A Dynamic Model for Venture Capitalists' Entry–Exit Investment Decisions*

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Abstract

In this paper we develop a dynamic model to study the entry and the exit decision of a VC facing the opportunity to invest and expand a start-up firm. Two settings are considered. A benchmark setting, where no time constrain for exiting is in place, is compared with the one where, realistically, the VC has a finite time-window for divesting. In both cases we consider the trade sale (M&A) as the exit route. The model returns the entry and the exit triggers, the optimal post-money ownerships, the expected cash multiple for the VC, and also proposes a new time-adjusted version of the cash multiple, useful for measuring, *ex ante*, the expected performance of the investment. The model aims to guide the VCs when analyzing their investment opportunities, considering the entire VC's business-cycle (entry–expand–exit). Finally, the model is applied to an hypothetical, but realistic, situation in order to understand the main outcomes. A comparative statics analysis is also performed.

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1 Introduction

Venture capital is a type of equity-financing with the purpose of investing in start-up firms with large potential. These young firms typically face serious constrains regarding the traditional sources of financing (e.g., bank financing) due to their typical risky nature. In addition to the financial resources, venture capitalists (VCs) also support the start-up firms with management coaching and monitoring, and offer access to new business contracts and alliances (Cumming et al. 2017).

In this context, venture capital plays an important role to promote innovative entrepreneurship, by supporting the pre-launch/launch of new products, or the expansion of businesses in an early stage of development, in a wide range of industries (e.g., IT, biotech, health, energy, among others). The venture capital activity has been continuously increasing over the past decades, reaching an annual record of USD 254 billion invested worldwide in 2018 (KPMG 2019).

The VCs investment process follows a sequence of fundamental steps. First, there is a pre-deal selection process, with initial screening, due-diligence activities and internal approvals. If accepted, the deal takes place under the contracted conditions and the cooperation between the parties begins. Once the start-up firm has been funded, the venture capitalist will pursue the necessary actions to develop the business as previously planned, while continuously monitoring the key operational milestones and value drives, with the final goal of exiting the start-up firm with profit, following a given exit route.[1]

There is a large body of research studying each of these phases. For instance, Petty and Gruber (2011) refer that the literature over the past four decades shows a continuous interest of academics in studying the VCs decision process, and their criteria for assessing potential deals. Some recent contributions on this matter can be found in Knockaert et al. (2010), Kollmann and Kuckertz (2010), and Minola et al. (2017). For the intermediate phase, during which the VC stays invested in the start-up firm, matters like the monitoring activities or the allocation of attention (e.g., Gompers 1995, Robbie et al. 1997, Shepherd et al. 2005, Bernstein et al. 2016), the staging investment (e.g., Bigus et al. 2006, Li 2008, Lukas et al. 2016), and the equity sharing (e.g., Vergara et al. 2016, Tavares-Gärtner et al. 2018a) already appear in the literature. As for the exit phase the literature is also vast, covering subjects like the exit routes (e.g., Giot and Schwienbacher 2007, Cumming and Johan 2008, Jenkinson and Sousa 2015), the timing of the exit (e.g., Neus and Walz 2005, Giot and Schwienbacher 2007, Hsu 2013), and the exit performance (e.g., Jääskeläinen et al. 2006, Dai et al. 2012, Jia and Wang 2017).

Differently from the previous literature, in this paper we connect two of the main decisions that must be taken by VCs. On the one hand, the entry decision, where, for instance, the timing of the deal and the equity share for each party need to be defined. On

[1] Please refer to Landström (2007) and Klonowski (2010) for further discussion about the VCs investment process.

the other hand, the exit decision, where the timing and the gains under a given exit route need also to be considered. In the end of the investment process, the performance for the VC will depend both on the conditions of entry and on the success of exit. As entry and exit are closely related decisions, we explicitly consider their joint effect on the investment process. Remarkably, to the best of our knowledge, this joint analysis has not yet been considered under the same model. Setting all the picture reveals particularly relevant due to the fact that investment decisions are taken in the context of high uncertainty, where the future – and so the performance – is, *ex ante*, not known.

Accordingly, the aim of our paper is to develop a model for guiding VCs on their investment decisions process considering the entire business-cycle (entry–expand–exit) and its interactions. Given the option nature of the investment, and given the important role played by uncertainty, we set our framework following the well established real option theory[2].

The literature that uses the real options lens to address VCs' decision problems has increased during the last two decade. For instance, Seppä and Laamanen (2001) tests if an option-based framework fits better than traditional models for explaining risk-return profiles of venture capital investments, which they reveal to be the case. The results imply that option-based methods have empirical relevance for the assessment of VCs investment opportunities. Li (2008) empirically shows that market uncertainty encourages VCs to postpone their rounds of financing, whereas competition, project-specific uncertainty and agency concerns have an opposite effect. The same deterrence effect of market uncertainty is found by Li and Mahoney (2011). Hsu (2010) studies the staging investment decisions of VCs in a principal-agent theoretical framework, and Li and Chi (2013) sets a real options view on VCs decision to withdraw from an investment project prior to its completion.

More recently, Lukas et al. (2016) develop dynamic model for start-up financing under uncertainty, combining compound option pricing with sequential non-cooperative contracting. They analyze how uncertainty drives the equity share the VC will get in the start-up firm and conclude that a higher uncertainty significantly increases the stake for the VC. These results are not entirely consistent with the ones we get in this paper, as we depict an effect of uncertainty that can be null (in line with Tavares-Gärtner et al. (2018a)) or much less dramatic.

Tavares-Gärtner et al. (2018a) studies how the heterogeneous beliefs regarding the future developments of the start-up firm may affect the optimal ownership structure and the exercise of the growth option, and Tavares-Gärtner et al. (2018b) present a real options model for entrepreneurial financing in the presence of contingent payment mechanisms and they show that optimal contingent payments have no influence in the timing of the investment.

[2]For some founding literature on real options theory please refer to Dixit and Pindyck (1994), and Trigeorgis (1996).

Our paper differs from the previous literature in different ways. Firstly, as already mentioned, we build a model that jointly considers the decisions of investing and exiting from the star-up firm. These two closely related decisions, given their mutual impact, need to be combined when formulating an appropriate decision model.

Secondly, an optimal exit decision (via trade sale) is defined. For a deep understanding of the investment dynamics, we develop a model with time restrictions on the exit (a requirement in the venture capital industry) which will be compared with a benchmark model, where no such restrictions are present. In addition to the purpose of guiding VCs decision, we will be able to study the impact of imposing a finite duration for the investment on the entry–exit process.

Thirdly, our model allows to set, *ex-ante*, a prospect of performance for the VCs, both by using an expected cash-on-cash multiple (a typical measure in the industry) and by proposing an adjusted version of that multiple that accounts for the stochastic time duration of the investment.

Finally, by applying the model to a hypothetical, but realistic case, and by performing a comparative statics analysis on the main value-drives, several empirical implication will be discussed. For instance, our results show that, under time restrictions for exiting, the VC will invest for a larger trigger, require a larger equity share on the start-up firm, and will optimally exit for a smaller trigger level. They also show that cash multiples tend to be smaller in the restricted case, but this effect realistically disappears when a time-adjusted cash multiple is considered. Additionally, consistently with the empirical evidence, we show that volatility has a positive effect on the entry and exit triggers, and also on the cash multiples. The model also suggests that in high synergistic industries (that influence the profit on the exit), the cash multiple tends to be smaller than that of a less synergistic sector; however, if we consider the adjusted multiple the effect is the opposite. Transposing this finding to a rate of return measure, the model predicts a positive relation between synergies and the IRR, even though a lower cash multiple could be observed. Finally, a relatively more optimistic VC will invest and exit sooner and will require a lower share on the start-up firm. Curiously, the level of optimism reveal no impact on the cash multiples.

The paper unfolds as follows. Chapter 2 contains the model setup; a perpetual setting is derived as a benchmark setting within Chapter 3; and the model is extended in Chapter 4 to account for the time restriction VCs face. Since the time restricted model finds no analytical solution, a numerical example is provided in Chapter 5 to help conceptualizing the entire framework and understand its economic intuitions. Chapter 6 concludes.

2 Model setup

The model comprises an investment and a later exit of a Venture Capitalist (VC) in an established entrepreneurial firm which is assumed to be owned by a single Entrepreneur

(E). The firm holds a growth opportunity defined by an expansion of its value, V_t, by a factor $\theta > 1$, subject to an investment cost of K. Due to capital constrains[3], only part of the investment can be financed by the Entrepreneur, $K_E(< K)$, which means the VC invests the remaining $K - K_E$, in a single round of financing.

The individual prospects of VC and E on the growth opportunity may differ, as they may have heterogeneous beliefs regarding the growth factor θ. In fact, the growth prospects of E, represented by θ_E, can be larger (or smaller) than those of the VC, θ_{VC}, showing the former as being more (or less) optimistic than the latter. Naturally, a particular situation may occur where $\theta_E = \theta_{VC} = \theta$, which reveal a case of homogeneous beliefs between the parties.

In addition, the VC is assumed to have no funding constraints neither burden any additional opportunity cost from other potential investments in other ventures. Also, the capital increase (either by the E or VC) is made at no premium or discount.

The post-equity round ownership on the firm held by the VC is $0 < Q_{VC} < 1$, while the E's ownership will decrease from 1 to $0 < Q_E < 1$, where $Q_E = 1 - Q_{VC}$.

The VC, after investing $K - K_E$, receives $Q_{VC}\theta_{VC}V_t$, which corresponds to the value of the firm after the expansion, according to his own growth beliefs. On the other hand, E spends K_E and receives an incremental value of $Q_E(\theta_E - 1)V_t$, based on his own beliefs.

Finally, it is assumed that the value of the start-up firm, given its current (*pre* expansion) scale is represented by V_t, which follows a geometric Brownian motion:

$$dV_t = \alpha V_t dt + \sigma V_t dz, \text{ with } V_0 > 0 \tag{1}$$

where σ is the volatility, α expected growth rate and dz as an increment of a Wiener process. It is also assumed that all agents are risk neutral and that the risk-free interest rate $r, (r > \alpha)$ controls for the time-value of money. Accordingly, α is a risk neutral drift. It is assumed that, after exercising the growth option the value of the firm becomes $\theta_i V(t)$, $i \in \{\mathrm{VC}, \mathrm{E}\}$, which does not modify the dynamics presented in Equation (1).

At this stage, two main issues emerge. On the one hand, it is crucial to find the appropriate stake each party is going to end-up with; in other words, VC and E both need to agree about how to share the venture after the expansion. On the other hand, both parties also need to agree about the timing of the expansion, i.e, about when the growth option should be optimally exercised. In a word, the interests of VC and E need to be aligned in order to jointly agree about the investment on the firm. Interestingly, even

[3]This capital constrain assumes that the start-up – Entrepreneur included – has no access to debt financing, assumption taken upon the facts that firms in this circumstances (i) typically do not operate for a length of time long enough to provide banks with a financial record to prove its credit worthiness; (ii) typically do not have collateral to provide against a debt facility; and also (iii) indebtedness could hardly damage the start-up's ability to burn cash aiming for exponential growth, due to the further responsibility of debt repayment and its service cost in the form of interests. Aside from these facts, debt facilities may simply be unavailable as a result of specific market and macroeconomic conditions.

when VCs and Es show different opinions about the growth prospects, it is possible for them to agree about the investment in the firm's growth.

Differently from what typically happens to the entrepreneur, the venture capitalist invests in the firm with the obligation to divest after a finite period of time. This exit obligation introduces an asymmetric perspective on the duration of the investment with an important impact on the investment dynamics for the venture capitalist and, although indirectly, for the entrepreneur.

For the complete development of the model, the entry decision (investment) needs necessarily to look forward and account for the moment when the exit takes place (divestment). The derivation follows the standard backwards procedure, starting with the exit decision, and then moving back to the initial decision of investing in the start-up firm. From all the different exit routs available for the VC, in this paper we only consider the trade sale (M&A) alternative, given the fact that it is the most common exit strategy used by the venture capitalists (e.g., Espenlaub et al. 2015a,b).

3 Benchmark setting: no time restrictions on the exit

For understanding the impact of the finite duration of the investment for the VC on the investment dynamics, let us start by presenting the case of an investor that has no time restrictions on the exit. This means that the investor will exit at the optimal moment, not being forced to sell the asset during a given pre-established time window. This general setting serves as a benchmark in our analysis.

3.1 Exit without time restrictions

For modeling the exit as a trade sale process, we follow the contributions of Lambrecht (2004) and Lukas and Welling (2012). The dynamic process unfolds as follows: (i) the VC initiates the exit (selling) procedure by offering his fraction on the start-up to a third-party firm (simply called Buyer and labeled as B); (ii) this stake in the start-up firm will benefit B, as it will be integrated in his current assets and will produce a given level of synergies ($\gamma > 0$); (iii) the VC firm requires a given premium ($\phi > 0$) for agreeing to sell his stake to B. Without time restrictions for the trade sale, the VC has an American perpetual option to divest.

The dynamic process has a Markovian Perfect Nash Equilibrium path to determine the optimal decision for each party. In our case, the VC (assuming the role of a seller at the exit moment) sets the offer and requires a given selling premium, whereas the Buyer, considering the conditions of the offer, defines the timing when the deal is accepted, i.e., when the trade sale takes place. Obviously, premium and timing are closed related. The VC, as an offering entity, defines a premium that maximizes its own position, taking in consideration, however, the reaction function of B. The derivation process follows this dynamics.

Let us start with the Buyer (firm B). Using similar arguments as those in Lambrecht (2004) and Lukas and Welling (2012), firm B, incorporates the terms of the offer (consisting in a given premium ϕ required by the VC) and decides upon the timing:

$$f_B(V_t) = \max_t E[((\gamma - \phi)Q_{VC}\theta_{VC}V_t - (1-\epsilon)C)e^{-rt}] \tag{2}$$

where $E[.]$ denotes the expectation operator, $\gamma > 0$ is a factor that captures the level of synergies (proportional to V_t), C represents the transactions cost, and $(1-\epsilon)$ is the fraction of those costs supported by firm B (the remaining fraction ϵ is supported by the VC). Notice that the dynamics of the trade sale phase is set from the VC perspective, and so his beliefs on the expansion is considered (θ_{VC}).

Using the standard arguments, Equation (2) can be redefined as follows:

$$f_B(V_t) = \max_{V_B(\phi)} \left[((\gamma - \phi)Q_{VC}\theta_{VC}V_B(\phi) - (1-\epsilon)C) \left(\frac{V_t}{V_B(\phi)} \right)^{\beta} \right] \tag{3}$$

where $\beta = \frac{1}{2} - \frac{(r-\delta)}{\sigma^2} + \sqrt{\left(\frac{r-\delta}{\sigma^2} - \frac{1}{2} \right)^2 + \frac{2r}{\sigma^2}} > 1$, corresponds to the positive root of a fundamental quadratic equation (for the details, see Dixit and Pindyck 1994), and $V_B^*(\phi)$ is the optimal trigger for B to accept the offer.

On the other side, the VC fully anticipates this behavior of firm B and asks a premium such that the following objective function is maximized:

$$f_{VC}(V_t) = \max_{\phi} \left[(\phi Q_{VC}\theta_{VC}V_B^*(\phi) - \epsilon C) \left(\frac{V_t}{V_B^*(\phi)} \right)^{\beta} \right] \tag{4}$$

Solving both objective functions recursively, we obtain the following results:

Proposition 1. ***Trade sale attributes.*** *The trade sale optimally takes place at $V_t = V_B^*$, as a result of a premium ϕ^* required by the VC. Both V_B^* and ϕ^* are as follows:*

$$\phi^* = \frac{\gamma(1 + (\beta - 2)\epsilon)}{\beta - \epsilon} \tag{5}$$

$$V_B^* \equiv V_B^*(\phi^*) = \frac{\beta(\beta - \epsilon)}{(\beta - 1)^2} \frac{C}{\gamma Q_{VC}\theta_{VC}} \tag{6}$$

From the VC's standing point, V_B^* and ϕ^* reveal to be a crucial piece of information, essential for fully equating the entry decision. In addition:

Proposition 2. ***VC's exit option value.*** *The value of the VC's exit option (when no* *time restrictions are set) is as follows:*

$$\begin{aligned} F_{VC}(V_t) &= (\phi^* Q_{VC}\theta_{VC}V_B^* - \epsilon C)\left(\frac{V_t}{V_B^*}\right)^{\beta} \\ &= \frac{(\beta-\epsilon)C}{(\beta-1)^2}\left(\frac{V_t}{V_B^*}\right)^{\beta} \end{aligned} \tag{7}$$

After modeling the exit phase, let us move backwards for deriving the entry decision.

3.2 Entry without time restrictions

Let us position now in the initial stage, where both the venture capitalist and the entrepreneur are facing the chance to invest in the expansion of the start-up firm.

From E's point of view, the decision consists in exchanging part of his stake on the start-up firm for a fraction of the incremental cash flows arising from the expansion. The decision needs to consider E's own beliefs about the growth prospects (θ_E), the dilution effect ($1-Q_{VC}$), and the investment cost (K_E). The incremental payoff for E is as follows:

$$\Pi_E(V_t) = ((1-Q_{VC})\theta_E - 1)\,V_t - K_E \tag{8}$$

Given its optionality, E agrees about the joint investment on the growth opportunity at the optimal trigger (V_E^{**}), the one that maximizes his position:

$$g_E(V_t) = \max_{V_E^{**}}\left[(((1-Q_{VC})\theta_E - 1)\,V_t - K_E)\left(\frac{V_t}{V_E^{**}}\right)^{\beta}\right] \tag{9}$$

where,

$$V_E^{**} = \frac{\beta}{(\beta-1)}\frac{K_E}{(1-Q_{VC})\theta_E - 1} \tag{10}$$

Considering the position of the VC, his payoff is as follows:

$$\Pi_{VC}(V_t) = Q_{VC}\theta_{VC}V_t + F_{VC}(V_t) - (K - K_E) \tag{11}$$

where the first term on the right-hand side captures his fraction on the firm after the expansion, considering his own beliefs (θ_{VC}); the second term, $F_{VC}(.)$, accounts for the fact that, after investing, the VC also receives the option to exit; and, finally, the last term represents the VC's investment cost. Following the same arguments, the VC's optimization problem is:

$$g_{VC}(V_t) = \max_{V_{VC}^{**}}\left[(Q_{VC}\theta_{VC}V_t^{**} + F_{VC}(V_{VC}^{**}) - (K - K_E))\left(\frac{V_t}{V_{VC}^{**}}\right)^{\beta}\right] \tag{12}$$

where $F_{VC}(V_{VC}^{**})$ denotes the value option to exit (Equation (7)) assessed at the entry trigger $V_t = V_{VC}^{**}$.

The optimization (12) leads to the following entry trigger for the VC:

$$V_{VC}^{**} = \frac{\beta}{\beta - 1} \frac{K - K_E}{Q_{VC}\theta_{VC}} \tag{13}$$

Aligning the investment trigger of each party (E and VC) through the optimal ownership results in the following proposition:

Proposition 3. ***Post-money ownership.*** *The post-money optimum ownership for the VC and E, respectively Q_{VC}^{opt} and Q_E^{opt}, comes as follows:*

$$Q_{VC}^{opt} = \frac{(K - K_E)(\theta_E - 1)}{K\theta_E + K_E(\theta_{VC} - \theta_E)} \tag{14}$$

$$Q_E^{opt} = 1 - Q_{VC}^{opt} \tag{15}$$

Interestingly, neither the triggers nor optimal ownership in the entry decision are affected by the exit option. This solution makes the model without time restrictions yielding results similar to those derived by Tavares-Gärtner et al. (2018a). In addition we can state the following:

Corollary 1. ***Sensitivities of the post-money ownership.*** *(i) Fixing θ_E, the lower (higher) the VC's expectations about the growth factor (θ_{VC}), the higher (lower) the share he requires; (ii) the larger the relative investment cost of the VC, the greater the share he demands. Furthermore, the volatility of the project (σ) and the exit potential synergies (γ) reveal not to affect the optimal ownership for the VC.*

Corollary 2. ***Homogeneous beliefs.*** *For the particular case where both VC and E agree about the growth prospects, i.e. $\theta_{VC} = \theta_E = \theta$, we have:*

$$Q_{VC}^{opt}(\theta_{VC} = \theta_E = \theta) = \frac{(K - K_E)}{K} \frac{(\theta - 1)}{\theta} \tag{16}$$

which has an intuitive economic interpretation. Under homogeneous beliefs, the VC optimally receives a fraction on the incremental growth which is proportional to his relative investment cost.

After computing the optimal ownership for the parties, it is possible to define the final solutions for the entry and exit triggers. Let us star by the former.

Proposition 4. ***The complete solution for the entry trigger.*** *The entry trigger for the VC is as follows:*

$$V^{**} \equiv V_{VC}^{**}(Q_{VC}^{opt}) = V_E^{**}(Q_E^{opt}) = \frac{\beta}{\beta - 1} \frac{(K - K_E)\theta_E + K_E\theta_{VC}}{(\theta_E - 1)\theta_{VC}} \tag{17}$$

In normal circumstances, the entry trigger is prior to the exit trigger, i.e., $V_{VC}^{**} < V_B^*$. However, if the synergies created to the buyer (i.e., firm B on the exit deal) are large enough, a reversed order of the triggers may occur, i.e., the exit trigger can be smaller than the entry trigger. This means that, for large synergies, the VC may find optimal to invest, divesting immediately after. This could be considered a rare situation[4] but need to be considered in our complete set of solutions.

Proposition 5. ***The case of high synergies.*** *In the case of synergies such that* $\gamma \geqslant \bar{\gamma} = \dfrac{\beta-\epsilon}{\beta-1}\dfrac{C}{K-K_E}$, *the exit trigger becomes smaller than the entry trigger. In such a case, the appropriate premium would not be as in Equation* (5), *but instead:*

$$\bar{\phi}(\gamma \geqslant \bar{\gamma}) = \gamma - (1-\epsilon)\frac{c}{K-K_E} < \phi^* \tag{18}$$

corresponding to the one that aligns both triggers.

Incorporating what was just stated, the complete solution for the exit trigger is now possible.

Proposition 6. ***The complete solution for the exit trigger.*** *The exit trigger comes as follows:*

$$V^* \equiv V_B^*(Q_{VC}^{opt}) = \begin{cases} \dfrac{\beta}{(\beta-1)^2}\dfrac{(\beta-\epsilon)C((K-K_E)\theta_E + K_E\theta_{VC})}{\gamma(K-K_E)(\theta_E-1)\theta_{VC}} & \text{if } \gamma < \bar{\gamma} \\ V^{**} & \text{if } \gamma \geqslant \bar{\gamma} \end{cases} \tag{19}$$

where V^{**} *is the entry trigger as defined in Equation* (17).

With the complete set of solutions it becomes straightforward to compute the VC's expected cash multiple, corresponding to the ratio between expected value at the exit and the initial capital investment. This multiple is a common measure of performance in the VC/PE industry and it is usually called as cash-on-cash multiple.

Proposition 7. ***Cash multiple.*** *The expected cash multiple for the VC,* M_{VC}, *which compares the expected in-flow at the exit, which corresponds to* $(1+\phi^*)Q_{VC}^{opt}\theta_{VC}V^* - \epsilon C$ *if* $\gamma < \bar{\gamma}$ *or* $(1+\bar{\phi}(\gamma))Q_{VC}^{opt}\theta_{VC}V^{**} - \epsilon C$ *if* $\gamma \geqslant \bar{\gamma}$, *to the initial investment,* $K - K_E$, *comes*

[4] A recent possible example of such a fast and profitable deal came to us by Blackstone Group Inc. which doubled the value of its investment in Refinitiv in about 10 months. The PE firm entered in the capital of Refinitiv, a Thomson Reuters Corp. company, at a highly levered \$20Bn investment of which about \$6.5 billion was equity. The business was almost immediately sold in a merger deal with London Stock Exchange, raising the value to \$27Bn, meaning an equity value worth over twice as much, given that the debt level has remained at similar levels (`https://www.bloomberg.com/news/articles/2019-07-29/blackstone-bets-on-further-refinitiv-gains-after-doubling-value`).

in the following simplified way:

$$M_{VC} = \begin{cases} \dfrac{(\beta-\epsilon)(\beta+\gamma)C}{(\beta-1)^2(K-K_E)\gamma} & \textit{if } \gamma < \bar{\gamma} \\[2ex] \dfrac{\beta(1+\gamma)}{\beta-1} - \dfrac{(\beta-\epsilon)C}{(\beta-1)(K-K_E)} & \textit{if } \gamma \geqslant \bar{\gamma} \end{cases} \tag{20}$$

Proposition 8. ***Adjusted cash multiple.*** *Equation* (20) *shows a raw cash multiple, where no time value is considered*[5]*. An adjusted cash multiple can be easily determined:*

$$Adj\,M_{VC} = \begin{cases} M_{VC}D & \textit{if } \gamma < \bar{\gamma} \\[2ex] M_{VC} & \textit{if } \gamma \geqslant \bar{\gamma} \end{cases} \tag{21}$$

where $D = \left(\dfrac{V^{**}}{V^{*}}\right)^{\beta} < 1$ *(for* $V^{**} < V^{*}$*) is the (stochastic) factor that discounts the expected in-flow at the exit back to the entry moment. Naturally, the adjusted cash multiple becomes a more adequate measure as it accounts for the time-value of money, through the stochastic length of time needed to achieve the exit trigger.*

Additionally, the following can be stated:

Corollary 3. ***Sensitivities of the cash multiples.*** *The sensitivities of both types of cash multiples in relation to the main value drivers (*σ*,* γ*, and,* θ_{VC}*) are as follows:*

- $\dfrac{\partial M_{VC}}{\partial \beta}\dfrac{\partial \beta}{\partial \sigma} > 0$*, independently from the level of* γ*;*
- $\dfrac{\partial M_{VC}}{\partial \gamma} < 0$ *if* $\gamma < \bar{\gamma}$*, but* $\dfrac{\partial M_{VC}}{\partial \gamma} > 0$ *if* $\gamma > \bar{\gamma}$*;*
- $\dfrac{\partial M_{VC}}{\partial \theta_{VC}} = 0$*, as* M_{VC} *does not depend on* θ_{VC}

as for the Adjusted Cash Multiple only for $\gamma < \bar{\gamma}$ *needs now to be considered:*

- $\dfrac{\partial Adj\,M_{VC}}{\partial \beta}\dfrac{\partial \beta}{\partial \sigma} > 0$*;*
- $\dfrac{\partial Adj\,M_{VC}}{\partial \gamma} < 0$*;*
- $\dfrac{\partial Adj\,M_{VC}}{\partial \theta_{VC}} = 0$*, given that* $Adj\,M_{VC}$ *does not depend on* θ_{VC} *neither directly nor through* D[6]*.*

[5] This is the normal approach in the VC/PE industry.

[6] Which becomes clear after simplifying D to $\left(\dfrac{\beta-\epsilon}{\beta-1}\dfrac{C}{(K-K_E)\gamma}\right)^{\beta}$.

4 Investment under time restriction on the exit

Consider now the setting where the venture capitalist has a limited time horizon for staying invested in the start-up firm. Let $T \ll \infty$ represent the investment maximum duration, which means the VC is obliged to exit until time $t_0 + T$. t_0 represents the entry date, but for simplifying the notation we set $t_0 = 0$.

A restricted time horizon for the investment will significantly impact the exit dynamics, namely when compared to that of the unrestricted case. In fact, after identifying the trade sale attributes, defined by the pair $\{\phi^*, V_B^{*R}\}$, where the superscript R labels the restricted case, two situations may unfold: (i) either V_B^{*R} is achieved before T, and in this case the trade sale occurs as presented in the previous section; or (ii) the limiting date T arrives without V_B^{*R} has been achieved, which means the trade sale takes place at an inefficient moment, as it only occurs because the VC is forced to exit at that date. Naturally, the time restrictions should impact negatively the value of the exit option since less flexibility is now present.

To compute the option to exit under time constrains, we use the concept of Forward Start Option (FSO), generally defined in Shackleton and Wojakowski (2007), and used in various contexts by Pereira and Rodrigues (2014), Azevedo et al. (2016), Azevedo et al. (2018) and Adkins et al. (2019).

Let $R_{VC}(V_t)$ represent the time-restricted option to exit for the VC. Its value is the same as that of a replicating portfolio, $P(V_t)$, that includes a long position in a perpetual option to exit and a short position on a FSO to exit, that initiates at moment T. The value of the former position can be given by Equation (7), where V_B^{*R} – yet to be determined – replaces V_B^*. The short FSO position is presented next.

Under risk-neutral expectations, the value of a forward start option is:

$$FSO(V_t) = E\left[F_{VC}(V_T)\mathbf{1}_{V_T < V_B^{*R}}\right] e^{-rT} \tag{22}$$

where $F_{VC}(V_T)$ represents the value of the trade sale option at time T, and $\mathbf{1}_{condition}$ equals 1 if the condition is met, and 0 otherwise.

From Shackleton and Wojakowski (2007) (see their Appendix A) we acknowledge that FSO can be expressed as:

$$FSO(V_t) = \frac{(\beta - \epsilon)C}{(\beta - 1)^2}\left(\frac{V_t}{V_B^{*R}}\right)^{\beta} N(-d3(V_t)) \tag{23}$$

where

$$d3(V_t) = \frac{\ln\left(\frac{V_t}{V_B^{*R}}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} + (\beta - 1)\sigma\sqrt{T} \tag{24}$$

The term $N(-d3(V_t))$ captures the (risk-neutral) probability for the state variable to hit the trigger later than time T, conditional to the fact that V_B^{*R} is not achieved at $t \in (0, T]$.

The aforementioned portfolio comes as follows:[7]

$$P(V_t) = F_{VC}(V_t) - FSO(V_t) = \frac{(\beta - \epsilon)C}{(\beta - 1)^2}\left(\frac{V_t}{V_B^{*R}}\right)^{\beta} N(d3(V_t)) \tag{25}$$

Finally, as time T represents the last moment for the exiting, we assume that, at this critical moment, the VC completely looses its bargaining power. In other words, it becomes unfeasible for him to ask for any premium when negotiating the trade sale. In fact, at T the VC receives simply the value of the firm (without any premium), while incurring in transactions costs. These circumstances lead to a negative net payoff for the VC corresponding to ϵC. The present value of the net expected payoff for the case where the exit happens at time T is given by:

$$\Pi_{VC}^{T}(V_t) = -\epsilon C e^{-rT} N(-d2(V_t)) \tag{26}$$

where

$$d2(V_t) = \frac{\ln\left(\frac{V_t}{V_B^{*R}}\right) + \left(\alpha + \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}} - \sigma\sqrt{T} \tag{27}$$

The factor $N(-d2(V_t))$ captures the risk-neutral probability of the option to expire unexercised, with $V_T < V_B^{*R}$.

Putting together all the components, the time-restricted investment payoff received by the VC at the entry moment is:

$$\Pi_{VC}^{R}(V_t) \quad = \quad Q_{VC}^{R}\theta_{VC}V_t + P(V_t) + \Pi_{VC}^{T}(V_t) - (K - K_E) \tag{28}$$

and it would be straightforward to show that $\Pi_{VC}^{R}(V_t) \to \Pi_{VC}(V_t)$ as $T \to \infty$.

Differently from the non-restricted case, the procedure for computing the complete solution is now numerical. In sequence, we need (i) to find the generic entry trigger; (ii) to align the entry trigger with that of the E, and determine the optimal post-money ownership; (iii) and, finally, to plug the optimal ownership into Equation (6) for determining the exit trigger. These steps are presented next in a formal manner.

Proposition 9. ***Entry triggers (restricted duration).*** *The entry trigger for the venture capitalist, V_{VC}^{**R}, when time constrains on the exit are in place, comes as the numerical solution of the following optimization problem:*

$$\max_{V_{VC}^{**R}}\left[\left(\Pi_{VC}^{R}(V_{VC}^{**R})\right)\left(\frac{V_t}{V_{VC}^{**R}}\right)^{\beta}\right] \tag{29}$$

[7]As previously mentioned, Equation (7) is used for computing $F_{VC}(V_t)$, but V_B^{*R} substitutes V_B^{*}.

In addition, the trigger for the entrepreneur remains as presented in Equation (10).

Aligning the investment triggers through the optimal ownership results in the following propositions:

Proposition 10. ***Post-money ownership (restricted duration).*** *The post-money ownership for the venture capitalist corresponds to the share Q_{VC}^{optR} such that $V^{**R} = V_{VC}^{**R} = V_E^{**}$, equations* (29) *and* (10), *respectively. The entrepreneur receives $Q_E^{optR} = 1 - Q_{VC}^{optR}$.*

Proposition 11. ***Exit trigger (restricted duration).*** *The exit trigger for a limited investment duration, $V^{*R} \equiv V_B^{*R}$, can be easily determined by incorporating Q_{VC}^{optR} into Equation* (6). *If this solution leads to a situation of inverted triggers (i.e., the exit trigger smaller than the entry one), a higher premium can be required by the VC, $\bar{\phi}^R > \phi^*$, such that $V^{*R} = V^{**R}$. This situation occurs in the case of high synergy, higher than a certain level (say $\bar{\gamma}^R$). Differently from the unrestricted situation, both $\bar{\gamma}^R$ and $\bar{\phi}^R$ need to be numerically determined.*

Finally, the cash multiples under time restrictions can be computed as follows:

Proposition 12. ***Cash multiples (restricted duration).*** *The cash multiple for the VC when time restrictions on the exit are in place is as follows:*

$$M_{VC}^R = \begin{cases} \dfrac{Q_{VC}^{optR}\theta_{VC}V^{*R} + \dfrac{(\beta-\epsilon)C}{(\beta-1)^2}N(d3(V^{**R})) - \epsilon C e^{-rT}N(-d2(V^{**R}))}{K - K_E} & \text{if } \gamma < \bar{\gamma}^R \\ \dfrac{(1+\bar{\phi}^R)Q_{VC}^{optR}\theta_{VC}V^{**R} - \epsilon C}{K - K_E} & \text{if } \gamma \geqslant \bar{\gamma}^R \end{cases} \tag{30}$$

where Q_{VC}^{optR} and V^{*R} are, respectively, the numerical solutions for the post-money ownership for the VC and the exit trigger. Notice that the density functions $d3(.)$ and $-d2(.)$ are measured at the entry timing, i.e., at $V_t = V^{**R}$.

Proposition 13. ***Adjusted Cash multiples (restricted duration).*** *The adjusted cash multiple that accounts for the time value of money is:*

$$AdjM_{VC}^R = \begin{cases} M_{VC}^R D^R & \text{if } \gamma < \bar{\gamma}^R \\ M_{VC}^R & \text{if } \gamma \geqslant \bar{\gamma}^R \end{cases} \tag{31}$$

where $D^R = \left(\dfrac{V^{**R}}{V^{*R}}\right)^\beta < 1$ is the (stochastic) factor that discounts the expected value at the exit back to the entry moment.

Given the nature of the solutions, the main sensitivities will be studied through a comparative statics analysis.

5 Numerical example and comparative statics analysis

In this section we apply the models to an hypothetical, but realistic, situation. We also perform a comparative statics analysis to main value drivers of the model. Consider the following base case parameters:

Parameter	Description	Value
K	Total investment of the growth opportunity	1,000
K_E	Entrepreneur's own funds	250
α	Drift	2%
r	Risk-free interest rate	4%
σ	Volatility	25%
θ_{VC}	Growth prospects of the VC	2
θ_E	Growth prospects of the Entrepreneur	3
γ	Exit synergies	15%
C	Transaction costs of the exit	100
ϵ	Proportion of C supported by the VC	50%
T	Investment maturity (years)	7 or ∞

Table 1: Base case parameters.

▷ **Outputs for the base case.** In order to understand the impact of the exit restrictions, i.e., the limited duration of the investment, the results, both for $T = 7$ and $T = \infty$, appear in Table 2.

Description	Restricted ($T = 7$)	Unrestricted ($T = \infty$)
Optimal share for the VC	$Q_{VC}^{optR} = 56.8\%$	$Q_{VC}^{opt} = 54.5\%$
Entry trigger	$V^{**R} = 3,435$	$V^{**} = 2,799$
Exit trigger	$V^{*R} = 6,059$	$V^{*} = 6,309$
Cash multiple	$M_{VC}^{R} = 9.6$	$M_{VC} = 10.2$
Adjusted cash multiple	$AdjM_{VC}^{R} = 4.5$	$AdjM_{VC} = 3.5$

Table 2: Main outputs for the base case parameters presented in Table 1.

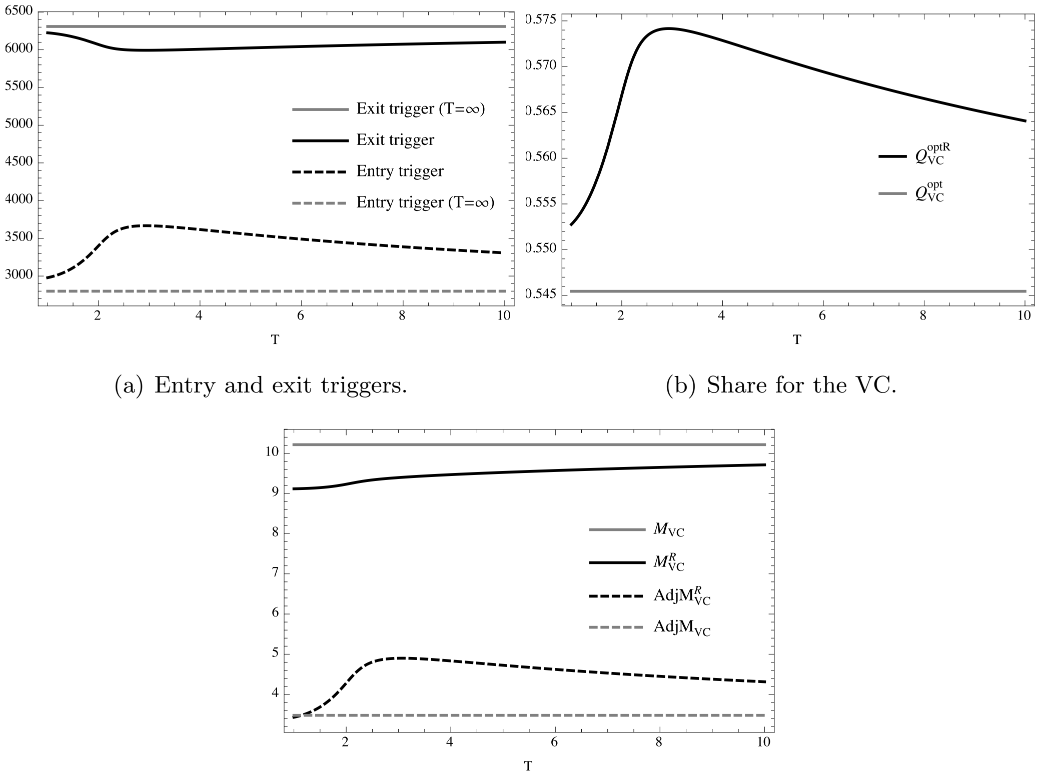
For a limited time duration of investment of 7 years, the VC optimally requires a share of 56.8% on the start-up firm, in exchange of an investment of $K - K_E = \$750$. It is optimal to invest for a level of V equal to \$3,435 and the exit occurs at $V^{*R} = 6,059$. Considering the firm growth expected by the VC, the exit trigger corresponds to $\theta_{VC}V^{*R} = \$12,118$. The expected cash multiple is about 9.6, while the time-adjusted multiple is 4.5.

It is important to deep understand the meaning of both measures in the context of time restrictions. The trade sale dynamics establishes the optimal exit decision, corresponding to a trigger ($V^{*R} = 6,059$) and an optimal trade sale premium $\phi^* = 12.0\%$. At the time of the investment, the VC does not know if the trigger is going to be achieved during the investment period (7 years, in our example). If it happens, the exit occurs and the VC benefits not only from the expected expansion ($\theta_{VC} = 2$) but also from the trade sale premium (after paying for the transaction costs). If the exit trigger is not hit during investment period, the VC sells his fraction on the firm, at year 7, only for its intrinsic value, not being able to extract any additional premium (as we said, under sale pressure, the VC is assumed to loose any bargaining power, ending up in a poor negotiating position). Since the final outcome is not known in advance, the numerators of Equation (30) gives the expected exit proceeds, weighted by the (risk-neutral) probabilities of occurrence. Naturally, the adjusted multiple is far smaller than the raw multiple due to the stochastic discount effect.

When comparing these outcomes to those of the unrestricted case ($T = \infty$), we see that, when no time restrictions are in place, the VC invests earlier, requires a lower fraction on the firm, and presents a higher exit trigger. Our results also show a larger (raw) cash multiple in the unrestricted situation. Interestingly, the adjusted cash multiple is smaller for the unrestricted case, when compared to that of the time-restricted situation. This is due to the effect of a higher discount factor, as the VC optimally invests sooner and exits later if $T = \infty$.

$\triangleright$ **Comparative statics.** Let us know analyze the impact of some key parameters (T, σ, γ, and θ_{VC}) on the main outputs (entry and exit triggers, VC's post-money ownership, and cash multiples).

Figure 1 shows the effect of the duration of the investment. For comparison, the figure also shows the outputs for the time unrestricted situation ($T = \infty$). It becomes clear that the time restriction deters the investment (it rises the entry trigger), despite the higher post-money ownership for the VC. Time restrictions present an opposite effect on the exit decision, reducing the optimal trigger and accelerating the divestment. Naturally, apart from a short non-monotonic period, we see that, as T increases, all outputs converge to the unrestricted solutions. The cash multiples show distinct behaviors: M_{VC}^R stays below M_{VC}, while $AdjM_{VC}^R$ stays above $AdjM_{VC}$. This shows that the adjustment made by the discount factor is smaller in the restricted case, due to the fact that time restrictions induce the firm to enter later and exit sooner than it would happen in an unrestricted case.



(a) Entry and exit triggers.

(b) Share for the VC.

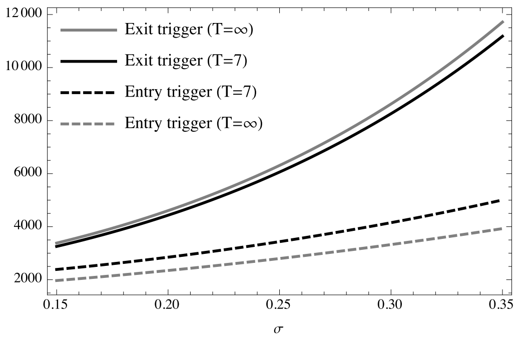
(c) Cash multiples.

Figure 1: The effect of the investment duration. All parameters as in Table 1, except the stressed one (T).

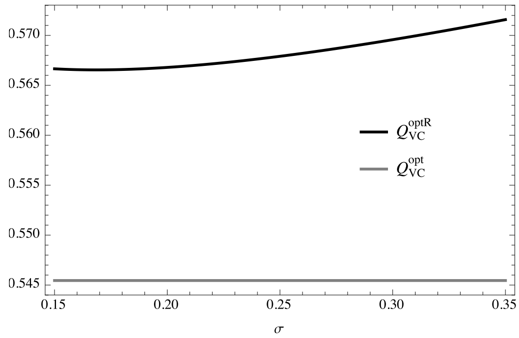
Apart from the well known positive relation between σ and the entry trigger, Figure 2(a) shows that the exit trigger is more sensitive to volatility than the entry trigger. Interestingly, we see that σ has no significant impact on the sharing rule of a time-restricted investment[8], however, time restrictions on the investment induce the VC to require a higher share on the venture. Finally, all cash multiples increase with volatility, which means that larger multiples are expected when investments take place in more uncertain industries. This finding is supported by recent empirical evidence, as for example, in Peters (2018). In addition, the outcomes for the (raw) cash multiple are plausible when compared to those empirically observed by Block et al. (2019)[9].

[8]From Equation (14) we already know that Q_{VC}^{opt} is independent from volatility.

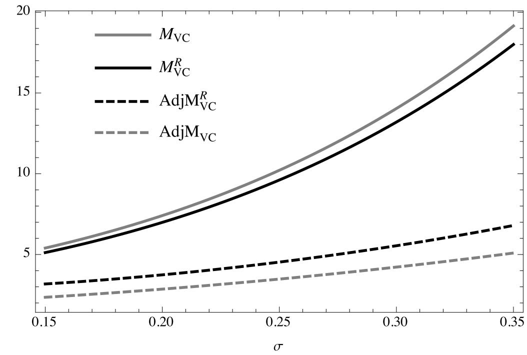
[9]Considering only the cash multiples larger than 1 (for comparability with our model that only returns profitable exits) they show that about 2/3 of the exits produced a cash multiple larger than 2×, and multiples larger than 5× occurred in 1/4 of the exits. Furthermore, almost 1/10 of the exits produced multiples of more than 10×.



(a) Entry and exit triggers.



(b) Share for the VC.

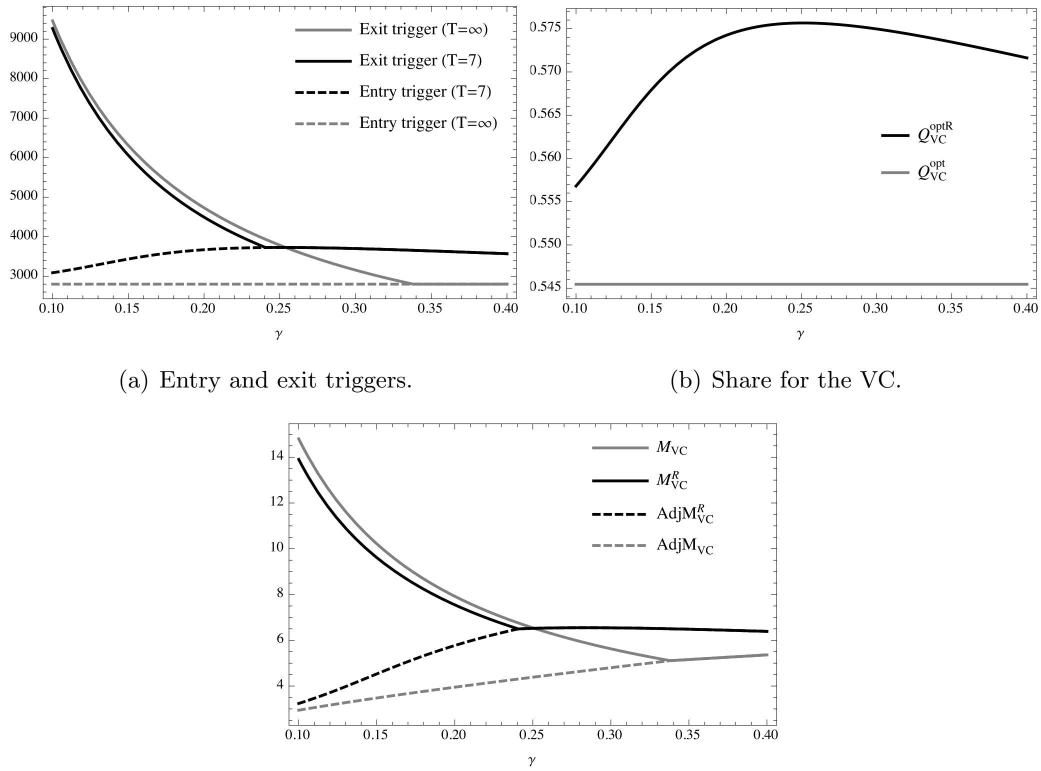


(c) Cash multiples.

Figure 2: The effect of volatility, with and without time restrictions. All parameters as in Table 1, except the stressed one (σ).

Figure 3 shows the effect of synergies arising on the exit. We see that γ impacts significantly the exit decision (the VC divests much sooner when synergies are high) while the impact on the entry decision is less relevant. Both effects produce a decrease of the (raw) cash multiple, unless a reversed order of the triggers occurs. As the exit optimally occurs for a lower value of the state variable, the exit cash-in-flow is smaller for the VC, producing a smaller cash multiple. Notice, however, that, as the exit occurs sooner in the case of high synergies (the trigger is smaller), the previous effect is corrected by the stochastic discount factor, revealing an adjusted cash multiple increasing with γ. For large synergies – in our example for $\bar{\gamma} \geqslant 33.8$ and $\bar{\gamma}^R \geqslant 24.0$, respectively in the unrestricted and restricted case – the VC will exit immediately after investing, so both triggers coincide. In this case, the discount factor does not operate any effect and it is shown that γ only marginally affects the multiples.

The model predicts that, in high synergistic industries, the (raw) cash multiple tends to be smaller, when compared with that of a less synergistic sector; however, the time adjusted multiple is, in fact, larger when synergies are high due to a smaller expected duration of the investment. Transposing this finding to the often used IRR measure, the model predicts a positive relation between γ and the IRR, even though a lower cash multiple may occur.



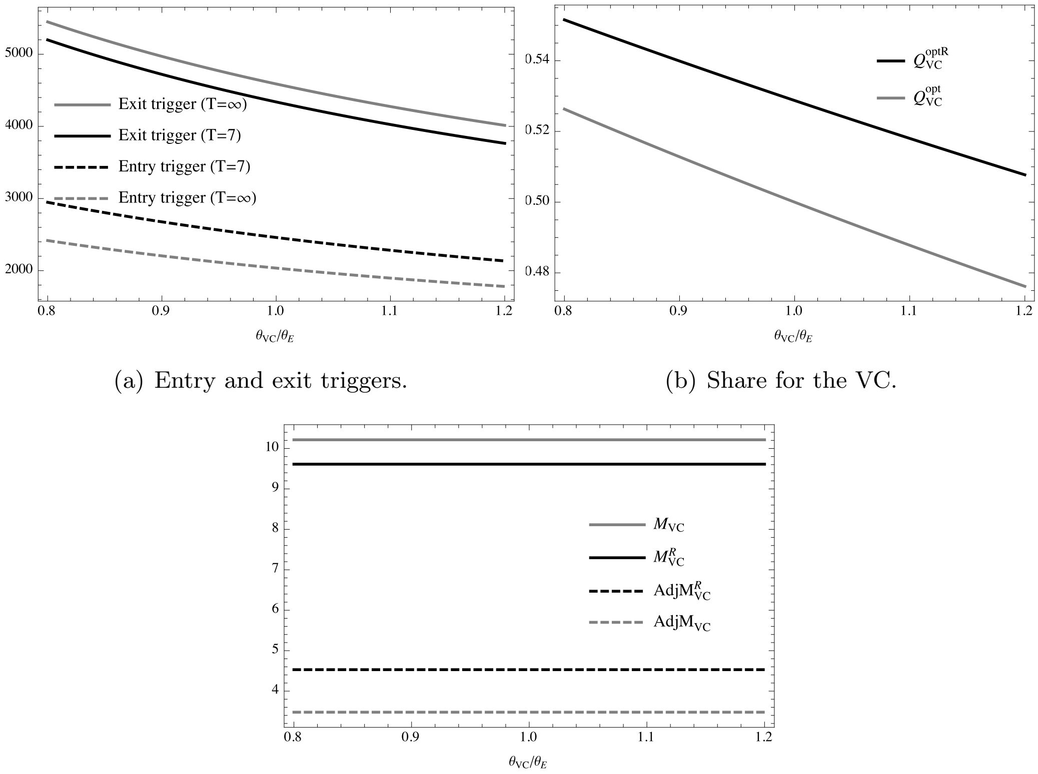
(a) Entry and exit triggers.

(b) Share for the VC.

(c) Cash multiples.

Figure 3: The effect of exit synergies, with and without time restrictions. All parameters as in Table 1, except the stressed one (γ). For these parameters $\bar{\gamma} = 33.8$, and $\bar{\gamma}^R = 24.0$.

Finally, Figure 4 depicts the effect of the VC's growth prospects on the decision, considering both a VC that is less optimistic ($\theta_{VC}/\theta_E < 1$) or a more optimistic ($\theta_{VC}/\theta_E > 1$) than the Entrepreneur. We see that more optimistic VCs invest sooner and require a smaller share on the firm. As already stated, the figure shows that cash multiples are independent from the VC's optimism.



(a) Entry and exit triggers.

(b) Share for the VC.

(c) Cash multiple.

Figure 4: The effect of VC growth prospects, with and without time restrictions. All parameters as in Table 1, except the stressed one (θ_{VC}).

6 Conclusions

In this paper we develop a dynamic model to study the entry and the exit decision of a VC that is facing the opportunity to invest and expand a start-up firm. The expansion cannot be fully financed by the Entrepreneur, due to assumed capital constrains, and so the VC enters as a partner for supporting the growth option. Two fundamental settings are considered: one, where no time constrain for exiting is in place (to serve as a benchmark); and another one, where the VC has a finite time-window for divest from the start-up firm. In both cases we consider the trade sale (M&A) as the exit route. The model (in both setting) returns the entry and the exit triggers, the optimal post-money ownership for each party, and the expected cash multiple for the VC (a typical measure of performance used in the industry). Additionally, the model proposes an adjusted cash multiple that accounts for the time-value of money, which can be useful for measuring, *ex ante*, the expected performance of the investment. The model aims to guide the VCs when analyzing their investment opportunities, considering the entire life-cycle of the business (entry–expand–exit).

The main findings are the following. With time restrictions for exiting, the VC will invest for a larger trigger (i.e., restricted durations deter the investment), require a larger share on the start-up firm, and will optimally exit for a smaller trigger. Additionally, the (raw) cash multiple tends to be smaller in the restricted case; but, after considering the time-value of money, the adjusted cash multiple becomes, in fact, larger. Additionally, we find that volatility has a positive effect both on the entry and exit trigger, as well as, on the cash multiples, which is in line with recent empirical evidence. Regarding the synergies on the exit, the model predicts that, in high synergistic industries, the (raw) cash multiple tends to be smaller than that of a less synergistic sectors; however, when the adjusted multiple is considered, the effect is the opposite. Transposing this finding to a rate of return measure, the model predicts a positive relation between synergies and the IRR, even though a lower cash multiple could be observed. Finally, relatively more optimistic VCs will invest and exit sooner and will require a lower share on the start-up firm, while optimism reveal to have no impact on the cash multiples.

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