MTRC's Novel Infrastructure Financing Model: Rationale Based on a Stackelberg Game of Timing under Uncertainty

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Abstract

Because of debt concerns, many cities face challenges in financing their infrastructure. Hong Kong's transit operator designed a novel scheme to exploit the positive externalities of public transport on real estate prices. We develop a Stackelberg leader-follower game of timing under uncertainty to explore the rationale of this scheme. Our main findings are that internalizing positive externalities provides additional revenue sources for defraying the overall costs of infrastructure investments, thereby accelerating the delivery of infrastructure; in a multi-player stopping game, the equilibrium investment times are not typical first-hitting times. This study provides theoretical insight into infrastructure planning and financing based on compound (growth) options.

Keywords: Stopping game, compound (growth) options, Stackelberg game, positive externalities, infrastructure investments

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1. Introduction

Well-functioning infrastructure is an important driver of a city's attractiveness and competitiveness in the present globalized economy. New infrastructure projects enhance connections among places of residence, work, and leisure and generate growth options for the economy as a whole.

Infrastructure projects should be affordable to users, high-quality, and self-sustainable. To ensure affordability, transit fares are often regulated and set below the total costs of capital investments, operations, and maintenance; this reduces the incentive to deliver high-quality and self-sustainable infrastructures and calls for subsidizing by a social planner. Yet, exacerbated by sovereign debt concerns, public subsidizing for infrastructure projects conflicts with other social and economic objectives (e.g., welfare programs). Financing is often a bottleneck in the provision of infrastructure. Innovative funding approaches to infrastructure investments are needed.

Dixit and Pindyck [1994] stress three common characteristics among investment projects (including infrastructure): irreversibility, uncertainty, and flexibility in timing.[1] In addition to these features, infrastructure projects are normally capital intensive, requiring substantial upfront costs as well as high (fixed) operations and maintenance expenses. The huge capital investments and periodic expenditures may make infrastructure investors reluctant to launch new projects—especially if capital providers are short-sighted whereas the benefits from a project are reaped over more than 30 years and/or if the

[1] First, start-up costs are (at least partially) irreversible in the sense that expenditures will not be recovered if the project is reversed. Second, rewards from infrastructure investments are uncertain. Third, investors have some flexibility about the timing of their infrastructure investments.

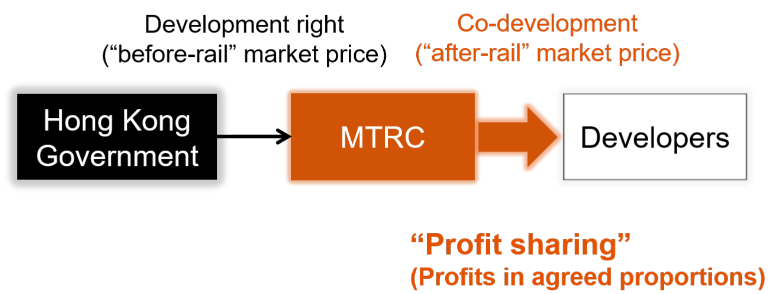

Figure 1: **R+P program: Interplay among the HK government, MTRC, and private real-estate developers.** Based on Cervero and Murakami [2009].

infrastructure investor does not internalize all the benefits to society. Because of these characteristics, an infrastrture provider is more prone to delay a project launch until its value is sufficiently larger than its initial investment costs (in the spirit of real options). To accelerate the delivery of infrastructure and to bolster the growth of the economy, the search for new revenue sources becomes a top priority.

On the above matters, a great deal can be learned from Hong Kong's (HK's) recent experience in public transportation. HK is among the few cities in which rail transit operations are highly profitable without government subsidizing. Its financial success rests on the "*Rail Plus Property*" (R+P) financing model developed by the HK's railway operator, the Mass Transit Railway Corporation ("MTRC"). Under the R+P program, the government grants MTRC exclusive property development rights of government-owned land around rail transit stations at a "before-rail" market price. MTRC then makes transit investments and captures the land appreciation created by the R+P model through further granting development rights to private developers at an "after-rail" market price, jointly developing the land and property, and sharing profits generated by property development in agreed proportions. The land premiums (i.e., the difference between the before-rail and the after-rail market prices) and the shared profits are used to recoup the capital, operations, and maintenance costs of railway projects.

Incorporating positive externalities into the income stream of a transit operator is an innovative approach to designing, planning, and financing capital-intensive infrastructure. Public transport improvements are known to boost the demand for housing nearby. However, under a traditional transit operations model, the operator (e.g., MTRC) rarely benefits from greater housing demand. If an infrastructure project is simply evaluated on the basis of fare revenues, a project which is environmentally and socially favorable but lacks financial profitability is likely to be rejected by private investors straight away, especially if the government does not subsidize or guarantee any minimum rewards. In that case, a springboard investment that would be socially beneficial would not be undertaken until its project value is "deep in the money." In other words, unless the future rewards considerably exceed the sunk costs, the financing issues will not be completely tackled.

It would be advised in the appraisal of an infrastructure investment project to value the follow-on options derived from an early investment. Infrastructure investments open up a series of valuable investment opportunities, also to external parties. The R+P development program presents a new perspective on infrastructure investments because MTRC takes the external economic benefits into consideration, internalizing the externality by granting development rights to private developers as well as sharing the profits with them provided by the subsequent property development in an agreed proportion. MTRC capitalizes on the commercial real estate options derived from the first-stage transit investment opportunity and captures the land value appreciation, viewing the value of property development as a part of the overall value of an infrastructure investment.

Uncertainty is a key driver of infrastructure investments; yet, the MTRC's novel financing model adds an element of strategic uncertainty because of the interactions and synergies with private property developers. This novel design leads naturally to a sequential game situation in which the leader and follower roles are predetermined. The follower is restrained from taking action until the leader has already done so. In this paper, the problem faced by MTRC is modelled as a Stackelberg leader-follower game of timing under uncertainty. More specifically, we model the rationale behind this innovative infrastructure financing scheme, using notions borrowed from real options (in particular compound options) and game theory (in the context of Stackelberg game of timing). We focus on continuous-time models of irreversible infrastructure investment under uncertainty, stress the factors that affect the timely delivery of infrastructure provisions, and derive the equilibrium investment rules by using dynamic programming.

2. Literature review

Myers [1984] recalls the connection between (a) capital budgeting, which is concerned with project assessment, typically based on the discounted cash flows (DCF), and (b) strategic planning, whose primary objective is to determine the investment decisions that best achieve a long-term objective. Both perspectives are embedded in Real Options Analysis (ROA). Real options are coined in analogy with financial options [Myers, 1977]: a real option gives its holder an opportunity to acquire a real asset (e.g., to invest in a project) at a prespecified cost if conditions turn favorably [Trigeorgis, 1996]. A key benchmark model is the seminal optimal investment timing problem in McDonald and Siegel [1986]. In our model, we deal with a problem where an early investment opens up a chain of further projects; this class of problems is coined "growth options," "options on options," or "compound options" in the literature (Kester [1984], Trigeorgis and Mason [1987]).

There have been some real options papers dealing with infrastructure investments. Smit and Trigeorgis [2009] illustrate practical application of the option games approach in the airport industry context by developing discrete-time models that involve two privatized European airports competing in the times to expand their own lumpy infrastructure capacity. Ukkusuri and Patil [2009] model the demand uncertainty as a scenario tree, analyzing the optimal transportation network investments decision problem over multiple time periods. Besides discrete-time cases, many papers model uncertainty in continuous time as a stochastic process (e.g., a geometric Brownian motion) (see, e.g., Gao and Driouchi [2013], Li et al. [2015]). Most of them specify the value of the investment as a function of the target strategy chosen, then determine the optimal transit investment timing given the investment payoff function. The investment timing problem is thus converted into the determination of the optimal investment trigger. Once a stochastic process followed by an underlying asset or factor (e.g., the project value, demand, and population) reaches a specified barrier selected by the investor ex ante, the investment project is implemented as well as the investment timing is found. In such cases, the optimal investment timing is precisely the first-hitting time as they address the optimal infrastructure investment issue in a monopoly situation over one decision period. In a multi-player stopping game, however, the equilibrium investment times are not typical first-hitting times. Guo et al. [2018] consider decision makers' time preferences and beliefs by modeling authorities' intertemporal choices on the basis that the planning horizon in public transit investment, such as 20 or 30 years, is excessively longer than the election cycles of government officers (e.g., 4-5 years). While they analyze the impact of intertemporal decisions, they do not consider the impact of positive externalities derived from strategic interactions and the duopoly model of Stackelberg leader-follower game in an irreversible capital-intensive infrastructure investment under uncertainty, which is a main focus of our study.

We here focus on the parties' investment decisions, allowing for two income sources for the infrastructure investor. The issue of debt instruments (e.g., straight debt vs project finance) is beyond the scope of this paper. More specifically, this paper elaborates upon a real-world case in the infrastructural industry based on the Stackelberg leader-follower game of timing under uncertainty, providing a compelling economic rationale for a successful infrastructure financing model in HK. Our approach is rooted in the real options analysis (ROA) and leverages on the dynamic programming in the operations research literature as well as the optimal control theory in the mathematics literature by solving variational inequalities [see Bensoussan and Lions, 1982]. We further extend our discussion in depth on the leader's problem that involves more complex continuation/stopping sets based on the research study by Bensoussan et al. [2010]. This is a key novel contribution versus the extant literature.

3. Traditional financing models

Before analyzing the novel R+P model, we set a couple of benchmarks by examining the most common infrastructure financing models, namely "user pays," "government pays," or a combination of these two. We have a greater emphasis on the user-pays model which MTRC uses in conjonction to their novel financing model.

3.1. "User-pays" model

Model setup. Under user-pays transit investments, fare revenues are the sole income stream for the transit operator (e.g., MTRC). For simplicity, we assume these revenues to be deterministic.[2] Specifically, we model the (perpetuity) value of transit operations at time t, Y_t^y, as the solution to a first-order ordinary differential equation (ODE), namely

$$Y_0^y = y, \tag{1a}$$

$$\mathrm{d}Y_t^y = \rho Y_t^y \mathrm{d}t, \tag{1b}$$

where ρ is a constant drift parameter. Equation (1b) implies that the project value grows compoundly at a constant rate of ρ per unit of time. At some fu-

[2]The assumption is reasonable because the demand for railways, metro, and buses services is hardly elastic and less influenced by business and economic industry cycles. In addition, transportation prices are highly regulated because transport is considered a common good. The fare revenue growth derives primarily from population growth, which is mostly predictable.

ture time (to be determined) the operator pays the construction cost I_1 to set up a new metroline. Because the construction of a large-scale infrastructure project generally requires substantial time from initiation until completion, we consider a time-to-build feature: $h_1 > 0$ is the lag in constructing a new metroline and the operator receives a payoff of $Y^y_{t+h_1}$ if he/she invests at time t. We let r stand for the constant discount rate.

The situation faced by an operator under the user-pays model is a comparatively simple (deterministic) problem of investment timing: the metro operator must determine the time $t \geq 0$ at which to incur a construction cost I_1 in return for a value $Y^y_{t+h_1}$ once construction is completed:

$$v(y) := \sup_{t \geq 0} \quad e^{-rt} \Big\{ e^{-rh_1} Y^y_{t+h_1} - I_1 \Big\}. \tag{2}$$

We assume that the discount rate $r > \rho$ to ensure that $v(y) < \infty$.

Dynamic programming. The problem (2) can be analyzed using dynamic programming.[3] We let

$$\varphi(y) := e^{-rh_1} Y^y_{h_1} - I_1 = e^{-(r-\rho)h_1} y - I_1 \tag{3}$$

denote the terminal payout received when investing. Here, the rail operator must decide whether to initiate ("stop") or delay the investment ("continue"). If the operator faces such an alternative, then its value—which corresponds to the optimal choice—must be no less than the payoff from

[3] Alternatively, we can view it as a static optimization problem. Using dynamic programming allows highlighting similarities when dealing with the stochastic game of timing in a later section.

either course of action. We now consider each alternative action in turn. Given flexibility in timing, the rail operator cannot be worse off than investing straight away; it must be that $v(y) \geq \varphi(y)$ for all $y \geq 0$. In addition, by Bellman's (1957) "principle of optimality," the value must exceed the payoff:

$$v(y) \geq e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big), \quad \varepsilon > 0.$$

If $v(\cdot) \in C^1(\mathbb{R}_+)$, we can let ε go to 0 in the above. At any given point y, one weak inequality must be strict and the other is an equality; this heuristic leads a "complementarity slackness" criterion. In short, the value function $v(y)$ must satisfy

$$\min\big\{rv(y) - \rho y v'(y); v(y) - \varphi(y)\big\} = 0, \quad \forall y > 0. \tag{4a}$$

The dynamic programming equation (4a) is called a variational inequality (VI) following the terminology introduced in Bensoussan and Lions [1982]. Economic arguments also lead to two additional conditions. The condition

$$\lim_{y \downarrow 0} \quad v(y) = 0 \tag{4b}$$

asserts that the project is worthless if the users' pool vanishes. We further assume that

$$\lim_{y \uparrow \infty} \quad \frac{v(y)}{\varphi(y)} = 1, \tag{4c}$$

which implies that when the real option is "very deep in the money," the real options value coincides with the net present value $\varphi(y)$.

We solve the problem (4a)–(4c) in Theorem 1 (see proofs in Appendix A).

Theorem 1. *The transit operator's value function* (2) *takes the form*

$$v(y) = \begin{cases} \left(\frac{y}{\bar{y}}\right)^{r/\rho} \left(e^{-(r-\rho)h_1}\bar{y} - I_1\right), & y < \bar{y}, \\ e^{-(r-\rho)h_1} y - I_1, & y \geq \bar{y}, \end{cases}$$

where the investment threshold $\bar{y}$ *is given by*

$$\bar{y} = \frac{re^{(r-\rho)h_1}}{r-\rho} I_1 > I_1.$$

Following Theorem 1, the (optimal) decision whether to invest relates to the relative positions of the project value y and of the threshold $\bar{y}$. Alternatively, we can express the optimal stopping rule as $\hat{t}(y, \bar{y}) = \frac{1}{\rho} \ln \left(\frac{y \vee \bar{y}}{y}\right)$. This form hightlights a relationship between the optimal time $\hat{t}(y)$ and the threshold $\bar{y}$. The firm should invest straight away $[\hat{t}(y, \bar{y}) = 0]$ if and only if $y \geq \bar{y}$. Besides, $\bar{y} > e^{(r-\rho)h_1} I_1$, which is the "zero NPV" threshold. Consequently, the transit operator will defer the investment until the fare revenues are "deep in the money" (not "at the money"). The effect—coined *hysteresis* in the literature—arises even though the transport operator is certain about the growth of its future revenue stream.

Unfortunately, in most cities (Murakami [2012]), fares are often regulated and small to keep transit affordable. It is thus difficult to recoup the investment costs, so that many infrastructure projects are postponed, which is not socially desirable.

3.2. *"Government-pays" and mixed models*

In the *government-pays* model, the sole revenue source of a private investor is governmental transfer payment, i.e., ultimately sourced from the public budget (through taxing or borrowing). Hereby, the public and private parties reach an agreement in which the public party promises to acquire an infrastructure asset from the private party at some specific time and price. It is analogous to a *forward* contract. Government pays delay public expenditures that would appear on the liability side of the government's "balance sheet." Practically, the government engages a private company to develop an infrastructure asset or render infrastructure services, yet the cost of infrastructure is ultimately met from the public purse. Moreover, private funds are not a source of revenues but a way of raising funds, similar to a loan committed to repay the lender. They are still needed to be paid by public budgets in the end. The financial burden on the government entity is likely to raise the sovereign debt level and lead to greater fiscal pressure on households, not necessarily living in the vicinity of the infrastructure.

Besides the (pure) user- and government-pays models, a mix of these two is also widely used in practice. Here, user charges will ultimately be the main revenue sources yet are supplemented by government transfers paid at specific construction milestones. The government guarantee helps securitize the future streams of revenue for the private investor when the prespecified downside events occur, providing strong support for the government to channel more private funds in infrastructure. Further, if the project value is assumed to be stochastic rather than deterministic, a guarantee approximates to a *put option* that protects the private investor from downside risks, at the expense of the public sector (and ultimately at the expense of households). If there is no expiry, such a payment mechanism for the private party can be approximately modeled as the perpetual American put option written on the value of transit operations. While the government guarantee protects the private entity's interests against unfavorable conditions and provides the solid basis for attracting private funds, it implicitly increases the governments fiscal exposures in the form of contingent liabilities. Therefore, the mixed model increases the risk of government bankruptcy particularly when the government is facing high leverage ratios.

In summary, the three widely-used traditional financing models have various constraints to a varying degree. To accelerate infrastructure delivery, more effective financing mechanisms are required. We next formalize MTRC's innovative financing model.

4. MTRC's R+P scheme

Besides fare revenues modeled by (1a)–(1b), MTRC derives an income from the trading of the development rights to a property developer (or a consortium of property developers). For simplicity, we assume that the private developer does not invest in properties unless they are located near metro-lines. The developer's decision whether to acquire those rights depends on an average after-rail *property value* $X^x : \Omega \times [0, \infty) \to \mathcal{X} := \mathbb{R}_+$, which is assumed to follow a geometric Brownian motion (GBM) of the form

$$X_0^x = x > 0, \qquad \mathbb{P} - \text{a.s.}, \tag{5a}$$

$$\mathrm{d}X_t^x = \mu X_t^x \,\mathrm{d}t + \sigma X_t^x \,\mathrm{d}Z_t, \qquad t > 0, \tag{5b}$$

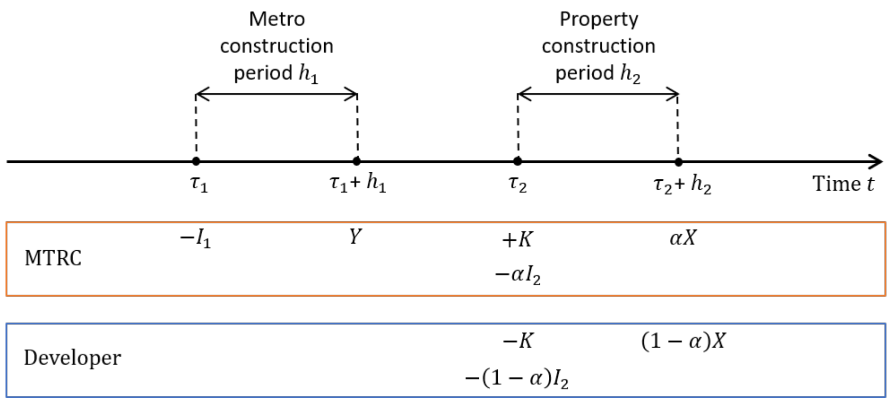

Figure 2: **Model timeline**

where $\mu > \frac{1}{2}\sigma^2$ and $\sigma > 0$ are the constant drift and volatility parameters respectively and Z is a standard Brownian motion.[4]

We model the MTRC's novel R+P scheme as a Stackelberg game in which the leader and follower roles are set ex ante. We depict the timeline in Figure 2 and use the index i to denote a particular party. MTRC ($i = 1$) starts the construction of the new metroline at a (stopping) time τ_1, while the property developer ($i = 2$) acquires the development rights and starts building the properties around the station at a time $\tau_2 \geq \tau_1$ for a cost I_2. MTRC agrees with the private developer on a fee K and a profit-sharing rule according to which a proportion $\alpha \in (0, 1)$ of the net proceeds accrues to

[4] In HK, land values are strongly driven by rapid urban population expansion and strong economic growth [Hong and Brubaker, 2010]. Besides population, property values depend on various macroeconomic factors such as housing scarcity, change in housing policies or regulations, the yield differential compared to other asset classes, and foreign-exchange rates. Modeling the feedback effect of population growth on real-estate prices is beyond the scope of this paper; for simplicity, we treat the parameters ρ in (1b) and μ in (5b) independently of one another. For the sake of illustration, a certain dependency is accounted for because we use estimates from real-world projects.

MTRC.[5] We further consider a time lag $h_2 > 0$ for the planning, building, and reselling of the properties; the property after completion is worth $X^x_{\tau_2+h_2}$. No party can perfectly forecast the development of the property price and form instead expectations (we use the operator $\mathbb{E}$ under the probability measure $\mathbb{P}$). Both parties are assumed to be risk-neutral for simplicity and discount at a constant rate $r > \min\{\rho, \mu\}$.

We now specify the two parties' *objective functionals*.

1. Follower: The property developer's objective is

$$J^x_2(\tau_1, \tau_2) := \mathbb{E}\Big[e^{-r\tau_2}\Big\{(1-\alpha)\big(e^{-rh_2}X^x_{\tau_2+h_2} - I_2\big) - K\Big\}\mathbf{1}_{\{\tau_2 \geq \tau_1\}}\Big], \quad (6)$$

   where the indicator $\mathbf{1}_{\{\tau_2 \geq \tau_1\}}$ accounts for the "Stackelberg constraint" $\tau_2 \geq \tau_1$.

2. Leader: MTRC has two income streams:
   (i) Fare revenues from *direct beneficiaries* (i.e., metro passengers) generate a net present value of $\varphi(y)$ as introduced in (3).
   (ii) MTRC internalizes a share of the positive externalities on *indirect beneficiaries*. The net present value of this second income stream is

$$\begin{aligned}\psi(x) &:= \alpha\big(e^{-rh_2}X^x_{h_2} - I_2\big) + K \\ &= \alpha\big(e^{-(r-\mu)h_2}x - I_2\big) + K. \quad (7)\end{aligned}$$

[5] The agreed proportion α may be thought of as the outcome (e.g., the Nash bargaining solution) of a negotiation between the leader and follower taking place at time τ_2. We instead assume it as a parameter, an assumption which is reasonable if MTRC auctions the development rights because, then, the terms are not negotiated.

MTRC's objective is

$$J_1^{x,y}(\tau_1,\tau_2) := \mathbb{E}\Big[e^{-r\tau_1}\varphi\big(Y_{\tau_1}^y\big) + e^{-r\tau_2}\psi\big(X_{\tau_2}^x\big)\Big]. \tag{8}$$

We specify the *solution concept* for this Stackelberg game of timing. We assume that players follow Markov strategies and use the superscripts x and y to highlight this ansatz. The leader anticipates that the follower will react optimally to its choice $\tau_1^{x,y}$ with a reaction $\mathcal{T}_2\big(\tau_1^{x,y}\big)$ assumed to be the unique solution to

$$J_2^x\Big(\tau_1^{x,y},\mathcal{T}_2\big(\tau_1^{x,y}\big)\Big) = \sup_{\tau_2^x \geq \tau_1^{x,y}} J_2^x(\tau_1^{x,y},\tau_2^x). \tag{9}$$

Given the investment sequence, the leader faces a decision-theoretic problem:

$$V_1(x,y) := J_1^{x,y}\Big(\hat{\tau}_1^{x,y},\mathcal{T}_2\big(\hat{\tau}_1^{x,y}\big)\Big) = \sup_{\tau_1^{x,y}} J_1^{x,y}\Big(\tau_1^{x,y},\mathcal{T}_2\big(\tau_1^{x,y}\big)\Big). \tag{10}$$

We call $\big(\hat{\tau}_1^{x,y},\mathcal{T}_2^x(\hat{\tau}_1^{x,y})\big)$ the game's Markov Stackelberg equilibrium. To solve for this, we proceed in the reverse investment order, determining the follower's reaction function in (9) first.

5. Property developer's decision

After some computations (see Appendix B), one shows that the follower's reaction function in (9) can be written in the form

$$\mathcal{T}_2^x(\tau_1^{x,y}) = \tau_1^{x,y} + \theta_2\big(X_{\tau_1^{x,y}}^x\big); \tag{11}$$

here, $\theta_2(x)$ is the solution to a "*myopic*" problem that becomes relevant once the leader starts construction works, namely the solution to[6]

$$\begin{aligned} V_2(x) := & \ \mathbb{E}\Big[e^{-r\theta_2(x)}\Big\{(1-\alpha)\big(e^{-(r-\mu)h_2}X^x_{\theta_2(x)} - I_2\big) - K\Big\}\Big] \\ = & \ \sup_{\tau_2^x} \ \mathbb{E}\Bigg[e^{-r\tau_2^x}\ \Big\{(1-\alpha)\Big(e^{-(r-\mu)h_2}X^x_{\tau_2^x} - I_2\Big) - K\Big\}\Bigg]. \end{aligned} \tag{12}$$

We now solve for $V_2(x)$ in the myopic problem (12) by using an approach similar to the one used to solve the deterministic problem (2). Given flexibility in timing, the private investor cannot be worse off than investing immediately: $V_2(x) \geq (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K$. Alternatively, the property developer can stay put for a period of time $\varepsilon > 0$ and then pursues the optimal stopping strategy $\theta_2(\cdot)$; this eventually yields the necessary condition $\mathcal{L}_2 V_2(x) \geq 0$, where $\mathcal{L}_2$ is a second-order operator given by[7]

$$\mathcal{L}_2 f := rf - \mu x \frac{\partial f}{\partial x} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 f}{\partial x^2}. \tag{13}$$

At a given point x, one weak inequality must be strict and the other is an equality; this heuristic leads a "complementarity slackness" criterion. In short, the value function V_2 must solve the dynamic programming equation given in Lemma 1.

[6]The term $V_2(x)$ in (12) should not be confused with $J_2^x\big(\hat{\tau}_1^{x,y}, \mathcal{T}_2^x(\hat{\tau}_1^{x,y})\big)$.

[7]The "waiting" stance leads to the inequality $V_2(x) \geq \mathbb{E}\big[e^{-r\varepsilon}V_2(X_\varepsilon^x)\big]$. As $\varepsilon \to 0$, then it obtains from (a generalized version of) Dynkin's formula [see Bensoussan and Lions, 1982, Theorem 8.5, pp.185-186] the inequality $\mathcal{L}_2 V_2(x) \geq 0$.

Lemma 1. *The value function V_2 in* (12) *must satisfy the VI* [8]

$$\min\Big\{\mathcal{L}_2 V_2(x); V_2(x) - (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) + K\Big\} = 0, \quad a.e.\ x \in \mathcal{X}, \tag{14a}$$

$$V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+) \quad and \quad V_2''(\cdot) \in \mathsf{L}^1_{loc}(\mathbb{R}_+), \tag{14b}$$

$$\lim_{x\to 0} \quad V_2(x) = 0, \tag{14c}$$

$$\lim_{x\to\infty} \quad \frac{V_2(x)}{(1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K} = 1. \tag{14d}$$

We introduce the positive and negative roots, γ_A and γ_B respectively, of the quadratic function

$$\mathcal{Q}(\gamma) := r - \mu\gamma - \frac{1}{2}\sigma^2\gamma(\gamma-1) \tag{15}$$

and provide an explicit functional representation of the value function V_2 in Theorem 2.[9]

Theorem 2. *We can express the value function V_2 in* (12) *as*

$$V_2(x) = \begin{cases} \left(\frac{x}{x_2}\right)^{\gamma_A}\Big[(1-\alpha)\big(e^{-(r-\mu)h_2}x_2 - I_2\big) - K\Big], & x < x_2, \\ (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K, & x \geq x_2, \end{cases} \tag{16}$$

[8] Compared to the regularity requirement $V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+)$, which we also had in the deterministic case, we introduce another condition $V_2''(\cdot) \in \mathsf{L}^1_{\text{loc}}(\mathbb{R}_+)$ in (14b) to ensure that the second-order term in the VI has a mathematical meaning. Smooth fit is a natural consequence of the assumption $V_2(\cdot) \in \mathsf{C}^1(\mathbb{R}_+)$.

[9] We omit a verification theorem. It is known [see, e.g., Bensoussan and Lions, 1982] that a value function of optimal stopping, e.g., (12), is the probabilistic representation of the solution to a variational inequality, e.g., (14a)–(14d).

where

$$x_2 := \frac{\gamma_A}{\gamma_A - 1}\Big(I_2 + \frac{K}{1-\alpha}\Big)e^{(r-\mu)h_2}. \tag{17}$$

To interpret Theorem 2, we recall that that one can associate to a first-hitting time $\inf\{t \geq 0 : X_t^x \geq \xi\}$ a discount factor (over states) given by

$$\mathbb{E}\Big[e^{-r\inf\{t\geq 0: X_t^x \geq \xi\}}\Big] = \Big(\frac{\min\{x;\xi\}}{\xi}\Big)^{\gamma_A}. \tag{18}$$

If the property value x is above x_2, the (myopic) property developer acquires the development rights, receiving the amount $(1-\alpha)(e^{-(r-\mu)h_2}x - I_2) - K$. If the property value is below that threshold, then the firm delays receiving the amount $(1-\alpha)(e^{-(r-\mu)h_2}x_2 - I_2) - K$ until the (first-hitting) time

$$\theta_2(x) := \inf\big\{t \geq 0 : X_t^x \geq x_2\big\}; \tag{19}$$

it discounts this amount using the factor $(x/x_2)^{\gamma_A}$.

In reality, the threshold x_2 is not the threshold above which the private developer develops the property. Indeed, the private investor wants to ensure MTRC has invested in urban rail infrastructure. In other words, we do not claim that the follower's reaction $\mathcal{T}_2^x(\tau_1^{x,y})$ is the first-hitting time in (19), but claim instead—because of the relation (11)—that it is of the form

$$\mathcal{T}_2^x(\tau_1^{x,y}) = \inf\Big\{t \geq \tau_1^{x,y} : X_t^{X^x_{\tau_1^{x,y}}} \geq x_2\Big\}. \tag{20}$$

To be able to specify the follower's equilibrium decision, we will need to solve the leader's problem (10) and determine the leader's optimal investment decision $\hat{\tau}_1^{x,y}$.

6. MTRC's problem

Lemma 2 specifies a VI for the leader's problem in (10). We interpret the term $G_1(x,y)$ given by

$$G_1(x,y) := \begin{cases} \varphi(y) + \left(\frac{x}{x_2}\right)^{\gamma_A} \psi\big(x_2\big), & x < x_2, \\ \varphi(y) + \psi(x), & x \geq x_2, \end{cases} \tag{21}$$

as the NPV from investing straight away.[10] If MTRC invests it receives net fare revenues $\varphi(y)$ as well as the (future) proceeds from a transaction with the property developer. These proceeds are $\psi(x)$ if the developer acquires the development rights straight away or are worth the present value $\left(\frac{x}{x_2}\right)^{\gamma_A} \psi\big(x_2\big)$ if the transaction is delayed until the process reaches the level x_2. Beside $\mathcal{L}_2$ in (13), we introduce the operator

$$\mathcal{L}_1 f := \mathcal{L}_2 f - \rho y \frac{\partial f}{\partial y}. \tag{22}$$

Lemma 2. *The leader's value function V_1 in* (10) *satisfies the VI*

$$\min\big\{\mathcal{L}_1 V_1(x,y); V_1(x,y) - G_1(x,y)\big\} = 0, \quad \textit{a.e. } x \in \mathcal{X}, \tag{23a}$$

$$V_1 \in \mathsf{C}^1(\mathbb{R}^2_+) \quad \textit{and} \quad \frac{\partial^2 V_1}{\partial x^2} \in \mathsf{L}^1_{loc}(\mathbb{R}_+). \tag{23b}$$

We recall that if a solution to the VI exists (and the solution to the VI coincides with the value function), then the continuation region is defined implicitly as $\mathcal{C}_1 := \big\{(x,y) \in \mathbb{R}^2_+ : V_1(x,y) > G_1(x,y)\big\}$, while the stopping region is $\mathcal{S}_1 := \mathbb{R}^2_+ \setminus \mathcal{C}_1$.

[10]The function G_1 is called the *obstacle* in line with the literature on optimal stopping.

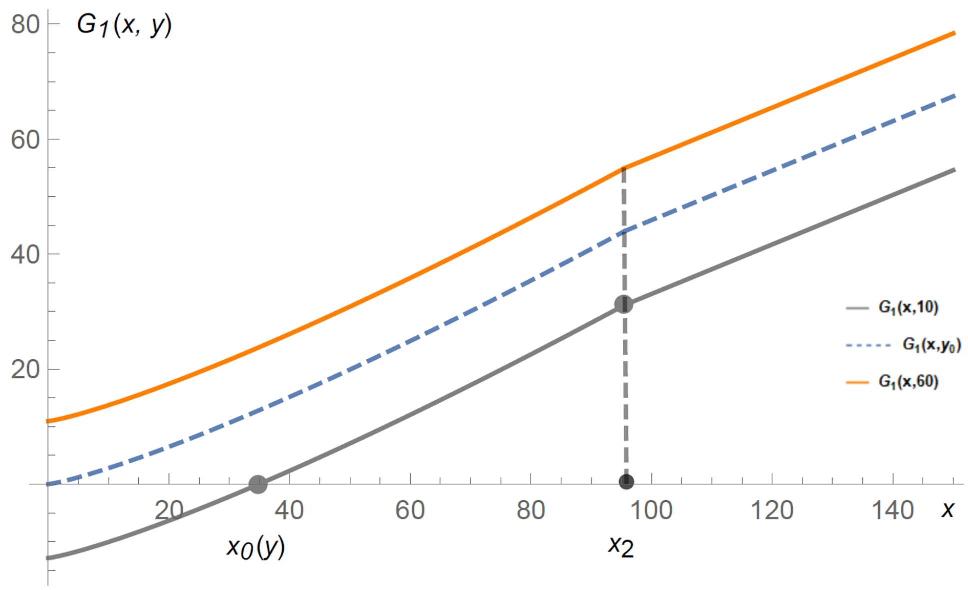

Figure 3: The dashed curve for $x \mapsto G_1(x, y)$ depicts the "real" case with $y_0 = e^{(r-\rho)h_1} I_1 = 36.9808$. The gray ($y = 10$) and orange ($y = 60$) curves are introduced for comparative statics. Cf. parameter values in Footnote 11.

We start by studying the obstacle G_1 in (21). We depict this function in Figure 3 using as parameter values those of the "SIL(E)" project in HK.[11] We can conclude after some computations that:

Lemma 3. *The obstacle* $x \mapsto G_1(x, y)$ *is continuously differentiable on* (x, ∞) *except at* x_2*. It is monotone increasing on* $(0, \infty)$ *from* $\varphi(y)$ *to* ∞*. It is convex on* $(0, x_2)$*, concave in the vicinity of* x_2*, and linear on* (x_2, ∞)*. We*

[11] The South Island Line (East) ("SIL(E)") project is a 7-km line costing $I_1 = 17.6$ HKDbn and financed under the R+P model. The construction commenced in May 2011 and completed in December 2016. In December 2017, MTRC awarded the property development package to a consortium for a land premium of $K = 5.2$ HKDbn. The development project is due for completion by 2023. The metropolitan area's population grew at 1.5% p.a. between 2010 and 2025, while the HK property price grows at 12% p.a. [Suzuki et al., 2015]. Other parameter values are $I_2 = 4.66$ HKDbn, as $\alpha = 0.5$, $r = 0.15$, $\mu = 0.12$, $\rho = 0.015$, $\sigma = 0.15$, $h_1 = 5.5$, and $h_2 = 5$.

introduce the parameters $y_0 := e^{(r-\rho)h_1} I_1$ *and* $y_\dagger := e^{(r-\rho)h_1}\big[I_1 - \psi(x_2)\big]^+$ *and define the curve* $x_0(\cdot)$ *by*

$$x_0(y) := \begin{cases} \left[\frac{I_1+\alpha I_2-K-ye^{-(r-\rho)h_1}}{\alpha}\right] e^{(r-\mu)h_2}, & y \in (0, y_\dagger), \\ x_2\left[-\frac{\varphi(y)}{\psi(x_2)}\right]^{\frac{1}{\gamma_A}}, & y \in (y_\dagger, y_0), \\ 0, & y \in (y_0, \infty). \end{cases}$$

The function $x \mapsto G_1(x, y)$ *is negative on* $(0, x_0(y))$ *and positive on* $(x_0(y), \infty)$*.*

Recall that $G_1(x, y)$ is the NPV from the infrastructure project. The parameter y_0 corresponds to the break-even point above which the project is worth undertaking based solely on the stream of fare revenues, while the level $x_0(y)$ is the cut-off above which the proceeds from the trading of development rights is sufficiently large to offset the shortfall in fare revenues and yield a positive NPV for MTRC overall.

The function $x \mapsto G_1(x, y)$ is not continuously differentiable at x_2. As we look for a solution to the VI (23a) that satisfies the regularity (23b), a set of state values (x, y) where the obstacle $x \mapsto G_1(x, y)$ is not continuously differentiable cannot be in the stopping region (in which $V_1 = G_1$). Consequently, because $x \mapsto G_1(x, y)$ is not continuously differentiable at x_2, the set $\{(x, y) : (0, x_1(y))\}$ with $x_1(y) < x_2$ cannot be the continuation region. We focus first on the case with a threshold policy such that $x_1(y) \geq x_2$. If the conjecture $x_1(y) \geq x_2$ holds, then MTRC and the private developer effectively decide on their own investments at the same time; this is because the fare revenues y are not sufficient to incentivize MTRC to develop the infrastructure. If the conjecture $x_1(y) \geq x_2$ does not hold, then MTRC will adopt a more subtle investment policy.

6.1. Case A with a threshold policy such that $x_1(y) \geq x_2$

We introduce the quadratic function

$$\mathcal{D}(\gamma) := r - \rho - (\mu - \rho)\gamma - \frac{1}{2}\sigma^2\gamma(\gamma - 1), \tag{24}$$

and denote by δ_A and δ_B its positive and negative roots, respectively. We further introduce

$$y_\star := e^{(r-\rho)h_1}\Big\{ - \Big[\frac{\gamma_A}{\gamma_A - 1}\frac{\delta_A - 1}{\delta_A}\alpha\Big(I_2 + \frac{K}{1-\alpha}\Big)\Big] + I_1 + \alpha I_2 - K\Big\} \tag{25a}$$

$$x_1(y) := \frac{\delta_A}{\delta_A - 1}\frac{(I_1 + \alpha I_2 - K) - e^{-(r-\rho)h_1}y}{\alpha}e^{(r-\mu)h_2}, \quad y \in (0, y_\star) \tag{25b}$$

$$\alpha_\star := \frac{(a+b+c) - \sqrt{(a+b+c)^2 - 4ac}}{2a} \in (0,1) \tag{25c}$$

where

$$a := I_2(\delta_A - \gamma_A) > 0, \quad b := \gamma_A K(\delta_A - 1) > 0, \quad \text{and} \quad c := \delta_A(\gamma_A - 1)\big(I_1 - K\big) > 0. \tag{25d}$$

We are able to establish the following theorem in Case A:

Theorem 3. *We make the assumptions*

$$\mu - \rho - \frac{1}{2}\sigma^2 > 0, \tag{26a}$$

$$I_1 + \alpha I_2 > K > \alpha I_2, \tag{26b}$$

$$0 < \alpha < \alpha_\star. \tag{26c}$$

Then, if

$$0 < y < y_\star, \tag{26d}$$

the value function V_1 *in* (10) *is given by*

$$V_1(x,y) = \begin{cases} \left(\frac{x}{x_1(y)}\right)^{\delta_A} \Big[\varphi(y) + \psi\big(x_1(y)\big)\Big], & x < x_1(y), \\ \varphi(y) + \psi(x), & x \geq x_1(y). \end{cases} \tag{26e}$$

According to Theorem 3, Case A arises when fare revenues y are limited—as per (26d)—and only a small share the proceeds accrues to the operator—as per (26c). Here, the operator would not invest in the metroline unless it ascertains a willingness by the property developer to invest, for $x \geq x_1(y)$. Other restrictions apply.[12] From Theorem 3, we characterize MTRC's optimal investment rule in Case A as the first-hitting time

$$\hat{\tau}_1^{x,y} = \inf\big\{t \geq 0 : X_t^x \geq x_1(y)\big\}, \quad \text{if } 0 < y < y_\star. \tag{27}$$

In other words, MTRC should invest if the property value exceeds the level $x_1(y) > x_2$. This case illustrated in Figure 4 yields interesting economic insights. First, when MTRC invests, the property developer immediately

[12] According to (26a), the excess return $\mu - \rho$ from property investment must be sufficient to compensate for the volatility σ in property prices. Further, from (26b), MTRC must charge a price K to the property developer that is sufficient to cover the operator's cost of participating in the real-estate business (αI_2), yet not too large to completely offset its own (total) investment cost, $I_1 + \alpha I_2$. Both conditions (26b) and (26c) impose a fair value redistribution between the two parties. This also relates to the notion of *co-opetition* (cf. Brandenburger and Stuart [2007], Trigeorgis and Reuer [2017]) because it mixes elements of cooperation and competition.

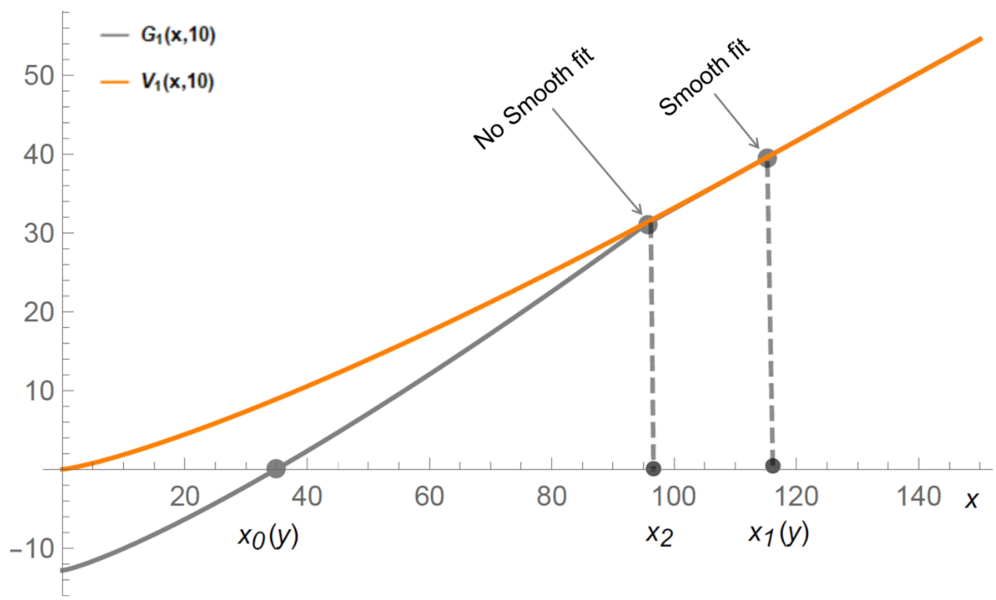

Figure 4: Graph of the MTRC's value function $x \mapsto V_1(x, y)$ for a given $y = 10$.

follows suit.[13] Such a pattern is often observed in reality. A case in point is the property development plan around Canary Wharf, a main financial center in London, and the urban planning scheme following the Jubilee line's extension.[14] Second, because of the difference between the thresholds $x_1(y)$ and x_2, the private developer will not invest when the real option is "deep in the money" (i.e, $x \geq x_2$) as a myopic investor would, but when it is even "deep*er* in the money" (i.e., $x_1(y) > x_2$). This delay is due to the Stackelberg nature of this timing game, with the leader investing only when its own compound real option is "deep in the money."

[13]Indeed, according to (19), $\theta_2\big(X^{x}_{\tau_1^{x,y}}\big) = 0$ if $x \geq x_1(y) \geq x_2$.

[14]An eastward extension of the Jubilee Line was proposed in the 1970s, yet not finalized until the late 1990s when the developers of Canary Wharf agreed to pay GBP500m to complete the extension.

7. Policy implications

This section stresses social implications from the R+P program. Ideally, fare revenues and/or government subsidies should cover the upfront capital investment and the recurrent operations and maintenance costs. Yet, transit fares are usually regulated and rarely sufficient to cover all costs, while government subsidies or guarantees lead to (contingent) liabilities for the sovereign. Without "strong" income streams, private developers and financiers will lack willingness to participate in, which would further delay the delivery of infrastructure. A capital-intensive infrastructure investment opportunity should not be viewed as a single option, but rather a series of options on options (i.e., compound options) combined with other commercial opportunities derived from infrastructure investments.

MTRC capitalizes on complimentary sources of revenues (i.e., fare revenues, land premiums, and the profit sharing with private developers in real estate sale and lease) to cover the overall costs of capital investment, operations, and maintenance. This novel financing approach is based on the "*beneficiary pays*" principle, which states that the beneficiaries of the infrastructure services or improvements that increase land values should partly bear public investment costs or return their benefits to the public (Suzuki et al. [2015]; Ardila-Gomez and Ortegon-Sanchez [2016]). In the R+P model, charging *indirect beneficiaries* (i.e., the private developer) allows MTRC to recoup the huge capital investments at the outset; charging the *direct beneficiaries* (i.e., metro passengers) helps MTRC cover the operations and maintenance costs that are relatively small-scale and periodic or ongoing. By blending the multiple financing instruments that respond to the time variations of expenditures, MTRC operates without government subsidizing and runs a highly profitable and efficient business with financial sustainability. The comprehensive and sustainable urban transport financing scheme also releases the HK government from the financial pressure to provide the public support for the rail operator.

The newly constructed metrolines benefit not only the newly developed corridors but the existing ones as well. A denser metro system has network externalities because it makes the use of alternative transport modes less relevant. A denser system also helps reduce traffic congestion and save passengers' costs and time, thereby achieving sustainability goals (e.g., on CO_2 emissions). By accelerating the delivery of infrastructure assets, the R+P scheme generates these network benefits at an earlier stage.

There is an ongoing debate in HK over housing affordability. The HK is one of the most densely populated metropoles in the world, thus land in such a case is naturally a scarce resource. Some critics argue that the R+P scheme contributes to an increase in property and rental prices. Private developers involved in the R+P program charge the new properties at a "mark-up" commensurate with the land premiums and benefits they pay to MTRC. In reality, however, the HK government requires developers to provide affordable housing and community facilities built close to metro stations in exchange for the additional rights and benefits, such as increasing the current floor-area ratio (FAR) or providing other regulatory incentives.[15] Moreover, the high-end development concept of R+P is not applied to all the metro stations

[15]The FAR measures the ratio of a building's total floor area to the size of the land on which it is built. In general, the higher the FAR, the higher the density.

managed by MTRC; under the R+P scheme most property sites are not far from the transit stations: a number of residential flats are within 500 meters of metro stations (Suzuki et al. [2015]). Although housing affordability has long been a public concern, the novel R+P scheme, instead of accumulating the financial burden of public debt and charging general purpose taxes to taxpayers or other charges or fees dedicated to infrastructure, enables the HK government to finance a wide range of high-quality local infrastructure and social welfare by transacting development rights around transit stations with private developers under the limited local land supply.

Besides helping MTRC internalize the benefits of its own investments on property prices, the R+P model leads to a fairer wealth redistribution compared to a financing via public subsidies. Public subsidies ultimately stem from taxpayers' money without discrimination among those who benefit from the insfrastructure or not (e.g., also borne by a herdsman in Tibet). Instead of using taxpayers' money, MTRC ultimately charges those who benefit from the infrastructure assets, either via transit fares or via a markup on property prices. Intermediation by tax authorities is not needed in this instance.

8. Conclusion

Our paper presents a new perspective on infrastructure investment under uncertainty, viewing it as a springboard that can generate follow-on growth opportunities in a multi-player stopping game. We examine the underlying rationale behind the two stakeholders' investment timing decisions in the context of HK's R+P program by using notions borrowed from compound (growth) options and Stackelberg games. We model MTRC's problem as a duopoly case of Stackelberg leader-follower game of timing in comparison with traditional infrastructure financing models. Much can be learned from the real-world novel financing model.

First, the infrastructure financing gap stems from insufficient sources of revenue. Only if a project can demonstrate favorable and sufficient revenue sources for both upfront capital investments and periodic operations and maintenance expenditures can issues of financing and delivery be addressed successfully. In other words, unless the adequacy of revenue streams is addressed, there remains a financing gap in infrastructure investments. To accelerate the delivery of infrastructure, one should account for positive externalities that derive from the prior infrastructure investment and turn these revenue streams into capital that can be used today to initiate a capital-intensive project.

Second, in a multi-agent stopping game, the equilibrium investment times are not typical first-hitting times. Most extant real options literature suggests that the exercising times are precisely first-hitting times (from above or below depending on the context); once the process reaches the critical threshold, the decision maker takes action. However, in a sequential Stackelberg stopping game, the follower's reaction is contingent upon the leader's action, and the leader must consider the effect of follower's anticipated entry. Therefore, the equilibrium stopping times are not necessarily the first-hitting times.

Our article contributes to extant literature by analyzing a real option game [see, e.g., Chevalier-Roignant and Trigeorgis, 2011] inspired by a practical example from the infrastructure industry. Embedding compound real options and strategic interactions with multiple stakeholders, our proposed model is not restricted to state leasehold system (e.g., as in Hong Kong) but can be applied to other land holding systems (e.g., the market freehold system in Tokyo).[16] Our approach is novel, rigorous, general and can be extended to other business settings where a investing party wants to internalize the benefits of its own investment on other parties, such as in infrastructure. Future research could further explore independencies among projects over time by considering portfolios of real options in combination with game-theoretic thinking or extend our lumpy entry investment decisions to incremental capacity expansion options in the context of the Stackelberg leader-follower game of timing in dynamic and uncertain environments.

[16]Tokyo runs a transit business successfully with a market freehold system in which the government and transit agency do not own land, yet can collaborate with private developers via land use regulations [see, e.g., Suzuki et al., 2015].

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Appendices

Appendix A. Proof of Theorem 1

Dynamic programming equation. The function $t \mapsto Y_t^y := ye^{\rho t}$ solves (1a) and (1b) and is thus a solution to the ODE (1a)–(1b). It is immediate that

$$v(y) \geq \varphi(y) = ye^{-(r-\rho)h_1} - I_1, \quad \forall y \geq 0. \tag{A.1}$$

Besides, by the "principle of optimality,"

$$v(y) \geq e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big), \quad \varepsilon > 0.$$

Noting that if $v(\cdot) \in C^1(\mathbb{R}_+)$, we then have by the fundamental law of calculus

$$\frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big(e^{-r\varepsilon} v(y^{\rho\varepsilon})\Big)\Big|_{\varepsilon=0} := \lim_{\varepsilon \downarrow 0} \quad \frac{e^{-r\varepsilon} v\big(ye^{\rho\varepsilon}\big) - v(y)}{\varepsilon} = -rv(y) + \rho y v'(y).$$

We thus conclude that

$$rv(y) - \rho y v'(y) \geq 0, \quad \text{a.e. } y \in \mathbb{R}_+.$$

Because the firm will either invest straight away or wait, then we conclude that the value function (2) satisfies

$$0 = \min\big\{v(y) - \varphi(y); rv(y) - \rho y v'(y)\big\} \tag{A.2}$$

provided the function $v(\cdot) \in C^1(\mathbb{R}_+)$ and $\lim_{y \downarrow 0} v(y) = 0$.

Boundary problem. We conjecture that if (A.2) admits a solution $v(y)$, then this solution solves the problem

$$rv(y) - \rho y v'(y) = 0, \qquad y < \bar{y}, \tag{A.3a}$$

$$v(y) = \varphi(y), \qquad y \geq \bar{y}, \tag{A.3b}$$

where the scalar $\bar{y}$ is an unknown.

We conjecture that (A.3a) admits a solution of the form $y \mapsto y^{\gamma}$, which holds true if $\gamma = r/\rho$. For $\lim_{y \downarrow 0} v(y) = 0$ we need to assume $r > \rho$. We have

$$v(y) = \begin{cases} c \times y^{\frac{r}{\rho}}, & y < \bar{y}, \\ y e^{-(r-\rho)h_1} - I_1, & y \geq \bar{y}. \end{cases}$$

Yet, we postulate that $v(\cdot)$ is $C^1(\mathbb{R}_+)$; it follows that

$$c = \left(\frac{1}{\bar{y}}\right)^{\frac{r}{\rho}} \left\{ \bar{y} e^{-(r-\rho)h_1} - I_1 \right\} \quad \text{and} \quad \bar{y} = \frac{r}{r-\rho} I_1 e^{(r-\rho)h_1}. \tag{A.4}$$

We thus obtained the function in Theorem 1.

Verification of the DP equation. For a solution to (A.3a)–(A.3b) to solve (A.2) it also needs to satisfy

$$v(y) \geq \varphi(y), \qquad y < \bar{y}, \tag{A.5a}$$

$$rv(y) - \rho y v'(y) \geq 0, \qquad y \geq \bar{y}. \tag{A.5b}$$

From Theorem 1, the inequality (A.5a) is equivalent to

$$\left(\frac{y}{\bar{y}}\right)^{\frac{r}{\rho}}\Big\{\bar{y}e^{-(r-\rho)h_1} - I_1\Big\} \geq ye^{-(r-\rho)h_1} - I_1, \tag{A.6}$$

$$\bar{y}^{-\frac{r}{\rho}}\varphi(\bar{y}) \geq y^{-\frac{r}{\rho}}\varphi(y), \quad y < \bar{y}. \tag{A.7}$$

which is satisfied if $\upsilon \mapsto \upsilon^{-\frac{r}{\rho}}\varphi(\upsilon)$ is monotone increasing on $(y, \bar{y})$. We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}\varphi(\upsilon)\big) &= \upsilon^{-\frac{r}{\rho}-1}\Big[\upsilon\varphi'(\upsilon) - \frac{r}{\rho}\varphi(\upsilon)\Big] \\ &= \upsilon^{-\frac{r}{\rho}-1}\Big[-\upsilon e^{-(r-\rho)h_1}\frac{r-\rho}{\rho} + \frac{r}{\rho}I_1\Big].\end{aligned}$$

From (A.4),

$$\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}\varphi(\upsilon)\big) = \upsilon^{-\frac{r}{\rho}-1}e^{-(r-\rho)h_1}\frac{r-\rho}{\rho}\Big[-\upsilon + \bar{y}\Big].$$

It is then immediate that $\frac{\mathrm{d}}{\mathrm{d}\upsilon}\big(\upsilon^{-\frac{r}{\rho}}\varphi(\upsilon)\big) > 0$ if $\upsilon \in (0, \bar{y})$, which proves the inequality (A.5a).

We now consider the inequality (A.5b), which from Theorem 1 is equivalent to proving that $\big[r-\rho\big]ye^{-(r-\rho)h_1} \geq rI_1$; this is immediate by definition of $\bar{y}$ in (A.4). This completes the proof of Theorem 1.

Appendix B. Proof of Theorem 2

We first want to establish the relation (11). Thanks to the law of iterated expectations, we can rewrite (6) as

$$J_2^x(\tau_1^{x,y}, \tau_2^x) = \mathbb{E}\Big[e^{-r\tau_2}\Big\{(1-\alpha)\Big(e^{-rh_2}\mathbb{E}\big[X_{h_2}^{X_{\tau_2}^x}\big] - I_2\Big) - K\Big\}\mathbf{1}_{\{\tau_2 \geq \tau_1\}}\Big].$$

Besides, X_t^x is lognormally distributed, so $\mathbb{E}\big[X_{h_2}^x\big] = xe^{\mu h_2}$. We define the function $G_2(\cdot)$ by

$$G_2(x) := (1-\alpha)\big(e^{-(r-\mu)h_2}x - I_2\big) - K. \tag{B.1}$$

It follows from the strong Markov property that

$$J_2^x(\tau_1^{x,y}, \tau_2^x) = \mathbb{E}\big[e^{-r\tau_2}G_2\big(X_{\tau_2}^x\big)\mathbf{1}_{\{\tau_2 \geq \tau_1\}}\big],$$

where the function $G_2(\cdot)$ is defined in (B.1). The relation (11) immediately follows.

We now want to solve the VI (14a)–(14d). We conjecture that the continuation set $\mathcal{C}_2 = \{x > 0 : V_2(x) > G_2(x)\}$ is of the form $(0, x_2)$. If this conjecture holds, the solution to (14a) solves the FBP

$$\mathcal{L}_2 V_2(x) = 0, \qquad x < x_2, \tag{B.2a}$$

$$V_2(x) = G_2(x), \qquad x \geq x_2, \tag{B.2b}$$

where x_2 is a free boundary. Given the conjectured regularity (14b), we also consider the smooth-fit conditions:

$$V_2(x_2) = (1-\alpha)\Big(e^{-(r-\mu)h_2}x_2 - I_2\Big) - K, \tag{B.2c}$$

$$V_2'(x_2) = (1-\alpha)e^{-(r-\mu)h_2}. \tag{B.2d}$$

The boundary conditions (14c) and (14d) are also supposed to be satisfied by the solution to the FBP (B.2a)–(B.2d).

We can easily show that $x \mapsto x^{\gamma_A}$ and $x \mapsto x^{\gamma_B}$ are independent solutions to the ODE (B.2a). More generally, any linear combination of these two functions are solutions to this ODE. Because of (14d), we focus on solutions of the form $V_2(x) = A_2 x^{\gamma_A}$ on $(0, x_2)$. We re-write the smooth-fit conditions (B.2c)–(B.2d) as:

$$\begin{aligned} A_2 x_2^{\gamma_A} &= (1-\alpha)\big(e^{-(r-\mu)h_2} x_2 - I_2\big) - K, \\ \gamma_A A_2 x_2^{\gamma_A - 1} &= (1-\alpha) e^{-(r-\mu)h_2}. \end{aligned}$$

These two conditions are sufficient to determine the two unknowns $\bar{x}_2$ and A_2. The free boundary x_2 is given in Theorem 2, while

$$A_2 = \Big(\frac{1}{x_2}\Big)^{\gamma_A} \Big[(1-\alpha)(e^{-(r-\mu)h_2} x_2 - I_2) - K\Big].$$

It remains to check the inequalities

$$V_2(x) \geq G_2(x), \qquad x < x_2, \tag{B.3a}$$

$$\mathcal{L}_2 V_2(x) \geq 0, \qquad x \geq x_2 \tag{B.3b}$$

to establish that the function $V_2(\cdot)$ in Theorem 2 solves the VI (14a)–(14d). We re-write (B.3a) as

$$x_2^{-\gamma_A} G\big(x_2\big) \geq x^{-\gamma_A} G_2(x), \quad x < x_2,$$

which holds true if $x \mapsto x^{-\gamma_A} G_2(x)$ is monotone increasing on $(0, x_2)$. We have

$$\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-\gamma_A}G_2(x)\Big) = (1-\alpha)x^{-\gamma_A-1}\Big\{(1-\gamma_A)e^{-(r-\mu)h_2}x + \gamma_A\big[I_2 + \frac{K}{1-\alpha}\big]\Big\}.$$

From Theorem 2, we have

$$\gamma_A\Big[I_2 + \frac{K}{1-\alpha}\Big] = (\gamma_A - 1)x_2 e^{-(r-\mu)h_2}.$$

Hence,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\Big(x^{-\gamma_A}G_2(x)\Big) &= (1-\alpha)x^{\gamma_A-1}(\gamma_A-1)e^{-(r-\mu)h_2}\big[x_2 - x\big]\\ &> 0 \text{ if } x \in (0, x_2).\end{aligned}$$

The inequality (B.3b) is equivalent to

$$x \geq \frac{r}{r-\mu}e^{(r-\mu)h_2}\Big[I_2 + \frac{K}{1-\alpha}\Big], \quad x \geq x_2.$$

For the above to hold, it suffices that

$$\frac{\gamma_A}{\gamma_A - 1} \geq \frac{\frac{r}{\mu}}{\frac{r}{\mu} - 1}. \tag{B.4}$$

From (15),

$$\mathcal{Q}\Big(\frac{r}{\mu}\Big) = -\frac{1}{2}\sigma^2\frac{r}{\mu}\frac{r}{r-\mu} < 0 \leq \mathcal{Q}(\gamma_A).$$

Besides,

$$\mathcal{Q}'(\gamma) = -\big(\mu - \frac{1}{2}\sigma^2\big) - \gamma\sigma^2,$$

which is negative under the (sufficient) assumption that $\mu > \frac{1}{2}\sigma^2$. We thus conclude that $\frac{r}{\mu} > \gamma_A$. Because the function $x \mapsto \frac{x}{x-1}$ is monotone decreasing, we have established (B.4) and completed the proof of Theorem 2.

If we use the notations $a \wedge b := \min\{a; b\}$ and $a \vee b := \max\{a; b\}$, we can re-write Theorem 2 as

$$V_2(x) = \Big(\frac{x \wedge x_2}{x_2}\Big)^{\gamma_A} G_2\big(x \vee x_2\big). \tag{B.5}$$

Appendix C. Proof of Lemma 2

By the law of iterated expectations, we can re-write (8) as

$$J_1^{x,y}\big(\tau_1, \tau_2\big) = \mathbb{E}\Big[e^{-r\tau_1}\Big\{e^{-rh_1} Y_{h_1}^{Y_{\tau_1}^y} - I_1\Big\} + e^{-r\tau_2}\Big\{\alpha\Big(e^{-rh_2}\mathbb{E}\big[X_{h_2}^{X_{\tau_2}^x}\big] - I_2\Big) + K\Big\}\Big].$$

Now given the semigroup nature of $Y_\cdot^y$ and $X_\cdot^x$, we can write

$$J_1^{x,y}\big(\tau_1, \tau_2\big) = \mathbb{E}\Big[e^{-r\tau_1}\Big\{e^{-(r-\rho)h_1} Y_{\tau_1}^y - I_1\Big\} + e^{-r\tau_2}\Big\{\alpha\Big(e^{-(r-\mu)h_2} X_{\tau_2}^x - I_2\Big) + K\Big\}\Big].$$

Given the reaction function (20), we obtain

$$\begin{aligned} J_1^{x,y}\Big(\tau_1^{x,y}, \mathcal{T}_2^x(\tau_1^{x,y})\Big) = \mathbb{E}\Big[&e^{-r\tau_1^{x,y}}\Big\{e^{-(r-\rho)h_1} Y_{\tau_1^{x,y}}^y - I_1\Big\} \\ &+ \Big(\frac{X_{\tau_1^{x,y}}^x \wedge x_2}{x_2}\Big)^{\gamma_A}\Big\{\alpha(e^{-(r-\mu)h_2}(X_{\tau_1^{x,y}}^x \vee x_2) - I_2) + K\Big\}\Big]. \end{aligned}$$

It is immediate that the function G_1 in (21) is a floor function for the value function V_1 in (10). Besides, by the principle of dynamic programming,

$$V_1(x,y) \geq \mathbb{E}\Big[e^{-r\varepsilon} V_1\big(X_\varepsilon^x, Y_\varepsilon^y\big)\Big]. \tag{C.1}$$

If V_1 is regular (in a sense that we shall specify next), then we can use Dynkin's formula (in a generalized form) to obtain

$$\mathbb{E}\Big[e^{-r\varepsilon} V_1\big(X_\varepsilon^x, Y_\varepsilon^y\big)\Big] = V_1(x,y) - \mathbb{E}\int_0^\varepsilon \mathcal{L}_1 V_1\big(X_s^x, Y_s^y\big)\mathrm{d}s,$$

where the operators $\mathcal{L}_1$ are given respectively by (22).

From (C.1) we then obtain as $\varepsilon \downarrow 0$ the inequality $\mathcal{L}_1 V_1(x,y) \geq 0$ almost every $x \in \mathcal{X}$. We also have a complementary slackness condition. The VI in this case reads (23a)–(23b).

Appendix D. Proof of Lemma 3

From the definition of the leader's obstacle G_1 in (21),

$$\frac{\partial G_1}{\partial x}(x,y) = \begin{cases} \gamma_A \left(\frac{x}{x_2}\right)^{\gamma_A} \frac{\psi(x_2)}{x}, & x < x_2, \\ \alpha e^{-(r-\mu)h_2}, & x \geq x_2, \end{cases}$$

$$\frac{\partial^2 G_1}{\partial x^2}(x,y) = \begin{cases} \gamma_A(\gamma_A - 1) \left(\frac{x}{x_2}\right)^{\gamma_A} \frac{\psi(x_2)}{x^2}, & x < x_2, \\ 0, & x \geq x_2. \end{cases}$$

It obtains after some computations from (7) and (17) that

$$\psi\big(x_2\big) = \frac{\alpha}{\gamma_A - 1}\Big[I_2 + \frac{K}{\alpha(1-\alpha)}(\gamma_A + \alpha - 1)\Big]. \tag{D.1}$$

As $\alpha \in (0,1)$, it is immediate that $\psi(x_2) > 0$. Therefore, $x \mapsto G_1(x,y)$ is monotone increasing and strictly convex on $(0, x_2)$ from $\varphi(y)$ to $\varphi(y)+\psi(x_2)$, while is monotone increasing and affine on (x_2, ∞) from $\varphi(y)+\psi(x_2)$ to $+\infty$. Consequently, the function $x \mapsto G_1(x,y)$ is continuous at x_2.

We have from (17) and (D.1)

$$\begin{aligned}\frac{\partial G_1}{\partial x}(x_2+,y) - \frac{\partial G_1}{\partial x}(x_2-,y) &= \frac{1}{x_2}\Big[\alpha x_2 e^{-(r-\mu)h_2} - \gamma_A \psi(x_2)\Big] \\ &= -\frac{\gamma_A}{x_2}\frac{K}{1-\alpha} \\ &< 0.\end{aligned}$$

It follows first that the obstacle $x \mapsto G_1(x,y)$ is not differentiable at x_2 and concave in the vicinity of x_2.

We study the sign of the obstacle $x \mapsto G_1(x,y)$. If $y < y_0 := e^{(r-\rho)h_1} I_1 \iff \varphi(y) < 0$, then by monotonicity and continuity of $x \mapsto G_1(x,y)$, there exists a unique point, say $x_0(y)$, such that

$$G_1(x,y) < 0 \text{ if } x < x_0(y),\ G_1(x_0(y),y) = 0 \text{ and } G_1(x,y) > 0 \text{ if } x > x_0(y).$$

We asssume that $y < y_0$ and note that

$$\varphi(y) + \Big(\frac{x}{x_2}\Big)^{\gamma_A}\psi(x_2) < 0 \iff x < x_2\Big[-\frac{\varphi(y)}{\psi(x_2)}\Big]^{\frac{1}{\gamma_A}}.$$

Further,

$$\Big[-\frac{\varphi(y)}{\psi(x_2)}\Big]^{\frac{1}{\gamma_A}} < 1 \iff \varphi(y)+\psi(x_2) > 0,$$

$$\iff y > y_\dagger := e^{(r-\rho)h_1}\big[I_1 - \psi(x_2)\big].$$

It is immediate that $y_\dagger < y_0$; we conclude that

$$x_0(y) := x_2\Big[-\frac{\varphi(y)}{\psi(x_2)}\Big]^{\frac{1}{\gamma_A}} \text{ if and only if } y \in (y_\dagger, y_0).$$

If $y < y_\dagger$, then $x_0(y)$ is the (unique) root of $x \mapsto \varphi(y)+\psi(x)$, which is

$$\Big[\frac{I_1+\alpha I_2 - K - ye^{-(r-\rho)h_1}}{\alpha}\Big]e^{(r-\mu)h_2}.$$

We conclude with

$$x_0(y) := \begin{cases} \Big[\frac{I_1+\alpha I_2-K-ye^{-(r-\rho)h_1}}{\alpha}\Big]e^{(r-\mu)h_2}, & y\in(0,y_\dagger), \\ x_2\Big[-\frac{\varphi(y)}{\psi(x_2)}\Big]^{\frac{1}{\gamma_A}}, & y\in(y_\dagger,y_0). \end{cases}$$

If $y > y_0$, then the obstacle is strictly positive on $(0,\infty)$. This completes the proof of the Lemma 3.

Appendix E. Proof of Theorem 3

Study of a free-boundary problem. We conjecture that $\mathcal{C}_1$ is $\{(x,y) : (0, x_1(y))\}$ with $x_1(y) \geq x_2$. If this ansatz holds, then the solution to (23a) also solves the FBP:

$$\mathcal{L}_1 V_1(x,y) = 0, \qquad x < x_1(y), \tag{E.1a}$$

$$V_1(x,y) = \varphi(y) + \psi(x), \qquad x \geq x_1(y). \tag{E.1b}$$

We conjecture that the ODE (E.1a) has a solution of the form $f(x,y) = x^{\gamma} y^{1-\gamma}$. We have then $\mathcal{L}_1 f(x,y) = \mathcal{D}(\gamma) x^{\gamma} y^{1-\gamma}$, where $\mathcal{D}(\cdot)$ is given in (24). We study the function $\mathcal{D}(\cdot)$. We have $\mathcal{D}(-\infty) = \mathcal{D}(+\infty) = -\infty$ and

$$\mathcal{D}'(\gamma) = -\big[\mu - \rho - \frac{1}{2}\sigma^2\big] - \gamma\sigma^2 \lesseqgtr 0 \iff \gamma \gtreqless \gamma^\star := -\frac{\mu - \rho - \frac{1}{2}\sigma^2}{\sigma^2}.$$

The function $\mathcal{D}(\cdot)$ is increasing on $(-\infty, \gamma^\star)$ from $-\infty$ to $\mathcal{D}(\gamma^\star)$ and decreasing on $(\gamma^\star, \infty)$ from $\mathcal{D}(\gamma^\star)$ to $-\infty$. After some calculations it obtains

$$\mathcal{D}(\gamma^\star) = r - \rho + \frac{\big[\mu - \rho - \frac{1}{2}\sigma^2\big]^2}{2\sigma^2} > 0.$$

If the assumption (26a) is satisfied, then $\gamma_\star < 0$. So the function $\mathcal{D}(\cdot)$ admits two roots, one positive noted δ_A and one negative noted δ_B. Further, $\mathcal{D}(1) = r - \mu > 0$ so that $1 \in \big(\delta_B, \delta_A\big)$. We have $\mathcal{D}(\gamma_A) = (\gamma_A - 1)\rho > 0$ so $\gamma_A \in (1, \delta_A)$. So we have:

$$\delta_B < 0 < 1 < \gamma_A < \delta_A.$$

We can now write the solution to the FBP (E.1a)–(E.1b) as

$$V_1(x,y) = \begin{cases} Ax^{\delta_A} y^{1-\delta_A} + Bx^{\delta_B} y^{1-\delta_B}, & x < x_1(y), \\ \varphi(y) + \psi(x), & x \geq x_1(y). \end{cases}$$

There remains to find three unknowns A, B, and $x_1(y)$. For $V_1(x,y)$ to be finite as $x \downarrow 0$, we set $B = 0$. We want to find a function V_1 that is regular in the sense of (23b). The smooth-fit conditions (in x) read

$$Ax_1(y)^{\delta_A} y^{1-\delta_A} = e^{-(r-\rho)h_1} y + \alpha e^{-(r-\mu)h_2} x_1(y) - (I_1 + \alpha I_2 - K), \tag{E.2a}$$

$$\delta_A A x_1(y)^{\delta_A - 1} y^{1-\delta_A} = \alpha e^{-(r-\mu)h_2}. \tag{E.2b}$$

The expression for $x_1(y)$ in (25b) follows from (E.2a)–(E.2b) after some computations; it also follows that

$$A = \frac{\varphi(y) + \psi(x_1(y))}{x_1(y)^{\delta_A} y^{1-\delta_A}}. \tag{E.3}$$

It is immediate that the function $x_1(\cdot)$ given in (25b) is monotone decreasing on $(0, \infty)$ from

$$x_1(0) = \frac{\delta_A}{\delta_A - 1} \frac{I_1 + \alpha I_2 - K}{\alpha} e^{(r-\mu)h_2}, \tag{E.4}$$

to $-\infty$. We now assume that

$$I_1 + \alpha I_2 - K > 0; \tag{E.5}$$

so $x_1(0) > 0$. It obtains after some (tedious) calculations that

$$x_1(0) - x_2 = \frac{e^{(r-\mu)h_2}}{(\delta_A - 1)(\gamma_A - 1)} \frac{f(\alpha)}{\alpha(1-\alpha)},$$

where

$$f(\alpha) := a\alpha^2 - \big(a + b + c\big)\alpha + c, \quad \alpha \in [0, 1].$$

where a, b, and c are defined by (25d) and positive.

Because $\operatorname{sgn}\{x_1(0) - x_2\} = \operatorname{sgn}\{f(\alpha)\}$, it follows $x_2 < x_1(0)$ iff $f(\alpha) < 0$. We study the function $f(\cdot)$. Because $f''(\alpha) = 2a > 0$, $f(\cdot)$ is decreasing on $(-\infty, \frac{a+b+c}{2a})$ from $f(-\infty) = +\infty$ to $f\big(\frac{a+b+c}{2a}\big)$ and increasing on $(\frac{a+b+c}{2a}, +\infty)$ from $f\big(\frac{a+b+c}{2a}\big)$ to $f(+\infty) = +\infty$. We have $f(0) = c > 0$ and $f(1) = -b < 0$. So necessarily the function $f(\cdot)$ admits a unique root in $(0, 1)$, $\alpha_\star$ defined in (25c). Further, $f(\alpha) \lesseqqgtr 0$ iff $\alpha \gtreqqless \alpha_\star$. It follows that $x_1(0) > x_2$ if $\alpha \in (0, \alpha_\star)$.

We recall that $x_1(\cdot)$ is monotone decreasing on $\mathbb{R}_+$ and note from (17) and (25b) that

$$x_1(y) > x_2 \iff 0 < y < y_\star,$$

with $y_\star$ defined in (25a). We can now conclude the function V_1 in (26e) solves the FBP (E.1a)–(E.1b) with the regularity (23b) if the conditions in Theorem 3 are satisfied.

Verification of the VI. We now want to prove that V_1 in (26e) solves the VI (23a). We drop the dependence of $x_1(\cdot)$ on y for conciseness. It remains to check that

$$x_1^{-\delta_A}\big[\varphi(y)+\psi(x_1)\big] \geq x^{-\delta_A}\Big[\varphi(y)+\big(\frac{x}{x_2}\big)^{\gamma_A}\psi(x_2)\Big], \qquad 0<x<x_2, \quad \text{(E.6a)}$$

$$x_1^{-\delta_A}\big[\varphi(y)+\psi(x_1)\big] \geq x^{-\delta_A}\big[\varphi(y)+\psi(x)\big], \qquad x_2 \leq x < x_1, \quad \text{(E.6b)}$$

$$\mathcal{L}_1\big[\varphi(y)+\psi(x)\big] \geq 0, \qquad x \geq x_1. \quad \text{(E.6c)}$$

We start by proving the inequality (E.6c), which is equivalent to proving that

$$\mathcal{L}_1\big[\varphi(y)+\psi(x)\big] = (r-\rho)e^{-(r-\rho)h_1}y+(r-\mu)\alpha e^{-(r-\mu)h_2}x-r(I_1+\alpha I_2-K) \geq 0, \quad x \geq x_1(y),$$

or that

$$x \geq \left[\frac{r}{r-\mu}(I_1+\alpha I_2-K)-\frac{r-\rho}{r-\mu}e^{-(r-\rho)h_1}y\right]\frac{e^{(r-\mu)h_2}}{\alpha}, \quad x \geq x_1(y).$$

By definition of $x_1(y)$ in (25b), it appears that the above inequality is satisfied because $\frac{\delta_A}{\delta_A-1} > \frac{r}{r-\mu}$.

To verify (E.6b), we study the function Λ given by

$$\Lambda(x,y) := x^{-\delta_A}\big[\varphi(y)+\psi(x)\big]. \quad \text{(E.7)}$$

We have

$$\begin{aligned}\frac{\mathrm{d}\Lambda}{\mathrm{d}x}\big(x,y\big) =& - x^{-\delta_A}\alpha e^{-(r-\mu)h_2}(\delta_A-1) \\ &+ \delta_A x^{-\delta_A-1}\big[(I_1+\alpha I_2-K)-e^{-(r-\rho)h_1}y\big] \\ =&(\delta_A-1)x^{-\delta_A-1}\alpha e^{-(r-\mu)h_2}\big[x_1(y)-x\big],\end{aligned}$$

where the second line comes from (25b). The derivative is positive if $x \in (0, x_1)$. It follows that $\Lambda(\cdot, y)$ is monotone increasing on $(0, x_1)$ for $y \in (0, y_\star)$, which proves (E.6b) directly.

We now consider the inequality (E.6a). We have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}x}\big[x^{-\gamma_A}\psi(x)\big] &= -\gamma_A x^{-\gamma_A-1}\psi(x) + x^{-\gamma_A}\psi'(x)\\ &= x^{-\gamma_A-1}\Big\{\alpha e^{-(r-\mu)h_2}x(1-\gamma_A) - \gamma_A(K-\alpha I_2)\Big\}\\ &= x^{-\gamma_A-1}\alpha e^{-(r-\mu)h_2}(1-\gamma_A)\Big[x + \frac{\gamma_A}{\gamma_A-1}\frac{K-\alpha I_2}{\alpha}e^{(r-\mu)h_2}\Big].\end{aligned}$$

If

$$K > \alpha I_2, \tag{E.8}$$

then $\frac{\mathrm{d}}{\mathrm{d}x}\big[x^{-\gamma_A}\psi(x)\big] < 0$. Therefore, if $x < x_2$, then $x^{-\gamma_A}\psi(x) > x_2^{-\gamma_A}\psi(x_2)$ by monotonicity. By definition of Λ in (E.7), it follows then that

$$\Lambda(x, y) > x^{-\delta_A}\Big[\varphi(y) + \big(\frac{x}{x_2}\big)^{\gamma_A}\psi(x_2)\Big]. \tag{E.9}$$

Finally, because we know that $\Lambda(\cdot, y)$ is monotone increasing on (x, x_1), we have $\Lambda(x_1, y) > \Lambda(x, y)$ for $x \in (0, x_1)$, which proves (E.6a).

The assumption (26b) combines both the parameter restrictions (E.8) and (E.5). To prove Theorem 3, it would remain to prove a verification theorem [for standard proofs see, e.g., Bensoussan and Lions, 1982].

Appendix F. Parameter values

	Symbols	Values
Cost parameters	I_1	17.6
	I_2	4.66
	K	5.2
	α	0.5
Growth and risk	r	0.15
	μ	0.12
	ρ	0.015
	σ	0.15
Time lags	h_1	5.5
	h_2	5
Static NPV threshold	y_0	36.9808
Case A	y	10
	$y_\star$	13.5696
Roots	γ_A	1.22426
	δ_A	1.25192
	δ_B	-9.58526
Thresholds	x_2	95.5191
	$x_0(y)$	34.9454
	$x_1(y)$	115.137