanting to motivate early investment, with large capacity to avoid congestion and crowding, could possibly achieve both objectives through appropriate price collars, but not through reducing volatility.

Keywords: Collars, Capacity, Thresholds, Real option value, Investment under price constraints.
JEL codes: D81, G31, H25.

*Paulo J. Pereira and Artur Rodrigues acknowledge that this research has been financed by Portuguese public funds through FCT - Fundação para a Ciência e a Tecnologia, I.P., in the framework of the projects UID/ECO/04105/2019 and UID/ECO/04105/2019, respectively.

1 Introduction

What is the optimal capacity and investment timing when there is a price floor and ceiling? We adapt previous solutions for an active firm with such a price collar, and provide a novel solution for an investment opportunity with simultaneous determination of optimal capacity and the price thresholds that justifies immediate investment.

In a world of demand uncertainty, a monopoly seeks to establish a facility having an optimally selected capacity to produce a product, or accommodate a demand such as toll traffic, where the toll rates are priced according to an iso-elastic demand function. An investment commitment is made when the market demand attains an optimal threshold. This project opportunity is subject to a single source of uncertainty due to demand variability. The investment cost has fixed and variable components, so it is more expensive to develop a plant having a greater capacity, but the marginal investment cost decreases with increasing capacity. After the plant is established, there are no options to abandon and temporary suspend operations, or make any changes in capacity.

Many authors have considered the real option aspects of capacity choice. Dangl (1999) considers choosing both the timing and upper boundary for capacity with uncertain demand. Huisman and Kort (2015) extend the Dangl approach using an inverse demand function for a duopoly. Huberts et al. (2015) summarize some capacity model developments, allowing for production suspension, and also for bounded capacity. Hagspiel et al. (2016) consider holding costs per unit of capacity, and also a linear demand structure. Chronopoulos et al. (2017) finds stepwise investment leads to greater optimal capacity. De Giovanni and Massabò (2018) focus on volume flexibility, but follow the Dangl's approach with utilization considerations. Balter et al. (2019) extend Huisman and Kort (2015) to finite project life, with different terminations for the follower and leader, and possible deferral/accommodation for the follower. Wen et al. (2019) look at a dedicated leader, and a possibly flexible follower, who could be useful for the leader.

There are now several articles on finite and perpetual collars, as cited in Adkins et al. (2019), but apparently none that study the effect of collars on both investment timing and optimal capacity. Extending previous collar models to optimal capacity choice allows us to study the implications of allowing a firm to choose optimal capacity when a collar is offered, and the impact of imposing a collar when the firm is free to choose its capacity.

We offer novel solutions for optimal timing and capacity choices. The next section outlines the basic economic model. Section 3 provides numerical illustrations, including an analysis of an active firm (having previously reached the appropriate price threshold), and then the value of an opportunity to invest in a project with price collars. The last section concludes with interpretation of the unique contribution and also limitation of this study, suggesting further research.

2 The model

Let us assume a monopolistic firm, for example a firm granted with a concession by the Government or with exclusive rights by another firm, in market a market where the demand function is iso-elastic:

$$P(t) = X(t)Q(t)^{-\gamma} \tag{1}$$

where $Q(t)$ is the total market output, $\gamma \in (0,1)$ is the price elasticity parameter, and $X(t)$ is an exogenous shock which affects the output price and follows a geometric Brownian motion (gBm) given by:

$$dX(t) = \alpha X(t)dt + \sigma X(t)dw(t) \tag{2}$$

where $X(t) > 0$, α (with $\alpha < r$) is the risk-neutral expected drift, r is the risk-free rate, σ is the instantaneous volatility, $dw(t)$ is the increment of a Wiener process.

Following Huisman and Kort (2015), let us also assume that the firm enters the market with a capacity (Q) and the investment cost comprises two components: a fixed cost (δ_0) and a cost per output unit (δ_1). The total investment is therefore given by: $\delta_0 + \delta_1 Q$. We also assume that after investing the firm operates at full capacity ($Q(t) = Q$).

Therefore, the firm's objective function is given by:

$$V(X) = \max_{T \geqslant 0, Q \geqslant 0} E\left[\int_{t=T}^{\infty} \left(QX(t)Q^{-\gamma}\right) e^{-rt} dt - (\delta_0 + \delta_1 Q)e^{-rT} | X(0) = X\right] \tag{3}$$

where T is the optimal time to invest, Q is the optimal entry capacity level.

The solution for Equation (3) is attained in two steps see (see Huisman and Kort, 2015). In a first step, we select the optimal capacity ($Q^*(X)$) for a given $X(t)$, through:

$$\max_{Q \geqslant 0} E\left[\int_{t=0}^{\infty} \left(QX(t)Q^{-\gamma})\right) e^{-rt} dt - (\delta_0 + \delta_1 Q) | X(0) = X\right] \tag{4}$$

which yields:

$$Q^*(X) = \left(\frac{(1-\gamma)X}{(r-\alpha)\delta_1}\right)^{\frac{1}{\gamma}} \tag{5}$$

In a second step, we replace Q in equation (3) by equation (5) and obtain the optimal investment threshold (X^*), given by:

$$\max_{X^*} \left[\left(\frac{Q^*(X^*)X^*Q^*(X^*)^{-\gamma})}{r-\alpha} - (\delta_0 + \delta_1 Q(X^*))\right) \left(\frac{X}{X^*}\right)^{\beta_1}\right] \tag{6}$$

where:

$$\beta_1 = \frac{1}{2} - \frac{\alpha}{\sigma^2} + \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} \tag{7}$$

Proposition 1. *In a market where a constant elasticity demand function ($P = XQ^{-\gamma}$) holds, a firm with a monopoly position over the decision to invest in a project whose cost comprises a fixed component (δ_0) and a variable component - per output unit of the entry capacity level - ($\delta_1 Q$), invests at the following optimal threshold:*

$$X^* = \left(\frac{\beta_1(1-\gamma)\delta_0}{(\beta_1\gamma - 1)\delta_1}\right)^{\gamma} \frac{r-\alpha}{1-\gamma}\delta_1 \tag{8}$$

with capacity:

$$Q^* \equiv Q^*(X^*) = \frac{\beta_1(1-\gamma)\delta_0}{(\beta_1\gamma - 1)\delta_1} \tag{9}$$

with $\beta_1\gamma > 1$, otherwise the firm would postpone investment forever (see Dixit and Pindyck, 1994) and $\delta_0 > 0$, otherwise the firm would invest immediately (see Huisman and Kort, 2015).

2.1 Perpetual collars

With the objective of incentivizing investment, a perpetual collar is offered to the firm, with a price floor and a price cap.

Active firm

The solutions for an investment opportunity with a perpetual collar for a price-taker firm can be found in Adkins and Paxson (2017) and Adkins et al. (2019). Following similar steps, for the current case of a firm facing an iso-elastic demand function, let $V_p(P)$ denote the value of an active project whose output price P is bounded by a price floor P_L and a price cap P_H. The solution for $V_p(P)$ satisfies the following non-homogeneous differential equation:

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 V_p(X)}{\partial X^2} + \alpha X \frac{\partial V_p(X)}{\partial X} - rV_p(X) + R(X) = 0, \tag{10}$$

where $R(X) = R(X, P_L, P_H, Q) = \min\{\max\{P_L, P\}, P_H\}Q$ for convenience, and $R(X) = QP_L$ for $X < X_L = P_L Q^{\gamma}$, $R(X) = QP_H$ for $X \geqslant X_H = P_H Q^{\gamma}$, and $R(X) = QP(X) = QXQ^{-\gamma}$ for $X_L \leqslant X < X_H$.

The general solution of (10) is:

$$V_p(P) = A_a P^{\beta_1} + A_b P^{\beta_2} \tag{11}$$

where

$$\beta_2 = \frac{1}{2} - \frac{\alpha}{\sigma^2} - \sqrt{\left(-\frac{1}{2} + \frac{\alpha}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}} < 0. \tag{12}$$

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The solutions for the non-homogeneous part (the particular solutions) depend on where $P\ (X)$ stands in relation to $P_L\ (X_L)$ and $P_H\ (X_H)$. Accordingly, the particular solution for $X < X_L$ is QP_L/r, for $X \in [X_L, X_H)$ is $QXQ^{-\gamma}/(r-\alpha)$, and for $X > X_H$ becomes QP_H/r. Considering that $V_p(0) = 0$, then $A_b = 0$ for $X < X_L$. Additionally, given that $V_p(X)$ has an upside limit of QX_H/r whenever $X \geqslant X_H$, then A_a must be set equal to 0 in this region. Putting together the solutions for all the regions we get:

$$V_p(X,Q) = \begin{cases} A_{11}X^{\beta_1} + \dfrac{QP_L}{r} & \text{for } X < X_L \\ A_{21}X^{\beta_1} + A_{22}X^{\beta_2} + \dfrac{QXQ^{-\gamma}}{r-\alpha} & \text{for } X_L \leqslant X < X_H \\ A_{32}X^{\beta_2} + \dfrac{QP_H}{r} & \text{for } X \geqslant X_H \end{cases} \tag{13}$$

The constants $A_{11}, A_{21}, A_{22}, A_{32}$ are found by ensuring that $V_p(P)$ is continuous and continuously differentiable along P. The solutions for the constants are as follows:[1]

$$A_{11} = \frac{\left(P_H^{1-\beta_1} - P_L^{1-\beta_1}\right)}{\beta_1 - \beta_2}\left(\frac{\beta_2 - 1}{\delta} - \frac{\beta_2}{r}\right)Q^{1-\beta_1\gamma} = a_{11}Q^{1-\beta_1\gamma} \tag{14}$$

$$A_{21} = \frac{P_H^{1-\beta_1}}{\beta_1 - \beta_2}\left(\frac{\beta_2 - 1}{\delta} - \frac{\beta_2}{r}\right)Q^{1-\beta_1\gamma} = a_{21}Q^{1-\beta_1\gamma} \tag{15}$$

$$A_{22} = -\frac{P_L^{1-\beta_2}}{\beta_1 - \beta_2}\left(\frac{\beta_1 - 1}{\delta} - \frac{\beta_1}{r}\right)Q^{1-\beta_2\gamma} = a_{22}Q^{1-\beta_2\gamma} \tag{16}$$

$$A_{32} = \frac{\left(P_H^{1-\beta_2} - P_L^{1-\beta_2}\right)}{\beta_1 - \beta_2}\left(\frac{\beta_1 - 1}{\delta} - \frac{\beta_1}{r}\right)Q^{1-\beta_2\gamma} = a_{32}Q^{1-\beta_2\gamma} \tag{17}$$

Investment for an idle firm with optimal capacity choice

Suppose an idle firm has a perpetual opportunity, without a holding cost, to invest in a project with a perpetual collar. The value of this opportunity, $F_p(X)$ must satisfy the following ordinary differential equation:

$$\frac{1}{2}\sigma^2 X^2 \frac{\partial^2 F_p(X)}{\partial X^2} + \alpha X \frac{\partial F_p(X)}{\partial X} - rF_p(X) = 0. \tag{18}$$

The general solution has the form $F_p(X) = B_a X^{\beta_1} + B_b X^{\beta_2}$. Considering that $F_p(0) = 0$ then we set $B_b = 0$. The arbitrary constant B_a is found using the value matching condition (VM):

$$F_p(X_p^*) = B_a X_p^{*\beta_1} = V_p(X_p^*) - K, \tag{19}$$

[1] These solutions are obtained equalizing the value and the derivatives of the first and second branches of Equation (13) at P_L, and of the second and third branches at P_H.

i.e.,

$$B_a = \left(V_p(X_p^*) - K\right)\left(\frac{1}{X_p^*}\right)^{\beta_1}, \tag{20}$$

and the investment trigger, X_p^*, is obtained by solving the smooth pasting condition (multiplying both sides by X_p^*):

$$\beta_1(V_p(X_p^*, Q_p^*) - K) = \frac{\partial V_p(X)}{\partial X}(X_p^*, Q_p^*)X_p^* \tag{21}$$

$$\frac{\partial V_p(X)}{\partial Q}(X_p^*, Q_p^*) = 0 \tag{22}$$

As shown by Adkins et al. (2019), the price floor must be lower than Kr/Q, otherwise it produces a risk-free profit, and the trigger X_p^* can be either below or above X_H (but above X_L), i.e. $X_p^* \in [X_L, \infty)$.

For the region $[X_H, \infty)$ the solutions to Equations (21) and (22) are closed-form:

$$Q_p^* = \frac{\beta_1(\beta_2\gamma - 1)\delta_0}{\beta_2(\beta_1\gamma - 1)\left(\frac{P_H}{r} - \delta_1\right)} \tag{23}$$

$$X_p^* = \left(\frac{Q_p^{*1-\beta_2\gamma}}{a_{32}(\beta_2\gamma - 1)}\left(\frac{P_H}{r} - \delta_1\right)\right)^{\frac{1}{\beta_2}} Q_p^{*\gamma} \tag{24}$$

where the condition $\delta_1 > \frac{P_H}{r}$ is needed for positive solutions.

For the region $[X_L, X_H)$, Equations (21) and (22) reduce to:

$$(\beta_1 - \beta_2)A_{22}X_p^{*\beta_2} + (\beta_1 - 1)\frac{Q_p^{*1-\gamma}X_p^*}{r-\alpha} - \tag{25}$$

$$(\beta_1\gamma - 1)A_{21}X_p^{*\beta_1} + (\beta_2\gamma - 1)A_{22}X_p^{*\beta_2} - (1-\gamma)\frac{Q_p^{*1-\gamma}X_p^*}{r-\alpha} + \delta_1 Q_p^* \tag{26}$$

that need to be solved numerically.

The condition that places the solution in the region $[X_H, \infty)$ is:

$$\delta_1 > (\beta_2\gamma - 1)\left(\frac{P_H}{r} - a_{32}{P_H}^{\beta_2}\right) \tag{27}$$

Investment for an idle firm without capacity choice

As a benchmark, we present the solution for the investment timing when the firm does not choose optimally its capacity (X_p^o), e.g. when a certain capacity is imposed. The solution if found solving Equation (21), with Q_p^* is replaced by a fixed capacity Q_p^o.

As before there are two regions. For the last region ($[X_H, \infty)$) the solution is closed-form:

$$X_p^o = \left(\frac{\beta_1}{a_{32} Q_p^o (\beta_1 - \beta_2)} \left(\delta_0 + \delta_1 Q_p^o - \frac{P_H Q_p^o}{r} \right) \right)^{\frac{1}{\beta_2}} Q_p^{o\gamma} \tag{28}$$

where the condition $\delta_0 > \left(\frac{P_H}{r} - \delta_1 \right) Q_p^o$ is needed for a positive solution.

For the region $[X_L, X_H)$, Equation (21) reduces to:

$$(\beta_1 - \beta_2) A_{22} X_p^{o\beta_2} + (\beta_1 - 1) \frac{Q_p^{o1-\gamma} X_p^o}{r - \alpha} - \beta_1 \left(\delta_0 + \delta_1 Q_p^o \right) \tag{29}$$

that need to be solved numerically.

The condition that places the solution in the region $[X_H, \infty)$ is:

$$\delta_0 < \left(\frac{P_H}{r} + \frac{(\beta_1 - \beta_2) a_{32} {P_H}^{\beta_2}}{\beta_1} \delta_1 \right) Q_p^o \tag{30}$$

3 Comparative statics

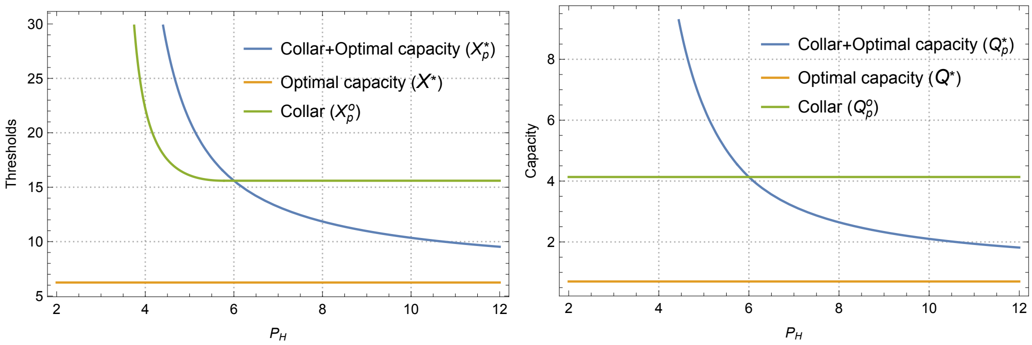
Table 1 shows the assumed base case parameter values for a firm with no operating cost but with a price floor and ceiling, translated into a demand floor and ceiling.

Parameter	Description	Value
P_L	Price floor	\$2
P_H	Price cap	\$6
γ	Price elasticity	0.7
σ	Volatility	0.2
r	Risk-free rate	0.04
α	Risk-neutral drift rate	0.01
δ_0	Fixed investment cost	\$20
δ_1	Variable investment cost	\$80

Table 1: The base case parameters.

Figures 1-7 illustrate the sensitivity of the investment triggers (X_p^*, X^*, and X_p^o) and capacity choice (Q_p^* and Q^*) to changes in some critical parameter values. For the purpose of comparability, we assume that when the firm is not allowed to choose capacity, the capacity imposed is the optimal capacity for the base case parameter values ($Q_p^o = Q^* \approx 4.134$).

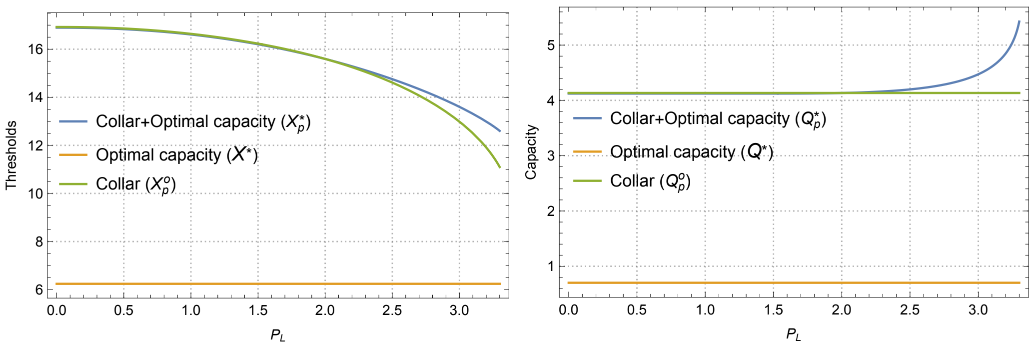
When the firm is allowed to choose capacity a higher price cap hastens investment but with in a smaller scale (Figure 1) and a collar with a price cap deters investment but induces larger scale investments ($X_p^* > X^*$, $Q_p^* > Q^*$). On the contrary, a higher price floor is capable of inducing earlier and larger investments (Figure 2). However, because of the price cap, imposing a collar has the same effect: investment is deterred but occurs in larger scales.



(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 1: The effect of the price cap

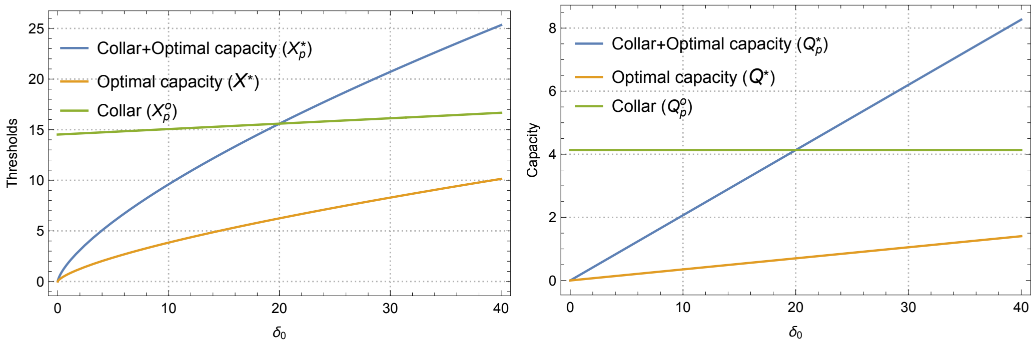


(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 2: The effect of the price floor

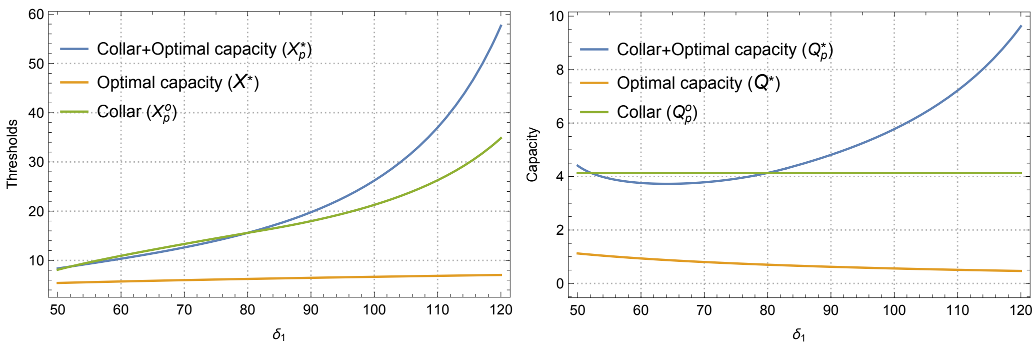
As expected, investment costs have, in general, the effect of deterring investment and, because of that, firms choose larger capacities (Figures 3 and 4). However a nonlinear effect can be observed for the effect of δ_1 on the capacity choice (Figure 4(b)). As before the introduction of the collar deters investment with the concurrent larger scale.



(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 3: The effect of δ_0

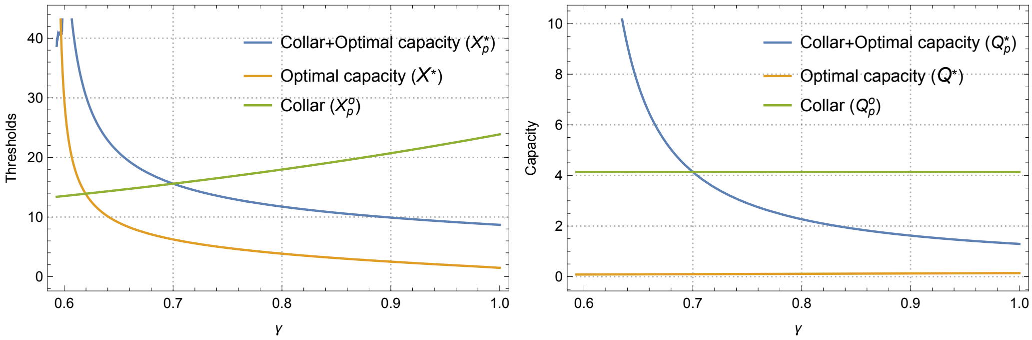


(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 4: The effect of δ_1

When demand is more elastic, it induces earlier investments, contrary to what happens when firms are not allowed to choose capacity (Figure 5). The introduction of the collar induces much larger scale investments the smaller the elasticity is. Additionally, a higher elasticity induces smaller scale investments when a collar is in place, whereas has mild opposite effect when it is not imposed.

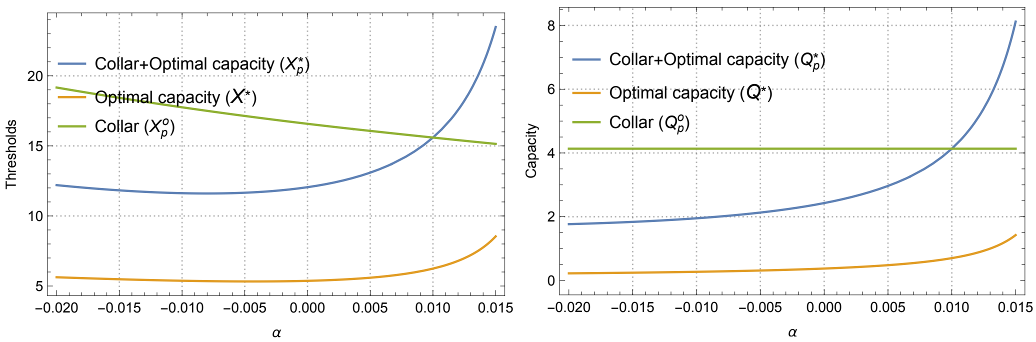


(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 5: The effect of γ

When offered a collar, a firm not allowed to choose an optimally its capacity invests earlier the higher the drift rate (Figure 6). On the contrary, when an optimal capacity is chosen, with or without a collar, that effect is limited to a negative drifts, and is reversed as the drift rate increases: a higher drift rate may induce later investments (Figure 6(a)). In any case, a higher drift rate induces larger investments (Figure 6(b)).

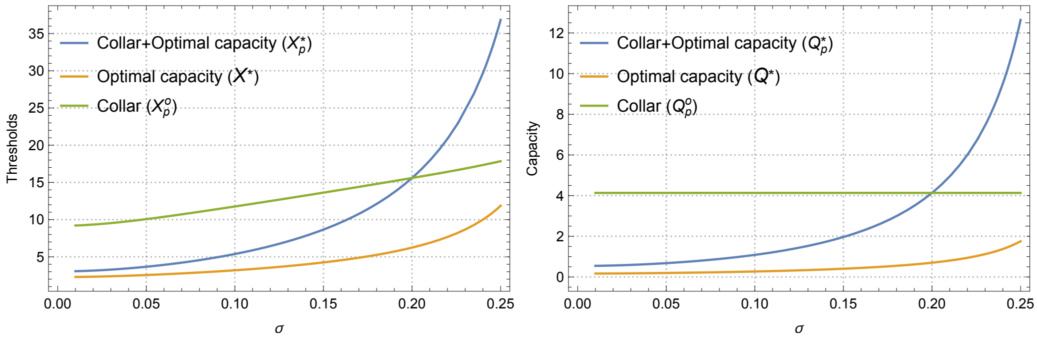


(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 6: The effect of α

Uncertainty has the standard real option effect on investment timing: a higher uncertainty deters in all cases investment (Figure 7(a)), and, as is generally the case for an iso-elastic demand, induces large scale investments (Figure 7(b)). The introduction of a collar as a higher impact the higher uncertainty is.



(a) Threshold (b) Capacity

$P_L = \$2; P_H = \$6; \sigma = 0.2; r = 0.04; \alpha = 0.01; \gamma = 0.7; \delta_0 = \$20; \delta_1 = \$80.$

Figure 7: The effect of σ

The primary findings are summarized in Table 2. When a collar if offered alongside the freedom to choose capacity, reducing the price floor induces the desirable outcome of earlier and larger investments. Any other parameter implies most of the times a trade-off between timing and scale of investment.

	X_p^*	X^*	X_p^o	Q_p^*	Q^*
P_H	$-$	0	$-$	$-$	0
P_L	$-$	0	$-$	$+$	0
δ_0	$+$	$+$	$+$	$+$	$+$
δ_1	$+$	$+$	$+$	$\pm$	0
γ	$-$	$-$	$+$	$-$	$+$
α	$\pm$	$\pm$	$-$	$+$	$+$
σ	$+$	$+$	$+$	$+$	$+$

Table 2: Summary of the comparative statics.

4 Conclusion

We propose a novel solution for an investment opportunity with simultaneous determination of optimal capacity and the price threshold when there is a price floor and ceiling. We show that higher price caps reduce the thresholds but also the optimal capacity. However, higher price floors also reduce the thresholds, and also raise the optimal capacity, producing a desirable outcome. Increased price volatility raises both the price threshold and the optimal capacity. Thus, a government wanting to motivate early investment, with large capacity to avoid congestion and crowding, could possibly achieve both objectives through appropriate price collars, but not through reducing volatility.

Several extensions are possible, namely extending the finite and retractable collars models of Adkins et al. (2019) to optimal capacity choice, or considering partial collars (caps or floors).

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