Optimal Project Upgrade under Uncertainty: Balancing Risk and Reward

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Abstract

We consider a firm that is currently producing an established product subject to a stochastic environment. The firm has the one-time opportunity to undertake an irreversible investment to switch to a new product, which changes the drift and volatility of the firm's underlying stochastic profit flow. We show that it is optimal to invest in the new product if an increase in the expected growth outweighs the increase in risk from switching. We find that the effect of uncertainty on the optimal investment strategy is not straightforward. The overall effect of uncertainty is determined by the interplay between the value of waiting and the effect of Jensen's inequality. Contrary to the standard real options result, that higher uncertainty delays investment, we show that an increase in volatility in the old market can both accelerate or delay the investment. We perform an analysis of the antecedents of this non-monotonic effect of uncertainty, and provide extensive economical reasoning for the results. Further, we show that an investment opportunity with changing characteristics of the stochastic environment and constant profit function, can be transformed to a case of changing profit function and constant parameters of the stochastic process.

Keywords: Switching options, Real options, Endogenous switching, Optimal stopping

1. Introduction

For a firm currently active in an established product market, switching to a new product is characterized by a change in the firm's stochastic environment. In deciding when to invest in the new product, the firm should take into account how the decision will affect the development of its expected profits, and the related uncertainty from changing the product market within which it operates. Highly innovative products might have the potential to become blockbusters, but also come with additional uncertainty stemming from risks related to how consumers respond to the new product. In the automotive industry, a pressing issue for traditional automobile manufacturers has been whether to enter the novel market for electric vehicles (EVs), and if so, when to enter. The industry incumbents have well-defined

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markets for their traditional products, but might be missing out on the growing profits from green transportation[1]. The EV market is growing faster than the market for traditional cars[2,3]. However, the younger EV market is exposed to higher uncertainty due to demanding technological challenges[4] and uncertain adoption rates of consumers[5]. Therefore, entering the EV market might change both the expected growth and uncertainty of the firm's profits.

This paper studies the investment problem of a monopolistic firm that has the one-time opportunity to switch from one stochastic profit flow to another. Such investment problems under uncertainty can be characterized as real options (Dixit & Pindyck, 1994; Trigeorgis, 1996), recognizing that the firm has discretion of the investment timing. However, a common assumption in modelling real options problems is that the uncertainty faced by the firm is both independent of the firm's actions and constant. In that, the fundamental uncertainty is modelled by an exogenous stochastic process, often a geometric Brownian motion with constant drift and volatility parameters. Acknowledging that the investment decisions of the firm can alter the characteristics of its uncertain profits, we need to account for the fact that the parameters of the stochastic process can be endogenously changed by the decision of the firm. We model the switch in products as a change from one stochastic regime to another. We model the underlying stochastic process as a geometric Brownian motion, but with different drift and volatility coefficients for each regime. Thus, the firm changes the characteristics of its stochastic environment from investing in the new product.

Most earlier work on investment decisions under uncertainty is limited to models where, upon investment, the functional form of the profit function change, but not the parameters of the underlying stochastic process (Dixit & Pindyck, 1994). One of the contributions here is Alvarez & Stenbacka (2001), who studies a setting where the profit function can be improved by adopting a new technology at a later stage. Their model does however assume that the long-term growth and the volatility driving the profits remain constant, and find that increased uncertainty delays the decision to invest. In contrast, our paper allows the parameters of the stochastic process driving the profits to be changed after investment. Under this assumption, increased uncertainty might not delay investment, but accelerate it. Another strand of the literature allows the stochastic process that the firm faces to be subject to different regimes, and that the process alternates between these regimes over time. Bensoussan et al. (2012) introduce a model where the firm is exposed to exogenous and random switches of regimes, in which the regimes differ in the growth and volatility

[1]The traditional German car manufacturers see a stagnation in their traditional markets, and set lofty goals for their EV portfolio. https://www.economist.com/news/business/21737534-coddled-successive-governments-industry-dogged-dieselgate-lagging-electric

[2]Boston Consulting Groups predictions on the EV market going forward: https://www.bloomberg.com/news/articles/2017-11-02/battery-powered-cars-to-be-half-of-global-auto-market-by-2030

[3]Statistics on the sales volumes of EVs: http://www.ev-volumes.com/

[4]How the technological uncertainty of the EV market affects the forecasts for the market: https://www.bcg.com/publications/2018/electric-car-tipping-point.aspx

[5]On the challenges in the EV market from consumer behaviour: https://www.bloomberg.com/news/features/2017-12-19/the-near-future-of-electric-cars-many-models-few-buyers

of the underlying stochastic process. Thus, the firm acknowledges that it might be facing different structures in its stochastic environment, but has no way of influencing the switching process itself. Like other studies on investment under uncertainty, Bensoussan et al. (2012) find that the optimal strategy of the firm is characterized by an investment threshold, where the firm invests as soon as it observes the exogenous process to reach the threshold. In a similar vein, Bollen (1999) studies the life-cycle problem of products, where the demand for a firm's output is randomly switched from an increasing to a decreasing regime. The author shows that the assumption of a constant regime for the stochastic demand can yield significant errors in the value of an investment opportunity. The aforementioned models allow the stochastic process to switch regimes, but do not allow the firm to affect these switches endogenously. Thus, they still assume that the firm is passive w.r.t. its stochastic environment. One attempt on making the underlying stochastic dependent on the firms actions is presented in Busch et al. (2013). They extend the setting of Bensoussan et al. (2012), by allowing the firm to alter the probability of a switch taking place through its investments. This represents an indirect influence of the firm on the stochastic environment, while in our paper we aim to model this explicitly.

For models considering a firm's possibility to directly influence the underlying stochastic process, the available literature is sparse. Kwon (2010) is one of the first to account for changing parameters of the underlying stochastic process in the investment decision. Kwon (2010) studies a firm currently producing an aging product, with the one-time opportunity to invest in a new product. He models the firm's initial profits by an arithmetic Brownian motion with negative drift. Investing in the new product increases the drift of the process. Assuming the resulting drift to still be negative, the paper also considers an exit option after investment, finding that increased uncertainty can both delay or speed up investment in the new product. Matomäki (2013, Article 1) generalizes the problem of Kwon (2010) by allowing for general specifications of the stochastic process, as well as including changes in volatility. He confirms the results of Kwon (2010), that increased uncertainty can both delay or expedite investment. However, both these papers include an exit option in addition to the investment option. Thus, the reason for the non-monotonic effect of uncertainty in these studies is that increasing the volatility increases the value of the exit option, which then makes it more attractive to invest sooner. Contrarily, the value of waiting from the option to invest in the new product increases in uncertainty, which therefore induces a delay in investment. Further, Hagspiel et al. (2016) extends Kwon (2010) by allowing for capacity discretion for the new product, while keeping the volatility constant and only switching the drift. This induces a monotonic effect of uncertainty, where increasing the volatility leads to an increase of the investment threshold. In the work of Kwon (2010), increased uncertainty makes it more likely that the exit option will be used, as the firm has a fixed capacity of production of the second product and cannot adjust to the increased uncertainty in the market. In Hagspiel et al. (2016), the possibility of optimizing the capacity restores the monotonic effect of uncertainty, which suggests that the firm can adjust the capacity to the change in volatility, and therefore making the increased value of the exit option smaller than the increased value of the option to invest in the second product. The aforementioned models with direct changes to the parameters of the stochastic process all include an exit option, which we do not consider. Our work is in line with Alvarez & Stenbacka (2003), as they look at a situation where the firm can invest in a new product, with changing parameters of the stochastic environment, and not including other embedded options. Upon investment, they assume that the volatility of the underlying stochastic process changes, while the drift is not affected. They find that uncertainty can indeed both delay and accelerate investment, under both concave and convex profit functions. This non-monotonic behaviour is a pure effect of the change in parameters of the underlying stochastic process, and does not arise from the presence of a compound option, like an option to exit.

In solving our problem, we utilize the theoretical framework of stochastic optimal control (Pham, 2009), relying on a viscosity solution approach to solving stochastic optimal control problems (Crandall et al., 1992). We apply the results of Ly Vath & Pham (2007) for a general switching problem, which permit multiple switches between various regimes. Our problem is a special case of the multidimensional problem studied therein. Thus, we employ well-defined mathematical results to investigate a novel investment problem, in addition analyzing how the optimal strategy depends on the underlying parameters. We show that it is optimal to invest in the new product if an increase in the expected growth from switching regimes outweighs the increase in risk. Our results confirm the findings of Alvarez & Stenbacka (2003) under cases where the drift is not changed from investment. Also allowing for a potential change in drift, we find that the effect of uncertainty on the optimal strategy is more complex and multifaceted than the results of Alvarez & Stenbacka (2003) imply. We show that increasing the uncertainty in the profits from the initial product can both delay or accelerate the investment. Further, we provide extensive economic reasoning for the determinants of this non-monotonic behaviour. Finally, we show that the posed problem can be transformed to a problem of constant parameters of the stochastic process, but with a change in the profit function after investment.

The rest of this paper is structured as follows. Section 2 presents the model, using the mathematical results of Ly Vath & Pham (2007) of optimal switching. Section 3 presents the comparative statics of the resulting investment threshold. Section 4 presents how a problem of changed profit function but constant parameters of the stochastic process can be transformed to a problem of constant profit function but changing process parameters. Section 5 provides some concluding remarks and possibilities for future work, while Appendix A presents proofs of all propositions.

2. Model

The firm currently operates in a market with an established product, denoted as regime $i = 1$. It holds a one time opportunity to invest in a new product, therewith switching to a regime $i = 2$. Switching incurs an irreversible investment cost. The profitability of the firm is uncertain, and assumed to follow a stochastic process given as a geometric Brownian motion with regime-dependent diffusion parameters. We define a stochastic process θ_i^{ϑ} taking values in $\mathbb{R}_+^* = (0, \infty)$ which satisfies the stochastic differential equation

$$d\theta_i(t) = \alpha_i \theta_i(t)dt + \sigma_i \theta_i(t)dW_t, \qquad \theta_i(0) = \vartheta,\ i = 1, 2, \tag{1}$$

where W is a Wiener process, and α_i and σ_i are drift and volatility coefficients of the process θ in regime i at time t, respectively. The drift parameter of each regime represents the general drift of the market of the given product, while the volatility parameter represents the uncertainty inherent in the given market. We assume that the driving Wiener process W is the same in both regimes, and only the growth and diffusion parameters change. This accounts for a case where the firm changes its product offering when switching regime, but stays within the same general market structure or consumer base for its products. The profit flow per time period for the firm in regime i is given by $\pi(\theta_i(t)) = \eta\theta_i^{\gamma}(t)$, where $\eta > 0$ and $\gamma \in (0, 1]$ are constants. We assume without loss of generality that there are no variable or fixed cost for producing the product. The profit flow is Hölder continuous and thus satisfy linear growth conditions with $\gamma \in (0, 1]$, where $\gamma = 1$ implies Lipschitz continuity[6]. This condition on the value function is necessary to utilize the approach of Ly Vath & Pham (2007), and ensures that the profit function is continuous and cannot explode in value.

The value function of the investment problem, given that the firm is currently in regime $i = 1, 2$, is denoted by v_i. These are the unique continuous viscosity solutions with linear growth conditions on $(0, \infty)$ and boundary conditions $v_i(0^+) = (-g_{ij})_+$, $i \neq j$ to the system of variational inequalities (Ly Vath & Pham, 2007, Thm. 3.4)

$$\begin{aligned} \min\{rv_1 - \mathcal{L}_1 v_1 - \pi_1,\ v_1 - (v_2 - g_{12})\} &= 0, \\ \min\{rv_2 - \mathcal{L}_2 v_2 - \pi_2,\ v_2 - (v_1 - g_{21})\} &= 0. \end{aligned} \tag{2}$$

Here r is the discount rate, g_{ij} is the cost of switching from regime i to j, and $\mathcal{L}_i$ the second-order operator associated with the diffusion θ when the system is in regime i, which for an arbitrary C^2-function φ on $(0, \infty)$ is given as $\mathcal{L}_i\varphi = \frac{1}{2}\sigma_i^2\theta^2\frac{\partial^2}{\partial\theta^2}\varphi + \alpha_i\theta\frac{\partial}{\partial\theta}\varphi$. We assume that $r > \alpha_i\ \forall i$, so that the drift in each region is not higher than the discount factor. This assures that investment might be optimal for the firm. Otherwise, the optimal strategy would always be to wait, as the profits would grow at a higher rate than the discount factor. The regions of θ where it is optimal for the firm to switch from regime i to j are denoted by $\mathcal{S}_i = \mathcal{S}_{ij}$ and defined as

$$\mathcal{S}_i = \{\vartheta > 0 \colon v_i(\vartheta) = v_j(\vartheta) - g_{ij}\}, \quad i, j = 1, 2, \quad i \neq j. \tag{3}$$

The firm would switch from regime i to j for any realizations of the diffusion process for which the optimal value function in the regime equals the value of the other regime, minus the switching cost. This is refereed to as the *stopping regions* of the investment problem. We define an upper and lower threshold of θ where it is optimal to switch from regime i as $\overline{\vartheta}_i^* = \sup \mathcal{S}_i \in [0, \infty]$ and $\underline{\vartheta}_i^* = \inf \mathcal{S}_i \in [0, \infty]$, where the convention $\inf \emptyset = \infty$ ensures that $\underline{\vartheta}_i^* = \infty$ if $\mathcal{S}_i = \emptyset$. I.e., if the stopping region is empty, it is never optimal to switch.

The first term of the maximization in the system of variational inequalities in Eq. (2)

[6]Hölder continuity implies that a function f satisfies $\mid f(x) - f(y) \mid^{\alpha} \leq C \mid x - y \mid^{\alpha}$ for some finite constant C and $\alpha \in (0, 1)$. Lipschitz continuity arises when $\alpha = 1$ in this expression. This also assures that the expectation of the form $\mathbb{E}\left[\int_t^T f(\theta(t), t)dt\right]$ is well defined (Pham, 2005), where $\theta(t)$ is a continuous-time stochastic process.

denotes the value function for $i = 1, 2$ when the firm would optimally stay in regime i and not switch, i.e. the *continuation region.* This term yields a second-order ordinary differential equation for $i = 1, 2$ given by

$$rv_i - \mathcal{L}_i v_i - \pi_i = 0. \tag{4}$$

Omitting the particular term stemming from the running profit function π_i, this ODE has the homogeneous solution given by

$$v_i(\vartheta) = A\vartheta^{\beta_1^i} + B\vartheta^{\beta_2^i}, \tag{5}$$

for some constants A and B, and

$$\beta_1^i = -\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2} + \sqrt{\left(-\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}}, \tag{6}$$

$$\beta_2^i = -\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2} - \sqrt{\left(-\frac{\alpha_i}{\sigma_i^2} + \frac{1}{2}\right)^2 + \frac{2r}{\sigma_i^2}}. \tag{7}$$

It can be shown that $\beta_1^i > 1$ and $\beta_2^i < 0$ (see e.g. Dixit & Pindyck (1994)). Further denote the particular solution to the ODE in Eq. (4) as $\hat{V}_i(\vartheta)$, subject to the boundary condition $\hat{V}_i(0^+) = \pi(0) = 0$. $\hat{V}_i$ is then defined as

$$\hat{V}_i(\vartheta) = \mathbb{E}\left[\int_0^\infty e^{-rt}\pi(\hat{\theta}_i^\vartheta(t))\right] = \frac{\eta\vartheta^\gamma}{r - \alpha_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} = K_i\eta\vartheta^\gamma, \tag{8}$$

where $\hat{\theta}_i^\vartheta$ is the solution to the SDE $d\hat{\theta}(t) = \alpha_i\hat{\theta}(t)dt + \sigma_i\hat{\theta}(t)dW_t$, with $\hat{\theta}(0) = \vartheta$, and we define the perpetual multiplier K_i as

$$K_i = \frac{1}{r - \alpha_i\gamma + \frac{1}{2}\sigma_i^2\gamma(1-\gamma)} > 0. \tag{9}$$

The particular solution for each regime represents the value of never switching, i.e. the value of the firm given that it produces product $i = 1, 2$ forever.

The model outlined above follows the work of Ly Vath & Pham (2007) and would generally allow for multiple switches between the two regimes. However, we focus on a single switch case from an initial regime $i = 1$ to a regime $i = 2$. Switching is costly and irreversible, so that $g_{12} > 0$. Switching back is not possible in our case. Using Theorem 4.1 in Ly Vath & Pham (2007), the existence of a finite switching threshold $\underline{\vartheta}_1^*$ depends on the difference of the value of staying in the regimes in perpetuity, i.e. $\hat{V}_2 - \hat{V}_1$, given that there is no salvage value of switching back. Formally, the result says that for $i, j = 1, 2$, $i \neq j$, assuming $g_{ij} > 0$ and $g_{ji} \geq 0$, if $\hat{V}_i = \hat{V}_j$, then $\mathcal{S}_i = \emptyset$ and $\underline{\vartheta}_1^* = 1$. However, if $\hat{V}_j > \hat{V}_i$, then $\mathcal{S}_i = [\underline{\vartheta}_i^*, \infty)$ with $\underline{\vartheta}_i^* \in (0, \infty)$ and $\mathcal{S}_j = \emptyset$. This assures that after switching from the initial regime $i = 1$, switching back is not optimal, as long as there is a non-negative cost of switching back. The intuitive explanation of this is that if the value of producing the two products forever are equal, the firm has no incentive to pay a positive investment cost to switch products, and the switching threshold is infinite. Conversely, if one of the products has a higher value of producing in perpetuity, there is a finite threshold at which the firm should pay an investment cost to obtain the more profitable profit stream. Given no salvage value there is never an incentive to switch back to the less valuable regime. We can now investigate the optimality of switching, for different combinations of the diffusion parameters. The following proposition presents the conditions on the diffusion parameters for the existence of an optimal finite switching threshold.

Proposition 2.1.

(i) *If* $\alpha_1 = \alpha_2$ *and* $\sigma_1 = \sigma_2$*, then* $K_1 = K_2$ *and* $\mathcal{S}_1 = \emptyset$*. Therefore, it is never optimal to switch regime in this case.*

(ii) *If* $\alpha_1 \neq \alpha_2$ *and* $\sigma_1 = \sigma_2$*, then it is optimal to switch for some threshold* $\underline{\vartheta}_i^* \in (0, \infty)$ *if* $\alpha_1 < \alpha_2$*, but never optimal otherwise.*

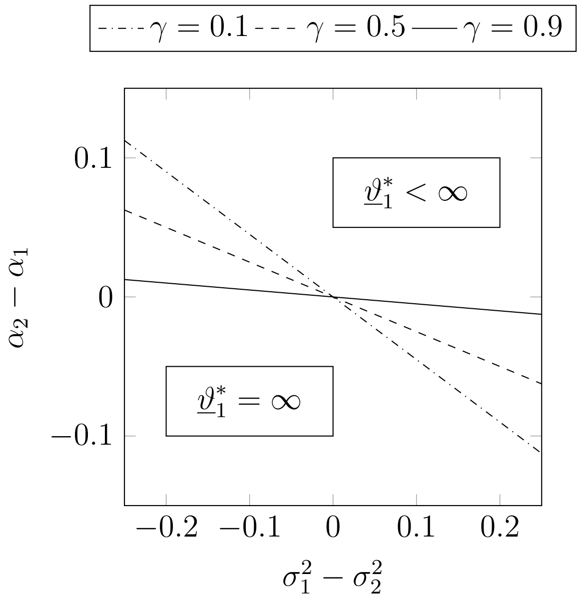
(iii) *If* $\alpha_1 = \alpha_2$ *and* $\sigma_1 \neq \sigma_2$*, then it is optimal to switch for some finite threshold if* $\gamma \in (0, 1)$ *and* $\sigma_1 > \sigma_2$*, but never optimal if* $\gamma = 1$ *or if* $\sigma_1 < \sigma_2$*.*

(iv) *If* $\alpha_1 \neq \alpha_2$ *and* $\sigma_1 \neq \sigma_2$*, the existence of an optimal switching threshold depends both on the difference in drift and volatility for the regimes. There exists a finite investment threshold* $\underline{\vartheta}_1^*$ *if* $K_2 > K_1$*, which is equivalent to the condition*

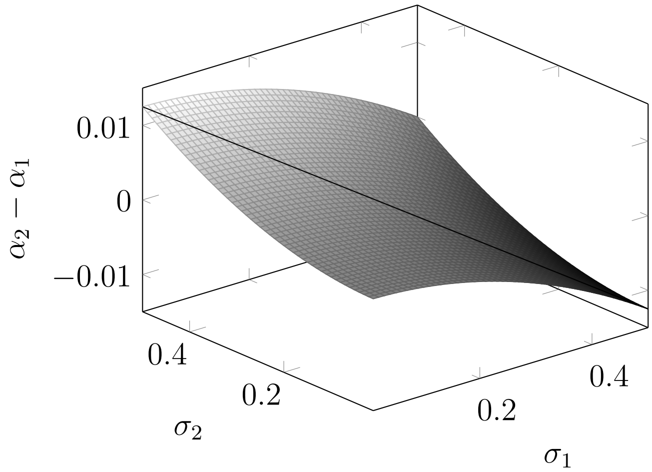
$$\alpha_2 - \alpha_1 + \frac{1}{2}(1 - \gamma)(\sigma_1^2 - \sigma_2^2) > 0. \tag{10}$$

We see from Proposition 2.1 (iii) that the effect of uncertainty is dependent on whether $\gamma = 1$ or $\gamma \in (0, 1)$. If $\gamma = 1$, the profit functions become linear in the diffusion process and the uncertainty of the regime does not affect the expected present value of the perpetuity. Thus, the existence of a finite switching threshold is solely dependent on the difference in drift, i.e. the sign of $\alpha_2 - \alpha_1$. However, if $\gamma \in (0, 1)$, the profit function becomes concave in the diffusion process, and a change in volatility alone can induce switching. The intuition for why a concave profit function leads to dependence on the volatility for the value of the perpetuity, is that a downward shock has a larger negative effect on the firms profit than the positive effect of an equal upward shock. Since the negative effect dominates, an increase in the uncertainty yields a lower value of the perpetuity since shocks (both negative and positive) are more likely with increased uncertainty of the diffusion process. In the case of $\gamma = 1$, the effects of downward and upward shocks of equal size on the profit are equal, so the net effect is zero, making the expected value independent of uncertainty.

If both the drift and the volatility in the regimes are different, Proposition 2.1 (iv) indicates that the optimality of switching depends on the interplay between the parameters of the two regimes. Even if the drift of the second regime is higher than for the first, it is still not optimal to switch if σ_2 is sufficiently high compared to σ_1. If $\alpha_1 > \alpha_2$, there is an



(a) Minimum change in drift $\alpha_2 - \alpha_1$ necessary for the existence of a finite investment threshold $\underline{\vartheta}_1^*$, as a function of the change in squared volatility $\sigma_1^2 - \sigma_2^2$, for $\gamma \in \{0.1, 0.5, 0.9\}$.



(b) Minimum change in drift $\alpha_2 - \alpha_1$ necessary for the existence of a finite investment threshold $\underline{\vartheta}_1^*$, as a function of the volatilities σ_1 and σ_2, assuming $\gamma = 0.9$. The black line represents $\gamma = 0.9$ in Fig. 1a.

Figure 1: Minimum required change in drift $\alpha_2 - \alpha_1$ for the existence of a finite investment threshold $\underline{\vartheta}_1^*$.

upper bound on σ_2 for switching to be optimal, given by

$$\sigma_2^2 < \sigma_1^2 - \frac{2(\alpha_1 - \alpha_2)}{1 - \gamma}. \tag{11}$$

Indeed, if $\alpha_1 > \alpha_2$ and $0 < \sigma_1 < \sqrt{\frac{2(\alpha_1 - \alpha_2)}{1-\gamma}}$, it can never be optimal for the firm to switch, for any positive volatility σ_2. Figure 1 shows the minimum required change in drift from investing necessary for the existence of a finite threshold $\underline{\vartheta}_1^*$, as a function of σ_1 and σ_2. We see that the required change in drift increases in σ_2 and decreases in σ_1, as a decrease in the change of volatility $\sigma_1 - \sigma_2$ from investing makes switching less attractive. In this, the firm requires a higher boost in drift from introducing the new product for switching to be optimal. Figure 1a shows how the steepness of the threshold for the required boost in drift increases when γ decreases. This is due to the fact that a lower γ yields a more concave profit function, so that the asymmetric effect of shocks discussed in the last paragraph is strengthened. Given that a finite optimal threshold exists, the following proposition presents the optimal value function of the firm and the investment threshold.

Proposition 2.2. *Assuming that the conditions in Proposition 2.1 for the existence of a finite switching threshold hold, i.e.* $K_2 > K_1$*, the value of the firm, given that it is currently* *in regime $i = 1$, is given by*

$$v(\vartheta) = v_1(\vartheta) = \begin{cases} A\vartheta^{\beta_1^1} + \hat{V}_1(\vartheta), & \vartheta < \underline{\vartheta}_1^*, \\ \hat{V}_2(\vartheta) - g_{12}, & \vartheta \geq \underline{\vartheta}_1^*, \end{cases} \tag{12}$$

where $\hat{V}_1(\vartheta)$ and $\hat{V}_2(\vartheta)$ are given by Eq. (8). The constant A and the threshold $\underline{\vartheta}_1^*$ are obtained from the value-matching and smooth-fit property of $v_1(\underline{\vartheta}_1^*)$. These boundary conditions give the explicit expressions

$$\underline{\vartheta}_1^* = \left[\frac{\beta_1^1}{\beta_1^1 - \gamma} \frac{1}{K_2 - K_1} \frac{g_{12}}{\eta}\right]^{\frac{1}{\gamma}}, \tag{13}$$

$$A = \frac{\eta\gamma}{\beta_1^1}(K_2 - K_1)(\underline{\vartheta}_1^*)^{\gamma - \beta_1^1}. \tag{14}$$

Proposition 2.2 states that the optimal investment strategy of the firm is to invest in the new product if the value of the stochastic process $\theta_1(t)$ is higher than a certain threshold, i.e. $\vartheta \geq \underline{\vartheta}_1^*$. Otherwise, the firm continues to produce the initial product, and holds on to the the option to invest at some later time, which is done as soon as $\theta_1(t)$ reaches $\underline{\vartheta}_1^*$. For any current value ϑ of the stochastic process $\theta(t)$, the value of the firm is given by Eq. (12). If $\vartheta < \underline{\vartheta}_1^*$, the firm's value is given as the value of producing in regime $i = 1$ forever, plus an option value from the opportunity to switch. If $\vartheta \geq \underline{\vartheta}_1^*$, the firm switches to regime $i = 2$ immediately, and the value of the firm is given as the expected discounted cash flows from producing in regime $i = 2$, minus the investment cost.

3. Comparative statics

This section presents a comparative static analysis on the optimal investment threshold $\underline{\vartheta}_1^*$ with respect to the drift and volatility of each regime. Throughout this section we assume that the conditions in Proposition 2.1 for the existence of a finite switching threshold hold.

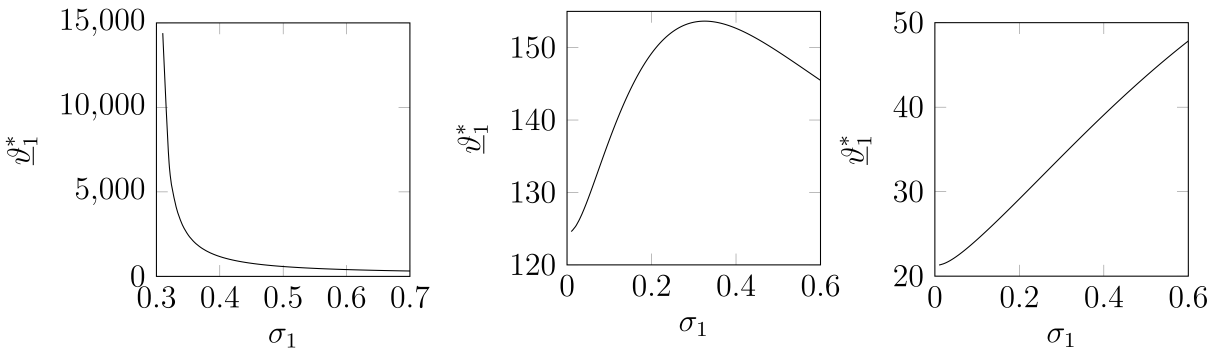
Proposition 3.1. *The optimal switching threshold $\underline{\vartheta}_1^*$ is decreasing in α_2 and increasing in α_1.*

Intuitively, if the positive change in drift from switching product is increasing (decreasing), the firm has more (less) incentive to switch, and thus the threshold where the firm would optimally pay the investment cost to obtain a more valuable profit flow decreases (increases). The following proposition presents the effect of uncertainty on the optimal investment threshold.

Proposition 3.2.

(i) The optimal switching threshold $\underline{\vartheta}_1^*$ is strictly increasing in σ_2. It is increasing in σ_1 if the following condition holds,

$$\frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} - (1 - \gamma)\frac{K_1^2}{K_2 - K_1} > 0, \tag{15}$$



(a) $\alpha_2 = \alpha_1 = 0.02$

(b) $\alpha_2 = 0.035$

(c) $\alpha_2 = 0.06$

Figure 2: Effect of initial uncertainty σ_1 on investment threshold $\underline{\vartheta}_1^*$ for three different values of α_2, capturing different levels of the change in drift from switching. General parameter values: $\alpha_1 = 0.02$, $r = 0.1$, $\sigma_2 = 0.3$, $g_{12} = 100$, $\gamma = 0.9$.

and decreasing otherwise.

(ii) If $\sigma_2 = \sigma_1 = \sigma$, then the optimal switching threshold $\underline{\vartheta}_1^*$ is increasing in σ.

We find that the investment threshold is always increasing in the post-investment volatility σ_2. If the volatility in the new market increases, the firm has less incentive to switch. Therefore, the firm demands a higher expected profit to switch to the new market and the threshold increases, which is a standard real options result. This result is similar to that of Alvarez & Stenbacka (2003) for a concave profit function. For the effect of uncertainty on investment timing, they point to the interplay of two opposing effects: (1) *the real options effect*, and (2) *the Jensen's inequality effect*[7]. Since increased post-investment uncertainty will decrease the expected cumulative profits after investment under a concave profit function (from Jensen's inequality), it will increase the value of waiting (the real options effect), and hence delay investment.

However, the effect of the initial volatility σ_1 is ambiguous, and a higher volatility can both accelerate or delay investment. The condition in Eq. (15) is rather involved, as it depends on most of the underlying parameters. Changing the initial volatility affects both the value of the option of switching and the value of producing in the initial regime in perpetuity, relative to the new regime. We refer to these two effects as the *option effect* and the *perpetuity effect*. An increase in the initial volatility would through the option effect increase the value of waiting, which increases the threshold. Reversely, it is also decreasing the attractiveness of the initial regime relative to the second regime through the perpetuity effect, which increases the incentive of switching and lowers the threshold. Figure 2 plots the threshold $\underline{\vartheta}_1^*$ as a function of pre-investment volatility σ_1 for three different levels of α_2, showing that for different values of α_2 the effect of initial volatility on the investment threshold can be various. In Figure 2c, we see that increasing σ_1 can delay investment

[7]From Jacod & Protter (2004): Jensen's inequality states that if X is a random variable and ϕ is a convex (resp. concave) function, then $\phi(\mathbb{E}(X)) \leq$ (resp. $\geq$) $\mathbb{E}(\phi(X))$

if the change in drift is relatively large. Figure 2b shows a non-monotonic behaviour for small levels of α_2 relative to α_1, so that marginally increasing σ_1 can delay investment for relative low levels of uncertainty, and accelerate investment for higher levels. In the case where the drift of the two regimes is equal, Figure 2a shows that increased initial volatility decreases the investment threshold monotonically, suggesting that the adverse change in value of staying in the initial regime is larger than the increased option value of switching. However, in Figure 2a we see that the investment threshold is very large when σ_1 is close to σ_2. In this case, the benefit of switching to the new product is small. When the change in volatility from switching becomes larger, the threshold drastically decreases in value, as the firm would get a larger benefit from switching to the second regime with lower volatility. In Alvarez & Stenbacka (2003), where the drift is equal for the two regimes, they also find that the effect of the initial uncertainty is unclear, but state that their numerical results suggest that the threshold should decrease in the initial uncertainty. Our analysis confirms the results of Alvarez & Stenbacka (2003). We show that also accounting for a potential change in drift leads to a more complex picture of the effect of uncertainty when the drift are changed from investing as well.

Investigating the antecedents of the two opposing effects, that is, the real options effect and the perpetuity effect, w.r.t. higher initial volatility, we look at how the left-hand side of Eq. (15) varies in the underlying parameters, *ceteris paribus*. In Figure 3, Eq. (15) is plotted as a function of the drift in each regime, the volatilities, the discount rate, and the level of concavity in the profit function. Note that we always assure that the conditions in Proposition 2.1 hold, so that a finite switching threshold exists. From Figure 3a and Figure 3b we find that the drift of the first and second regime have the opposite effect, as this represents changing the difference in drift obtained by switching regime, in each direction. An increase in the change of drift, i.e. $\alpha_2 - \alpha_1$, makes the value of staying in the first regime relatively lower than being in the second regime. From this, the perpetuity effect is relatively weaker and the option effect is more dominant. In Figure 3c we see that increased level of the initial uncertainty makes the marginal effect of a further increase in volatility smaller. Increasing the initial volatility strengthens both the option effect and the perpetuity effect. However, Figure 3c shows that increasing initial volatility would make the perpetuity effect more dominant. Economically, this suggests that for a higher level of uncertainty, a change in the initial uncertainty has a larger effect on the value of continuing to produce the first product, than on the value of waiting inherent in the option to invest in the new product. This explains the non-monotonic behaviour observed in Figure 2b. For the post-investment uncertainty, we see from Figure 3d that this also makes the perpetuity effect more dominant. Increasing the uncertainty in the second regime makes the attractiveness of the new product lower, and therefore weakens the option effect. The initial regime is not affected by the post-investment volatility, and therefore becomes relatively more attractive. Thus, increasing the initial uncertainty has more effect from the impact it has on the value of producing the initial product, the perpetuity effect, than on the value of waiting. In Figure 3e, we see that increasing the discount rate makes the perpetuity effect relatively more dominant compared to the options effect. A higher discount rate makes future profits less significant than profits today. Thus, the options effect from increased initial volatility is less dominant, as this represents the value of waiting from the option to invest in the new product in the future. For γ, which represents the degree of concavity of the profit function, Figure 3f shows that the perpetuity effect is less dominant when γ increases. This is intuitive, as the initial uncertainty has less effect on the value of producing the first product for a lower degree of concavity, resulting from a higher γ, and therefore the option effect is more dominant.

4. Improved profit flow and identical diffusion

Now assume that the initial profit flow is similar to before, but with different multiplicative constants η_i so that $\pi_i(\theta(t)) = \eta_i \theta^\gamma(t)$ for the two regimes $i = 1, 2$. The profit flow in the second regime has a structural improvement, so that $\eta_2 > \eta_1$ in this regime. This can be seen as an upward boost in the profit flow from investing, as this yields that $\pi_1(\vartheta) = \eta_1 \vartheta^\gamma < \pi_2(\vartheta) = \eta_2 \vartheta^\gamma$. However, now we assume that the diffusion parameters are identical in the two regimes, i.e. $\alpha_1 = \alpha_2$ and $\sigma_1 = \sigma_2$. Proposition 4.1 presents the optimal switching strategy for a firm, given that the drift and volatility coefficients are constant and denoted by α_1 and σ_1, respectively.

Proposition 4.1. *Given identical diffusions for the two regimes, but a structural multiplicative improvement of the profit flow in regime $i = 2$ over regime $i = 1$, the value of the firm is given as in Eq. (12) in Proposition 2.2, with constant A and optimal switching threshold $\underline{\vartheta}_1^*$ given by*

$$\underline{\vartheta}_1^* = \left[\frac{\beta_1^1}{\beta_1^1 - \gamma} \frac{g_{12}}{K_1(\eta_2 - \eta_1)} \right]^{\frac{1}{\gamma}}, \tag{16}$$

$$A = \frac{(\eta_2 - \eta_1)\gamma}{\beta_1^1} K_1 (\underline{\vartheta}_1^*)^{\gamma - \beta_1^1}. \tag{17}$$

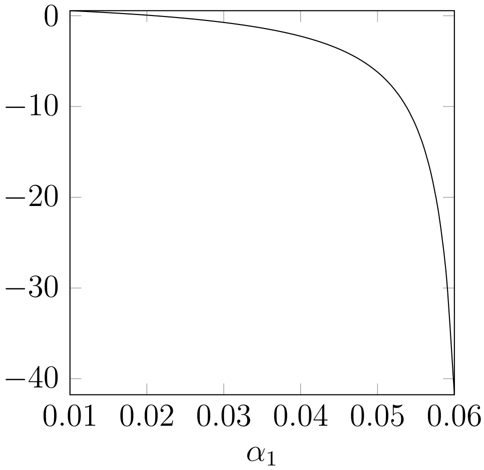
From Proposition 4.1, we can see that an investment problem of investing to boost the profit function can be transformed to a problem of investing to change the coefficients of the stochastic process. We refer to the former problem as a *profit switching* problem, and to the latter as a *regime switching* problem. The following corollary denotes how one problem can be transformed into the other.

Corollary 4.2. *For a regime switching problem with $\pi_i(\theta(t)) = \eta_1 \theta_i^\gamma(t)$ and $\theta_i(t)$ defined as in Eq. (1) with $i = 1, 2$, the equivalent profit switching problem with $\pi_i(\theta(t)) = \eta_i \theta_1^\gamma(t)$, with $\eta_2 > \eta_1$ and $\theta_1(t)$ defined as in Eq. (1) with $i = 1$ is obtained by setting*

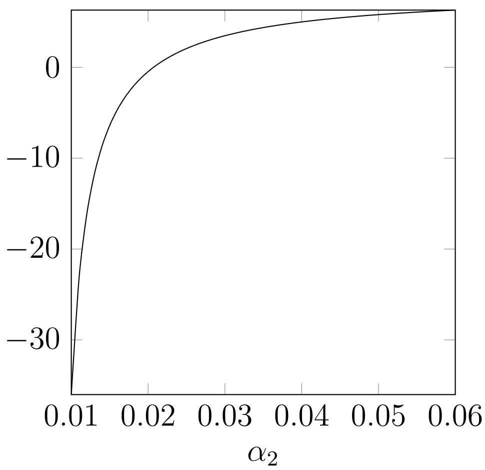
$$\frac{\eta_2}{\eta_1} = \frac{K_2}{K_1}, \tag{18}$$

where K_i is the perpetual multiplier in regime $i = 1, 2$ as defined by Eq. (9).

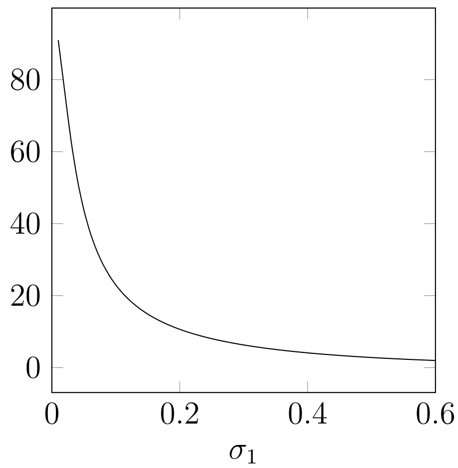
Corollary 4.2 implies that a firm is indifferent between the opportunity of investing to change the coefficients of the stochastic regime and keeping the profit structure unchanged,



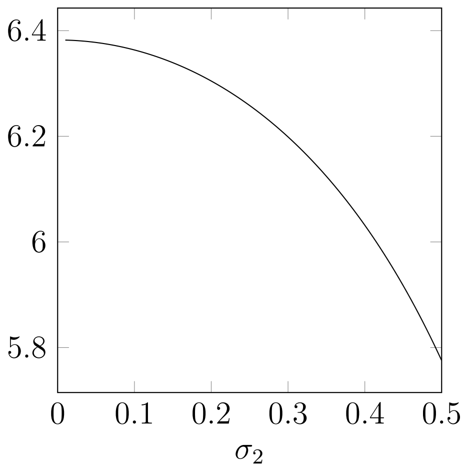
(a) Effect of initial drift, α_1.



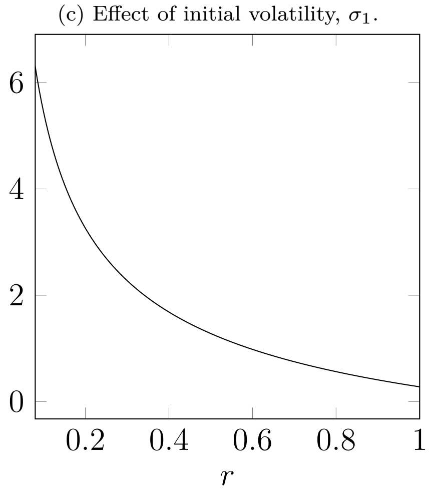
(b) Effect of second drift, α_2.



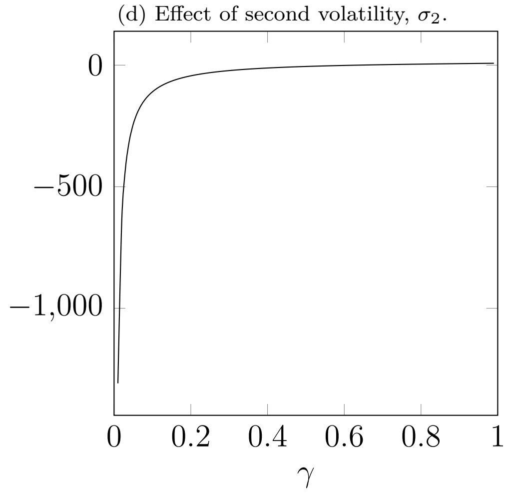
(c) Effect of initial volatility, σ_1.



(d) Effect of second volatility, σ_2.



(e) Effect of discount rate, r.



(f) Effect of concavity factor, γ.

Figure 3: Value of the left hand side of Eq. (15) as a function of the underlying parameters. General parameter values: $\alpha_1 = 0.01$, $\alpha_2 = 0.06$, $\sigma_1 = 0.3$, $\sigma_2 = 0.2$, $r = 0.08$, $\gamma = 0.9$. These plots show the dependence on the underlying parameters for the condition that increased initial uncertainty will increase the investment threshold. A value above (below) zero indicates that increased initial uncertainty increases (decreases) the investment threshold for the given parameters.

and the opportunity to boost the profit structure without the ability to affect the stochastic process, as long as the problem parameters satisfy Eq. (18). This result shows that a problem of singular optimal switching with structurally identical profit flows and different diffusion parameters can be transformed to a problem with changing profit structure but identical coefficients of the stochastic process. Here structurally equal profit flows means that for any realized value of the stochastic process the profit flow is equal, i.e. $\pi_1(\vartheta) = \pi_2(\vartheta)$. This suggests that one could specify the effect of an investment as either changing the coefficients of the underlying stochastic process, or changing the form of the profit functions, and obtain the same results, as long as the earlier assumptions hold. These two different problem specifications have different reasoning. Therefore, the fact that these problem statements are connected could give important insight into similarities between different rationales of investing. More rigorous investigation should be performed on this connection in future work.

5. Conclusions

This paper studies the investment problem of a monopolistic firm subject to a concave profit function, for which the profits are dependent on a stochastic process, specified by a geometric Brownian motion. The firm is currently active in the market with an established product, with the opportunity to invest in a new product. The two products represents different regimes for the firm's stochastic environment, which we model by different coefficients for the drift and volatility of the geometric Brownian motion in each regime. We utilize the results of Ly Vath & Pham (2007) to solve the problem, using as a viscosity solution approach.

We find that increased uncertainty in the second market always delays investment, but that increasing the uncertainty of the initial market can both delay or speed up the investment, dependent on the change in the drift it obtains in the new market relative to the current one. Further, the effect of initial uncertainty is shown to be dependent on two effects: (1) the option effect and (2) the perpetuity effect, which arises from the concave profit function of the firm. We study how the drift and volatility in the two regimes, the discount rate, and the degree of concavity in the profit function influence these two effects, and give economic reasoning for which effect dominates in each case. We confirm the results of Alvarez & Stenbacka (2003), while also showing that the dependence of the optimal investment strategy on the initial uncertainty is more multifaceted when we account for a concurrent change in drift as well as in volatility. Alvarez & Stenbacka (2003) remark that the non-monotonic effect of uncertainty might explain why empirical studies that assume the drift and volatility to be constant have difficulties assessing empirically if the relationship between uncertainty and investment is positive or negative.

A possible extension of this work is to account for strategic interactions in the investment problem. For an exogenous switching problem, Bensoussan et al. (2017) study the case of a duopoly game of Stackelberg competition. In the endogenous switching option that we study in this paper, the decisions of the firms in the game to invest might also affect the regime for the other firms in the market.

This paper is an initial effort of extending the study of investment problems under endogenous effects on the stochastic process, similar to the problems studied in Kwon (2010), Hagspiel et al. (2016), and Matomäki (2013, Article 1). Including decision-dependent changes in the stochastic process yields novel results that partly contradict the well established results of earlier work on investment under uncertainty, and give exciting new possibilities for future research. Especially looking into the connection between changes in the stochastic process and changes in the profit functions is promising. Our results indicate that these two classes of problems for a singular investment decision can be transformed into one another. If this result holds in general, it might give further insight into the reasoning for investments under changing profit regimes.

Appendix A. Proofs

Appendix A.1. Proof of Proposition 2.1

Defining the perpetual discount factor K_i as Eq. (9), Theorem 4.1 in Ly Vath & Pham (2007) states that the condition for the existence of a finite investment threshold in a two-regime case depends on the sign of $K_2 - K_1$, and the sign of the switching costs. Part (i) follows from Theorem 4.1 (1) in Ly Vath & Pham (2007) directly, while part (ii)–(iv) are found by solving for $K_2 > K_1$. □

Appendix A.2. Proof of Proposition 2.2

Noting that the profit functions in each regime are structurally equal, i.e. $\pi_1(\vartheta) = \pi_2(\vartheta) = \eta\vartheta^\gamma$, and satisfying Hölder continuity and linear growth ($\gamma \in (0, 1]$), the assumptions of Ly Vath & Pham (2007) are fulfilled. Thus, the value function are directly found by Theorem 4.1 in Ly Vath & Pham (2007), as $g_{12}, g_{21} \geq 0$ (i.e. there is no salvage value). The constant A and threshold $\underline{\vartheta}_1^*$ is found by direct derivations of the continuity and smoothness of the value function at the threshold. □

Appendix A.3. Proof of Proposition 3.1

Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (13), the partial derivatives w.r.t. α_2 and α_1 are given by

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \alpha_2} = -\frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}}{\eta} \times \left(\frac{K_1}{K_2 - K_1}\right)^2 \times \frac{\beta_1^1}{\beta_1^1 - \gamma} < 0, \tag{A.1}$$

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \alpha_1} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12} \beta_1^1}{(K_2 - K_1)\eta} \times \left[\frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} + \frac{1}{\beta_1^1 - \gamma} \frac{K_1^2}{K_2 - K_1}\right] > 0, \tag{A.2}$$

which always hold as long as the conditions for existence in Proposition 2.1 are fulfilled, i.e. $K_2 > K_1$ and $g_{12} > 0$. □

Appendix A.4. Proof of Proposition 3.2

(i) Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (13), the partial derivatives w.r.t. σ_2 and σ_1 are given by

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \sigma_2} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}(1-\gamma)\sigma_2}{\eta} \times \left(\frac{K_1}{K_2 - K_1}\right)^2 \times \frac{\beta_1^1}{\beta_1^1 - \gamma} > 0 \tag{A.3}$$

$$\frac{\partial \underline{\vartheta}_1^*}{\partial \sigma_1} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}\sigma_1\beta_1^1}{(K_2 - K_1)\eta(\beta_1^1 - \gamma)} \times \left[\frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma_1^2(2\beta_1^1 - 1) + \alpha_1} - \frac{(1-\gamma)K_1^2}{K_2 - K_1}\right]. \tag{A.4}$$

Equation (15) arises from the bracketed term in Eq. (A.4), which determines the sign of the derivative.

(ii) Given that the threshold $\underline{\vartheta}_1^*$ is given by Eq. (13), and $\sigma_1 = \sigma_2 = \sigma$ the partial derivative w.r.t. σ is given by

$$\begin{aligned}\frac{\partial \underline{\vartheta}_1^*}{\partial \sigma} = \frac{(\underline{\vartheta}_1^*)^{1-\gamma} g_{12}\sigma\beta_1^1}{(K_2 - K_1)\eta(\beta_1^1 - \gamma)} \times \Bigg[&(1-\gamma)(K_2 + K_1) \\ &+ \frac{\beta_1^1 - 1}{\beta_1^1 - \gamma} \times \frac{1}{\frac{1}{2}\sigma^2(2\beta_1^1 - 1) + \alpha_1}\Bigg] > 0,\end{aligned} \tag{A.5}$$

which always hold as long as the conditions for existence in Proposition 2.1 are fulfilled.

□

Appendix A.5. Proof of Proposition 4.1

Define the profit functions for each regime to be $\pi_i(\theta(t)) = \eta_i\theta(t)$, $i = 1, 2$, with $\eta_2 > \eta_1$, and that the stochastic process following a geometric Brownian motion has constant parameters, α and σ. Then the expected discounted value of producing in perpetuity in a regime i, for any realized value ϑ of the process $\theta(t)$, is given by Eq. (8) and given as $\hat{V}_i(\vartheta) = K\eta_i\vartheta^\gamma$, with K defined as in Eq. (9). In the stopping region, the value of the firm is given by $\hat{V}_2(\vartheta)$, while the value in the continuation region is given by the Bellman equation

$$rV(\vartheta)dt = \eta_1\vartheta^\gamma dt + \mathbb{E}[dV(\vartheta)]. \tag{A.6}$$

Expanding the expectation using Itô's Lemma, we obtain the differential equation

$$\frac{1}{2}\sigma^2\vartheta^2\frac{\partial^2 V(\vartheta)}{\partial \vartheta^2} + \alpha\vartheta\frac{\partial V(\vartheta)}{\partial \vartheta} - rV(\vartheta) + \eta_1\vartheta^\gamma = 0. \tag{A.7}$$

The particular solution to this second-order differential equation is given by $V^P(\vartheta) = \hat{V}_1(\vartheta)$, while the homogeneous part (excluding the last term) is given by $V^H(\vartheta) = A_1\vartheta^{\beta_1} + A_2\vartheta^{\beta_2}$, where β_1 and β_2 are the positive and negative solution to the quadratic equation $\frac{1}{2}\sigma^2\beta(\beta - 1) + \alpha\beta - r = 0$. The boundary condition that $V^H(\vartheta \to 0^+) = 0$ dictates that $A_2 = 0$.

Given that the firm is currently in regime $i = 1$ and there is a switching cost g_{12} to switch to regime $i = 2$, the value of the firm is given as in Eq. (12). Noting that the value function $v(\vartheta)$ should be continuous and smooth at the investment threshold $\vartheta = \underline{\vartheta}_1^*$ (Dixit & Pindyck, 1994; Alvarez & Stenbacka, 2003), we obtain the stated values of the constant A_1 and threshold $\underline{\vartheta}_1^*$ by direct derivation. □

Appendix A.6. Proof of Corollary 4.2

We compare the solution for the regime switching problem given in Proposition 2.2 and the solution for the profit switching problem given in Proposition 4.1. Equate the expressions for the investment threshold in each problem, given for the regime switching problem by Eq. (13), with $\eta = \eta_1$, and for the profit switching problem by Eq. (16). By straightforward rearrangement, we find that the stated condition in Eq. (18) makes the investment thresholds equal. Next, comparing the constant term for the solution of each problem, given by Eq. (14) and Eq. (17), we find directly by substitution that these are equal when Eq. (18) holds. □

References

Alvarez, L. H., & Stenbacka, R. (2001). Adoption of uncertain multi-stage technology projects: a real options approach. *Journal of Mathematical Economics*, *35*, 71–97.

Alvarez, L. H., & Stenbacka, R. (2003). Optimal risk adoption: a real options approach. *Economic Theory*, *23*, 123–147.

Bensoussan, A., Hoe, S., Yan, Z., & Yin, G. (2017). Real options with competition and regime switching. *Mathematical Finance*, *27*, 224–250.

Bensoussan, A., Yan, Z., & Yin, G. (2012). Threshold-type policies for real options using regime-switching models. *SIAM Journal on Financial Mathematics*, *3*, 667–689.

Bollen, N. P. (1999). Real options and product life cycles. *Management Science*, *45*, 670–684.

Busch, M., Korn, R., & Seifried, F. T. (2013). Optimal consumption and investment for a large investor: An intensity-based control framework. *Mathematical Finance*, *23*, 687–717.

Crandall, M. G., Ishii, H., & Lions, P.-L. (1992). User's guide to viscosity solutions of second order partial differential equations. *Bulletin of the American mathematical society*, *27*, 1–67.

Dixit, A. K., & Pindyck, R. S. (1994). *Investment under uncertainty*. Princeton university press.

Hagspiel, V., Huisman, K. J., Kort, P. M., & Nunes, C. (2016). How to escape a declining market: Capacity investment or exit? *European Journal of Operational Research*, *254*, 40–50.

Jacod, J., & Protter, P. (2004). *Probability essentials*. Springer Science & Business Media.

Kwon, H. D. (2010). Invest or exit? Optimal decisions in the face of a declining profit stream. *Operations Research*, *58*, 638–649.

Ly Vath, V., & Pham, H. (2007). Explicit solution to an optimal switching problem in the two-regime case. *SIAM Journal on Control and Optimization*, *46*, 395–426.

Matomäki, P. (2013). *On two-sided controls of a linear diffusion*. Ph.D. thesis Turku School of Economics.

Pham, H. (2005). On some recent aspects of stochastic control and their applications. *Probability Surveys*, *2*, 506–549.

Pham, H. (2009). *Continuous-time stochastic control and optimization with financial applications* volume 61. Springer Science & Business Media.

Trigeorgis, L. (1996). *Real options: Managerial flexibility and strategy in resource allocation*. MIT press.