Financing, investment, liquidation, and costly reversibility

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Abstract: We examine the interaction between financing and investment decisions under the condition that debt holders have the option of maximizing the debt-collection amount if a firm is liquidated during financial distress. We add to the literature by incorporating debt holders' optimization considerations related to the debt-collection amount. We show that if the debt-collection amount increases ex post when the firm is liquidated, the firm increases the amounts of debt issuance and investment quantity ex ante, delaying the corporate investment. This relationship is based on the fact that an increase in the debt-collection amount decreases the credit spread of debt holders. These results fit well with those of existing empirical studies.

Keywords: Real options pricing; Financing; Debt-collection maximization; Liquidation; Costly reversibility.

JEL classification: G32; G33; D21.

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1 Introduction

Using contingent claim models in corporate finance, numerous studies have analyzed the interactions between financing and investment decisions. Such studies include those of Brennan and Schwartz (1984), Mauer and Triantis (1994), Mauer and Sarkar (2005), Sundaresan and Wang (2007), Wong (2010), and Shibata and Nishihara (2015). In these contingent models, after the firm exercises its option to invest, bankruptcy becomes identical to liquidation (i.e., Chapter 7 of the US bankruptcy code). Once a firm enters a period of financial distress, it is liquidated immediately (i.e., the firm stops operating).

In practice, most companies in financial distress try to drive a turnaround by restructuring their outstanding debt by transferring corporate ownership to the debt holders (i.e., Chapter 11 of the US bankruptcy code). In addition, even if the firm fails to do so at the time of liquidation, the debt holders try to collect as much of the debt as possible, for example, by selling the used facility in the second-hand market. Thus, in a contingent claim model, we should incorporate optimization considerations at the time of default and shutdown (liquidation). At the time of default, the option is to drive a turnaround by transferring ownership to the debt holders. At the time of liquidation, the option is to maximize the amount of debt collection.

These two considerations are not incorporated in prior studies. For example, assume a model in which the amount of debt collection is not decided endogenously at the time of liquidation (i.e., the amount is given). Now, when this amount is expected to increase, the amount of debt issuance will decrease and the investment quantity will be invariant at the time of investment.[1] These theoretical results are not fully consistent with the empirical results of Riordan and Williamson (1985), Choate (1997), Vilasuso and Minkler (2001), Acharya et al. (2007), and Sibikov (2009), where the amounts of asset liquidity and debt issuance have a positive relationship. As shown in Section 4, without the option of costly reversibility, the theoretical results are as follows: the correlation between the debt-collected amount and face value is negative (-0.8206), the correlation between the debt-collected amount and investment trigger is negative (-0.9491), and the correlation between the debt-collected amount and investment quantity is 0.1566, which indicates

[1] These results correspond to those of our model without costly reversibility and to those of Shibata and Nishihara (2018) without financing constraints.

almost no correlation.[2]

In this study, we assume that debt holders make optimal decisions for default and shutdown (liquidation) after investment. Then, we examine the interactions between the financing and investment decisions of equity holders. To be more precise, once a firm goes into a period of financial distress, the debt holders get the ownership of corporate management transferred to them from the equity holders. Then, they decide whether the firm should continue or should cease operating after comparing the payoffs of continuation and shutdown. In addition, at the time of shutdown, the debt holders maximize their payoffs by recovering as much of their debt as possible. This extension provides theoretical results that are consistent with existing empirical results.

Mathematically, as shown in Table 1, the model has eight control variables. Two of these variables are determined automatically when the other six are chosen. The first indicates whether to adopt risk-free debt or risky debt at the time of investment, and the second indicates whether to continue or shut down at the time of default. These two variables are decided by choosing the optimal coupon payment and shutdown trigger, respectively. Thus, we have six control variables determined at the three times of investment and financing, default, and shutdown (liquidation). We derive six optimal decisions by working backward in three steps. First, we assume that four control variables (investment trigger, investment quantity, coupon payment, and default trigger) are fixed. We also assume that debt is risky and debt holders decide to continue operation at the time of default.[3] Then, we define the value function after default, and derive the optimal shutdown trigger and optimal liquidation value (which depend on the investment quantity). Two decisions are made by the debt holders to maximize their payoffs (i.e., the amount of debt collected). Second, we assume that three control variables (investment trigger, investment quantity, and coupon payment) are fixed. We then define the value functions after investment and derive the optimal default trigger (which depends on the coupon payment and investment quantity). The decision on whether to continue or to shut down is made automatically according to the magnitude of the relationship between the default and shutdown triggers. Third, we formulate the value before investment, and

[2]These figures correspond to those on the right-hand side of Table 2.

[3]When debt is risk free, the debt holders will make no decision because they will obtain the face value of the debt at the time of shutdown (liquidation).

then derive the optimal investment trigger, investment quantity, and coupon payment. Here, the decision on whether to adopt a risk-free debt or risky debt is made automatically according to the magnitude of the optimal coupon payment.

[Insert Table 1 about here]

Importantly, the debt holders in our model make two decisions after investment.[4] These two decisions are based on two assumptions: partial reversibility and costly reversibility. The former yields one control variable, namely the shutdown trigger decision (liquidation time). To be more precise, if the investment is partially reversible, the liquidation value is defined by some portion of the investment cost corresponding to the resale price of the used facility when the firm is liquidated. Then, the debt holders obtain ownership of the corporate management at the time of default, and have the option to continue operating or to stop. Several papers incorporate the notion of partial reversibility for investment, including those of Abel and Eberly (1994), Abel and Eberly (1996), Abel et al. (1996), and Wong (2010). However, these papers do not consider debt financing because of the assumption of all-equity financing. The latter (the other control variable) is due to the assumption of costly reversibility, which is observable in practice. For example, assume that a facility (such as a plant or equipment) is industry-specific, implying that the resale price is zero unless the firm incurs some cost. However, by changing the industry-specific facility to a more general one with incurred costs (i.e., costly reversibility), the resale price may be positive in the second-hand market. Ramey and Shapiro (2001) and Chirinko and Schaller (2009) show the existence of costly reversibility.[5] Shibata and Wong (2019) incorporate the notion of costly reversibility. However, they do not consider debt financing because all-equity financing is assumed.

The contribution of this study is that we consider the optimization problem of debt holders with control variables during the financial distress period, as well as the interactions between the financing and investment decisions of equity holders. To be more precise, equity and debt holders have four and two control variables, respectively. How-

[4] In fact, debt holders have three control variables, although one of the three decisions is determined automatically by choosing the other two decisions.

[5] For instance, Chirinko and Schaller (2009) estimated the irreversible premium that is both economically and statistically significant.

ever, in previous models, debt holders have no control variables,[6] and so the interactions between the financing and investment decisions are considered only through the decision-making of the equity holders. Thus, the difference between this and previous studies is that we investigate the interactions by considering the decision-making of both equity and debt holders. In particular, we consider how the debt holders' maximization strategies during financial distress affect the interactions between the financing and investment decisions.

We provide a novel result by incorporating the debt holders' optimization considerations. As a benchmark, we assume that debt holders are not allowed to maximize the amount of debt collected at the time of liquidation (i.e., the firm does not have the option of costly reversibility). Then, if the liquidation value increases, the amount of debt issuance might decrease, the investment quantity will remain invariant, and the investment is accelerated (the investment trigger decreases). These results do not fully fit with the empirical findings of Riordan and Williamson (1985), Choate (1997), and Vilasuso and Minkler (2001).[7] By contrast, assume that the firm is allowed to maximize the amount of debt collected at the time of liquidation (i.e., the firm has the option of costly reversibility). A key finding is that our theoretical results are at least consistent with existing empirical results (previously cited). To be more precise, if the liquidation value increases, the debt issuance amount and investment quantity increase, and the investment is delayed (the investment trigger increases). These results are obtained through the following mechanism. An increase in the amount of debt collected decreases the credit spread of the debt holders, leading to an increase in the amount of debt issuance for the equity holders. Then, the investment quantity is increased, and the investment is delayed. These results fit well with those of the empirical studies of Riordan and Williamson (1985) and Vilasuso and Minkler (2001).

The remainder of the paper is organized as follows. Section 2 describes our model, derives the value functions after investment, and formulates the optimization problem. Section 3 provides the solution to the problem and analyzes the properties of the solution. Section 4 discusses the implications of the model. Section 5 concludes the paper. Technical developments are included in two appendices. Appendix A provides the proof of the

[6]This is because investment is assumed to be completely irreversible.

[7]In these papers, firms with relatively specific assets incur little or no debt.

lemmas and propositions in our model. Appendix B derives the solution in the absence of costly reversibility.

2 Model

This section describes the model from three perspectives. First, we describe the model setup. Second, we provide the value functions after investment. Third, we define the value function before investment and formulate the financing and investment decision problem.

2.1 Setup

Consider a firm with an option to invest in a project facility. Throughout our analysis, we assume that the firm is risk neutral and aims to maximize the expected firm value.

When the option to invest is exercised, the firm receives an instantaneous cash inflow $(1-\tau)qX(t)$ after investment, where $\tau > 0$ represents the tax rate, q represents the investment quantity, and $X(t)$ represents the stochastic price given by the geometric Brownian motion

$$\mathrm{d}X(t) = \mu X(t)\mathrm{d}t + \sigma X(t)\mathrm{d}z(t), \quad X(0) = x,$$

where $\mu > 0$ and $\sigma > 0$ are positive constants, and $z(t)$ denotes the Brownian motion defined by a risk-neutral probability space $(\Omega, \mathcal{F}, \mathbb{Q})$.[8] For convergence, we assume that $r > \mu$, where r is a constant risk-free interest rate.[9] We also assume that the current value $X(0) = x > 0$ is sufficiently low that the firm does not undertake the investment option immediately.

When the investment option is exercised, the firm incurs an investment cost $I(q) > 0$. We assume that $I(q)$ is a function of the investment quantity q, and that it satisfies three conditions: $I(0) > 0$, $I'(q) > 0$, and $I''(q) > 0$. From an economic viewpoint, $I(q)$ is interpreted as the purchase price of installing the production facility, which increases with the magnitude of the investment quantity.

[8]This assumption is the same as that in Goldstein et al. (2001) and Sundaresan and Wang (2007).

[9]The assumption $r > \mu$ ensures that the value of the firm is finite. See Dixit and Pindyck (1994) for details.

After investment, the firm has the option to shut down the project and sell the used production facility at a resale price of $(s+k)I(q)$, where $s \in [0,1]$ and $k \in [0,1) \cap [0,1-s]$ represent the reversible ratios without any cost ("costless reversibility") and with a cost ("costly reversibility"), respectively. Thus, the purchase price is decomposed into three components: the costless reversible component $sI(q)$, the costly reversible component $kI(q)$, and the depletable (disposal) component $(1-s-k)I(q)$, which is completely depleted once the project is abandoned.

Let $g(k) \geq 0$ denote the cost function of costly reversibility. The function $g(k) \geq 0$ is interpreted as the cost of the transformation from an industry-specific facility to a more general facility.[10] We assume that $g(k)$ satisfies four conditions: $g(0) = 0$, $g'(k) > 0$, $g''(k) > 0$, and $\lim_{k \to 1} g(k) = +\infty$. The first condition, $g(0) = 0$, means that the cost is zero if the firm does not undertake the costly reversibility (i.e., $k = 0$). The second and third conditions state that $g(k)$ is strictly increasing and convex with k, respectively. The final condition, $\lim_{k \to 1} g(k) = +\infty$, implies that $k < 1$ when $s = 0$ (i.e., completely restoring the zero-valued capital using the costly reversibility, which incurs a huge cost that the firm cannot pay).[11] In other words, the reason $k < 1$ is that it is too costly to restore the used facility at the zero resale price to that at the purchase price. Most importantly, the ratio of costly reversibility, $k \in [0,1) \cap [0,1-s]$, is determined endogenously, implying that the liquidation value is decided endogenously.

In addition, following Leland (1994) and Fan and Sundaresan (2000), we assume that the firm incurs a (bankruptcy) liquidation cost $\alpha((s+k)I(q) - g(k)) \geq 0$, representing some portion of the liquidation value, where $\alpha \in (0,1)$. Thus, in this study, the liquidation value is defined as $(1-\alpha)((s+k)I(q) - g(k)) \geq 0$.

2.2 Value functions after investment

This subsection provides the value functions after investment. When deriving these value functions, we assume that $X(t)$, $q \geq 0$, and $c \geq 0$ are given. The value functions are obtained by working backward.

[10] It can also be regarded as the marketing cost or search cost of finding a matching buyer for a highly specialized facility.

[11] The final condition means that it is impossible to obtain $k = 1$ under $s = 0$, although we can have $s + k = 1$ under $s > 0$.

Now, suppose that the firm stops operating. At the time of liquidation, the optimization problem is formulated as

$$L(q) := \max_{k \in [0,1-s]} (s+k)I(q) - g(k). \tag{1}$$

Note that the optimal ratio k is constrained by the regions of $k \in [0, 1-s]$. As shown in Appendix A, we have the following lemma.

Lemma 1 *The optimal costly reversible ratio, $k(q)$, is obtained by*

$$k(q) = \begin{cases} 0, & \text{if } I(q) \in (0, g'(0)), \\ {g'}^{-1}(I(q)), & \text{if } I(q) \in [g'(0), g'(1-s)], \\ 1-s, & \text{if } I(q) \in (g'(1-s), +\infty). \end{cases} \tag{2}$$

In Lemma 1, $I(q)$ and $g'(k)$ correspond to the marginal revenue and cost of costly reversibility, respectively. Importantly, the investment cost $I(q)$ turns out to be the marginal revenue for the costly reversibility at the time of shutdown (liquidation). If $I(q) < g'(0)$, the firm does not exercise the costly reversibility. Otherwise, the firm exercises the costly reversibility.

Substituting the optimal ratio $k(q)$ into (1) gives

$$L(q) = \big(s + k(q)\big)I(q) - g(k(q)) \geq 0, \tag{3}$$

where $L(q)$ represents the endogenous liquidation value and is a function of q.

We assume that the firm issues a perpetual debt to finance the investment cost $I(q) > 0$. For analytical convenience, we limit the condition such that the firm issues a perpetual debt. This assumption, as in Black and Cox (1976) and Leland (1994), simplifies the analysis without substantially altering the key economic insights. Then, the face value of the debt is given as $c/r \geq 0$, where $c \geq 0$ indicates the coupon payment. There are two kinds of debts: risky debt and risk-free (riskless) debt. If the face value of the debt is larger than the liquidation value after the bankruptcy costs, that is,

$$\frac{c}{r} > (1-\alpha)L(q) \geq 0$$

is satisfied, then the debt is risky. Otherwise, the debt is risk free. We define the threshold of the coupon payment, which determines whether the debt is risky or risk free, as

$$\theta_1(q) := r(1-\alpha)L(q) \geq 0. \tag{4}$$

From (4), we have the following definition.

Definition 1 *Debt is risk free if* $c \in (0, \theta_1(q)]$. *Debt is risky if* $c \in (\theta_1(q), +\infty)$.

Let $D(X(t), q, c)$, $E(X(t), q, c)$, and $V(X(t), q, c) := D(X(t), q, c) + E(X(t), q, c)$ denote the values after investment of the debt, equity, and total firm, respectively. Importantly, each value differs according to whether the debt is risk free or risky. That is, each value is defined as

$$f := \begin{cases} f_1, & \text{if } c \in [0, \theta_1(q)], \\ f_2, & \text{if } c \in (\theta_1(q), +\infty), \end{cases}$$

where $f \in \{D, E, V\}$. More precisely, subscripts 1 and 2 represent "risk-free debt and equity" financing and "risky debt and equity" financing, respectively.[12]

We denote the (stopping) time of investment (indicated by superscript "i"), default (indicated by superscript "d"), and shutdown (indicated by superscript "s") as T^{i}, T^{d}, and T^{s}_m, respectively ($m \in \{1, 2\}$). Mathematically, these times are defined as $T^{\mathrm{i}} := \inf\{t \geq 0 | X(t) \geq x^{\mathrm{i}}\}$, $T^{\mathrm{d}} := \inf\{t \geq T^{\mathrm{i}} | X(t) \leq x^{\mathrm{d}}\}$, and $T^{\mathrm{s}}_m := \inf\{t \geq T^{\mathrm{i}} | X(t) \leq x^{\mathrm{s}}_m \wedge x^{\mathrm{d}}\}$,[13] respectively, where x^{i}, x^{d}, and x^{s}_m denote the associated investment, default, and shutdown triggers, respectively.

2.2.1 Value functions of a firm financed by risk-free debt and equity

Assume the case of $c \in [0, \theta_1(q)]$, that is, a firm financed by risk-free debt and equity and an all-equity-financed firm.

Consider any time $t \geq T^{\mathrm{i}}$ after investment. The value of the risk-free debt, $D_1(X(t), q, c)$, is equal to the face value of the debt, that is,

$$D_1(X(t), q, c) = \frac{c}{r} \geq 0. \tag{5}$$

The value of the equity, $E_1(X(t), q, c)$, is given by

$$E_1(X(t), q, c) := \sup_{T^{\mathrm{s}}_1 (\geq t)} \mathbb{E}^{X(t)} \left[\int_t^{T^{\mathrm{s}}_1} \mathrm{e}^{-r(u-t)} (1-\tau)(qX(u) - c) \mathrm{d}u + \mathrm{e}^{-r(T^{\mathrm{s}}_1 - t)} \left((1-\alpha)L(q) - \frac{c}{r} \right) \right],$$

[12] More precisely, the cases of $c = 0$ and $c \in (0, \theta_1(q)]$ correspond to "all-equity" and "risk-free debt and equity" financing, respectively. Here, by substituting $c = 0$ into the functions under risk-free debt and equity financing, we obtain the functions under all-equity financing. Thus, to simplify the notation, we denote the subscript "1" for $c \in [0, \theta_1(q)]$.

[13] Specifically, default does not exist for $m = 1$, but does for $m = 2$.

where $\mathbb{E}^{X(t)}$ denotes the expectation operator conditional on $X(t)$. The first term represents the discounted value of an instantaneous cash flow after tax, $(1-\tau)(qX(u)-c)$. The second term indicates the discounted residual value (the liquidation value minus the face value of the debt). Using the standard arguments of Dixit and Pindyck (1994), $E_1(X(t), q, c)$ is given by

$$E_1(X(t), q, c) = vqX(t) - (1-\tau)\frac{c}{r} + \left((1-\alpha)L(q) - vqx_1^{\mathrm{s}}(q,c) - \tau\frac{c}{r}\right)\left(\frac{X(t)}{x_1^{\mathrm{s}}(q,c)}\right)^{\gamma}, \tag{6}$$

where $v := (1-\tau)/(r-\mu)$, $\gamma := 1/2 + \mu/\sigma^2 + ((\mu/\sigma^2) - 1/2)^2 + 2r/\sigma^2)^{1/2} < 0$, and the optimal shutdown trigger $x_1^{\mathrm{s}}(q,c) := \mathrm{argmax}_y\{(1-\alpha)L(q) - vqy - \tau c/r\}(X(t)/y)^{\gamma}$ is

$$x_1^{\mathrm{s}}(q,c) = \frac{\varepsilon}{q}\left((1-\alpha)L(q) - \tau\frac{c}{r}\right) \geq 0, \tag{7}$$

where $\varepsilon := \gamma/((\gamma-1)v) > 0$. Note that $x_1^{\mathrm{s}}(q,c)$ is a negatively linear function of $c \geq 0$.

2.2.2 Value functions of a firm financed by risky debt and equity

Assume the case of $c \in (\theta_1(q), +\infty)$, that is, a firm financed by risky debt and equity.

The firm begins the corporate operation by issuing a risky debt at the time of investment, to obtain $(1-\tau)(qX(t)-c)$. As long as $X(t)$ maintains a level satisfying $qX(t) > c$, the firm continues to generate a cash flow. Once $X(t)$ decreases to a lower level, it will become difficult to pay $c \geq 0$, and the firm will declare default. At the time of default, following the absolute priority rule (APR), the debt holders will gain ownership of the corporation (as new equity holders) and decide whether to continue to operate or to liquidate the corporation. This decision depends on the magnitude of the coupon payment and on the liquidation value.

On the one hand, if the debt holders continue to operate the corporation after default, the instantaneous cash inflow reduces to $(1-\alpha)(1-\tau)qX(t)$ by $(1-\alpha)$, where $\alpha > 0$. This assumption follows Mella-Barral and Perraudin (1997).[14] The parameter $\alpha > 0$ is the same as that of the bankruptcy cost $\alpha L(q)$. Setting the same parameter enables us to easily compare our results with the findings of Leland (1994) if we assume that $(s+k) \downarrow 0$. On the other hand, if the debt holders stop operation at the time of default,

[14] They suppose that default impairs the efficiency of corporate management.

the firm is liquidated and the debt holders obtain $(1-\alpha)L(q)$. To summarize, default ("operating concern bankruptcy") is defined as the transfer of management rights from the equity to debt holders. Shutdown ("liquidation bankruptcy") is defined as the cessation of operation.

Consider any time $t > T^{\mathrm{i}}$ after investment. The equity value after investment, $E_2(X(t), q, c)$, is given by

$$
\begin{aligned}
E_2(X(t), q, c) :=& \sup_{T^{\mathrm{d}}(\geq t)} \mathbb{E}^{X(t)}\left[\int_t^{T^{\mathrm{d}}} \mathrm{e}^{-r(u-t)}(1-\tau)(qX(u)-c)\mathrm{d}u\right] \\
=& vqX(t) - (1-\tau)\frac{c}{r} + \left((1-\tau)\frac{c}{r} - vqx^{\mathrm{d}}(q,c)\right)\left(\frac{X(t)}{x^{\mathrm{d}}(q,c)}\right)^{\gamma},
\end{aligned}
\tag{8}
$$

where the optimal default trigger $x^{\mathrm{d}}(q,c) := \mathrm{argmax}_y\, vqX(t) - (1-\tau)c/r + ((1-\tau)c/r - vqy)(X(t)/y)^{\gamma}$ is obtained by

$$
x^{\mathrm{d}}(q,c) = \frac{\varepsilon}{q}\frac{1-\tau}{r}c > 0. \tag{9}
$$

Note that $x^{\mathrm{d}}(q,c) > 0$ is a linear function of $c > 0$, as originally shown by Black and Cox (1976). We denote the optimal default time by $T^{\mathrm{d}}(q,c) := \inf\{t \geq T^{\mathrm{i}} | X(t) \leq x^{\mathrm{d}}(q,c)\}$ for fixed $q > 0$ and $c > 0$.

We derive the debt value by working backward. We denote the value function following default by $W(x^{\mathrm{d}}(q,c), q)$. The value $W(x^{\mathrm{d}}(q,c), q)$ is defined by

$$
\begin{aligned}
& W(x^{\mathrm{d}}(q,c), q) \\
& := (1-\alpha) \sup_{T_2^{\mathrm{s}}(\geq T^{\mathrm{d}}(q,c))} \mathbb{E}^{X(T^{\mathrm{d}}(q,c))}\left[\mathrm{e}^{rT^{\mathrm{d}}(c)}\left(\int_{T^{\mathrm{d}}(q,c)}^{T_2^{\mathrm{s}}} \mathrm{e}^{-ru}(1-\tau)qX(u)\mathrm{d}u + \mathrm{e}^{-rT_2^{\mathrm{s}}}L(q)\right)\right] \\
& = \begin{cases} (1-\alpha)L(q), & c \in (\theta_1(q), \theta_2(q)], \\ (1-\alpha)\left(vqx^{\mathrm{d}}(q,c) + \left(L(q) - vqx_2^{\mathrm{s}}(q)\right)\left(\dfrac{x^{\mathrm{d}}(q,c)}{x_2^{\mathrm{s}}(q)}\right)^{\gamma}\right), & c \in (\theta_2(q), +\infty), \end{cases}
\end{aligned}
\tag{10}
$$

where $x_2^{\mathrm{s}}(q) := \mathrm{argmax}_y\, vqx^{\mathrm{d}}(q,k) + (L(q) - vqy)(x^{\mathrm{d}}(q,c)/y)^{\gamma}$ and $\theta_2(q)$ are obtained as

$$
x_2^{\mathrm{s}}(q) = \frac{\varepsilon}{q}L(q) \geq 0 \tag{11}
$$

and

$$
\theta_2(q) := r(1-\tau)^{-1}L(q)(\geq \theta_1(q) \geq 0), \tag{12}
$$

respectively. Note that $\lim_{c\to\theta_2(q)} x^{\rm d}(q,c) = x_2^{\rm s}(q) \geq 0$; that is, $W(x^{\rm d}(q,c),q)$ is continuous at $c=\theta_2(q)$.

Assume that $c \in (\theta_1(q), \theta_2(q)]$ (derived from $x^{\rm d}(q,c) \leq x_2^{\rm s}(q)$) is satisfied. After investment, when $X(t)$ reaches $x_2^{\rm s}(q)$, the firm does not shut down because $X(t)$ does not reach $x^{\rm d}(q,c)$.[15] When $X(t)$ decreases further and reaches $x^{\rm d}(q,c)$, the firm defaults. At the time of default, the debt holders gain the rights of management and ownership, and shut down the firm. Thus, the firm exercises the default and shutdown options simultaneously, which we call the "simultaneous default and shutdown" strategy.

Alternatively, suppose that $c \in (\theta_2(q), +\infty)$ (i.e., $x^{\rm d}(q,c) > x_2^{\rm s}(q)$) is satisfied. Then, when $X(t)$ decreases and reaches $x^{\rm d}(q,c)$, the firm defaults. At the time of default, the debt holders obtain the rights of management and ownership, and continue to operate the firm. If $X(t)$ decreases further and reaches $x_2^{\rm s}(q)$, the debt holders shut down the firm. Thus, the firm exercises the default and shutdown options sequentially, which we call the "sequential default and shutdown" strategy.

The market value of a risky debt after investment, $D_2(X(t),q,c)$, is given by

$$\begin{aligned} D_2(X(t),q,c) :=& \mathbb{E}^{X(t)}\left[\int_t^{T^{\rm d}(q,c)} {\rm e}^{-r(u-t)} c{\rm d}u + {\rm e}^{-r(T^{\rm d}(q,c)-t)}(1-\tau)W(x^{\rm d}(q,c),q)\right] \\ =& \frac{c}{r} + \left(W(x^{\rm d}(q,c),q) - \frac{c}{r}\right)\left(\frac{X(t)}{x^{\rm d}(q,c)}\right)^{\gamma}. \end{aligned} \tag{13}$$

Note that $D_2(X(t),q,c) < c/r = D_1(X(t),q,c)$. This property is obtained by the following two results: $W(x^{\rm d}(q,c),q) \leq (1-\alpha)L(q) < c/r$ and $(X(t)/x^{\rm d}(q,c))^{\gamma} \leq 1$. That is, the market value of the risky debt is less than the face value of the debt.

2.2.3 Two value functions after investment

In the previous two subsections, we derived the value functions after investment for two cases: $c \in [0, \theta_1(q)]$ and $c \in (\theta_1(q), +\infty)$. In this section, we have the following lemma (see Appendix A for the proof).

Lemma 2 *The value functions after investment, $E_m(X(t),q,c)$ and $D_m(X(t),q,c)$, are continuous at $c=\theta_1(q)$.*

[15] When $X(t)$ reaches $x_2^{\rm s}(q)$, the debt holders will want to shut down the firm. However, the shutdown is not exercised because the debt holders do not have the right to manage the firm.

To summarize, in Subsection 2.2, we define the value functions after investment (i.e., $E_m(X(t), q, c)$ and $D_m(X(t), q, c)$ for $m \in \{1, 2\}$) and derive the optimal reversible ratio and bankruptcy (default and shutdown) triggers (i.e., $k(q)$, $x_1^{\mathrm{s}}(q)$, $x_2^{\mathrm{s}}(q)$, and $x^{\mathrm{d}}(q, c)$).

2.3 Financing and investment decisions problem

This subsection provides the value function before investment.

The firm's option value *before investment* is given as

$$\sup_{T^{\mathrm{i}}>0, q\geq 0, c\geq 0} \mathbb{E}^x \left[\mathrm{e}^{-rT^{\mathrm{i}}} \left\{ V_m(X(T^{\mathrm{i}}), q, c) - I(q) \right\} \right], \tag{14}$$

where $X(0) = x$, $m \in \{1, 2\}$, and $V_m(x, q, c) := E_m(x, q, c) + D_m(x, q, c)$. Here, whether $m = 1$ or $m = 2$ is decided by the magnitude of $c \geq 0$. Under the standard argument, as in (6), the discount factor of (14) is rewritten as

$$\mathbb{E}^x [\mathrm{e}^{-rT^{\mathrm{i}}}] = \left(\frac{x}{x^{\mathrm{i}}} \right)^\beta, \tag{15}$$

where $x < x^{\mathrm{i}} := X(T^{\mathrm{i}})$ and $\beta := 1/2 - \mu/\sigma^2 + ((\mu/\sigma^2 - 1/2)^2 + 2r/\sigma^2)^{1/2} > 1$.

The firm's optimization problem is formulated as

$$O(x) = \max_{x^{\mathrm{i}}>0, q\geq 0, c\geq 0} J_m(x^{\mathrm{i}}, q, c) x^\beta, \tag{16}$$

where $x < x^{\mathrm{i}}$ and

$$J_m(x^{\mathrm{i}}, q, c) := (x^{\mathrm{i}})^{-\beta} \left\{ V_m(x^{\mathrm{i}}, q, c) - I(q) \right\}. \tag{17}$$

Note that the control variables before investment are the triple (x^{i}, q, c).

Before analyzing the optimal financing and investment strategies with the option of costly reversibility (i.e., the amount of debt collected is determined endogenously at liquidation), we first review two extreme cases: the optimal financing and investment strategies without the option (i.e., the amount of debt collected is determined exogenously).

2.4 Investment quantity in the absence of costly reversibility

In this subsection, as a benchmark, we assume that the firm does not have the option of costly reversibility (i.e., $k = 0$), which gives the liquidation value $(1 - \alpha)L_{\mathrm{N}}(q_{\mathrm{N}})$, where $L_{\mathrm{N}}(q_{\mathrm{N}}) := sI(q_{\mathrm{N}})$. Here, the subscript "N" indicates the optimum without the option of costly reversibility. In addition, we consider the special case of the all-equity-financed firm as a benchmark, where the subscript "U" indicates the optimum for the all-equity-financed firm.

As shown in Shibata and Nishihara (2018), we have the following lemma (the superscript "$*$" represents the optimum).

Lemma 3 *Suppose that the firm does not have the option of costly reversibility (i.e., $k = 0$). Then, the investment quantity q_{N}^* is obtained by solving the following equation:*

$$\frac{q_{\mathrm{N}} I'(q_{\mathrm{N}})}{I(q_{\mathrm{N}})} = \frac{\beta}{\beta - 1}. \tag{18}$$

In addition, we obtain $q_{\mathrm{N}}^* = q_{\mathrm{UN}}^*$, where q_{UN}^* indicates the optimal investment quantity for the all-equity-financed firm without costly reversibility.

To ensure the existence and uniqueness of q_{N}^*, $(qI'(q)/I(q))$ is increasing with q.[16] In addition, see Appendix B for the two other solutions $(c_{\mathrm{N}}^*, x_{\mathrm{N}}^{\mathrm{i}*})$. Lemma 3 provides two important properties. First, q_{N}^* is independent of $s \geq 0$, with the liquidation value $(1-\alpha)L_{\mathrm{N}}(q_{\mathrm{N}}^*)$ increasing with s. Thus, if the firm does not have the option of costly reversibility, an increase in $(1-\alpha)L_{\mathrm{N}}(q_{\mathrm{N}}^*)$ with $s \geq 0$ does not change q_{N}^*.[17] Second, when the firm does not have the option of costly reversibility, the investment quantity of the debt-equity-financed firm is the same as that of the all-equity-financed firm.

3 Model solution

In this section, we derive the solution to the firm's optimization problem.

As shown in Appendix A, we have the following propositions.

Proposition 1 *We obtain $c(x^{\mathrm{i}}, q) > \theta_1(q)$, where $c(x^{\mathrm{i}}, q) = argmax_c V_2(x^{\mathrm{i}}, q, c)$. Thus, we have*

$$O(x) := J_2(x^{\mathrm{i}*}, q^*, c^*)x^{\beta}. \tag{19}$$

[16] See Cui and Shibata (2017) for details.

[17] This theoretical result is not consistent with the empirical results of Riordan and Williamson (1985), Choate (1997), and Vilasuso and Minkler (2001).

Proposition 1 implies that the firm always prefers risky debt and equity financing to risk-free debt and equity financing in equilibrium. In other words, even when the debt holders maximize the amount of debt collected at liquidation, the firm always prefers a risky debt issuance.

We provide a graphical proof of Proposition 1. First, we consider the shape of $V(x^{\mathrm{i}}, q, c)$ with c. This is because the component dependent on c in $O(x)$ is $V(x^{\mathrm{i}}, q, c)$. Figure 1 depicts $V(x^{\mathrm{i}}, q, c)$ with c. The basic parameters are assumed to be

$$r = 0.06, \sigma = 0.2, \mu = 0.005, s = 0.3, \tau = 0.15, \alpha = 0.4, F = 5, \text{and } a = 20.$$

Then, the two other solutions are assumed to be $x^{\mathrm{i}} = 1$ and $q = 10$. Under these parameters, we have $k(q) = 0.3099$ and $\theta_1(q) = 2.2531$, implying that $V(x^{\mathrm{i}}, q, c)$ is given as

$$\begin{cases} V_1(x^{\mathrm{i}}, q, c), & c \in [0, \theta_1(q)], \\ V_2(x^{\mathrm{i}}, q, c), & c \in (\theta_1(q), +\infty). \end{cases}$$

We see four properties.[18] First, $V_1(x^{\mathrm{i}}, q, c)$ is a linear function with c. Second, $V_1(x^{\mathrm{i}}, q, \theta_1(q)) = \lim_{c \downarrow \theta_1(q)} V_2(x^{\mathrm{i}}, q, c)$. Third, $\mathrm{d}V_2(x^{\mathrm{i}}, q, c)/\mathrm{d}c|_{c \downarrow \theta_1(q)} > 0$. Fourth, $V_2(x^{\mathrm{i}}, q, c)$ is convex with c. These four properties imply that there exists $c(x^{\mathrm{i}}, q)$ such that $c(x^{\mathrm{i}}, q) > \theta_1(q)$. Numerically, under the basic parameters, we obtain

$$c(x^{\mathrm{i}}, q) = 5.5 > \theta_1(q) = 2.2531.$$

In addition, $V_{\mathrm{N}}(x^{\mathrm{i}}, q, c)$ indicates the total firm value for $k = 0$.[19] We see that costly reversibility increases the value; that is, $V(x^{\mathrm{i}}, q, c) > V_{\mathrm{N}}(x^{\mathrm{i}}, q, c)$.

[insert Figure 1 about here]

Thus far, we have shown that the optimal coupon payment $c(x^{\mathrm{i}}, q)$ is larger than $\theta_1(q)$. We now derive the solution $(x^{\mathrm{i}*}, q^*, c^*)$ to the problem in the following proposition.

[18] See Appendix A for details.

[19] We have $\theta_{1\mathrm{N}}(q) = 1.89$ for $k(q) = 0$, where $\theta_{1\mathrm{N}} := r(1-\alpha)L_{\mathrm{N}}(q)$.

Proposition 2 *For $c^* \in (\theta_1(q^*), \theta_2(q^*)]$, the triple (x^*, q^*, c^*) is obtained by solving the three simultaneous equations*

$$(\beta-1)vqx^{\mathrm{i}} + \beta\tau\frac{c}{r} + (\beta-\gamma)\left\{(1-\alpha)L(q) - \left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c}{r}\right\}\left(\frac{\kappa qx^{\mathrm{i}}}{c}\right)^{\gamma} - \beta I(q) = 0, \tag{20}$$

$$vqx^{\mathrm{i}} + \left(\gamma\left\{(1-\alpha)L(q) - \left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c}{r}\right\} + (1-\alpha)qL'(q)\right)\left(\frac{\kappa qx^{\mathrm{i}}}{c}\right)^{\gamma} - qI'(q) = 0, \tag{21}$$

$$1 - \left(\frac{\kappa qx^{\mathrm{i}}}{c}\right)\left(1 - \frac{\gamma}{\tau}\left(\frac{\theta_1(q)}{c} - 1\right)\right) = 0, \tag{22}$$

where $\kappa := r/((1-\tau)\varepsilon) > 0$. For $c^* \in (\theta_2(q^*), +\infty)$, the pair $(x^{\mathrm{i}*}, q^*)$ is obtained by solving the two simultaneous equations

$$(\beta-1)\frac{v}{\psi}qx^{\mathrm{i}} + (\beta-\gamma)\frac{1-\alpha}{1-\gamma}L(q)\left(\frac{qx^{\mathrm{i}}}{\varepsilon L(q)}\right)^{\gamma} - \beta I(q) = 0, \tag{23}$$

$$\frac{v}{\psi}qx^{\mathrm{i}} + \frac{1-\alpha}{1-\gamma}\big(\gamma L(q) + (1-\gamma)qL'(q)\big)\left(\frac{qx^{\mathrm{i}}}{\varepsilon L(q)}\right)^{\gamma} - qI'(q) = 0, \tag{24}$$

where $\psi := (1+\tau/(h(1-\tau)))^{-1} \leq 1$ and $h := (1-\gamma(1+\alpha(1-\tau)/\tau))^{-1/\gamma} \geq 1$. In addition, we obtain $c^* = c(x^{\mathrm{i}*}, q^*)$, where $c(x^{\mathrm{i}}, q)$ is given as

$$c(x^{\mathrm{i}}, q) = \frac{\kappa}{h}qx^{\mathrm{i}}. \tag{25}$$

On the one hand, for $c^* \in (\theta_2(q^*), +\infty)$, the pair $(x^{\mathrm{i}*}, q^*)$ is derived by solving the two simultaneous equations (23) and (24) and c^* is obtained by $c^* = c(x^{\mathrm{i}*}, q^*)$, as originally obtained in Leland (1994), where $c(x^{\mathrm{i}}, q)$ is a linear function of x^{i}. On the other hand, for $c^* \in (\theta_1(q^*), \theta_2(q^*)]$, the triple $(x^{\mathrm{i}*}, q^*, c^*)$ is obtained by solving the three simultaneous equations (20)–(22). Here, $c(x^{\mathrm{i}}, q)$ is not derived explicitly.

Recall that q_{N}^* is independent of $s \geq 0$, as in Lemma 3. In contrast, when the firm has the option of costly reversibility, q^* depends on $s \geq 0$. We summarize this result as the following proposition (see Appendix A for the proof).

Proposition 3 *Suppose that the firm has the option of costly reversibility. First, the investment quantity is increased, that is, $q^* \geq q_{\mathrm{N}}^*$. Second, q^* is not always independent of $s \geq 0$. Third, we obtain $q^* \neq q_{\mathrm{U}}^*$, although $q_{\mathrm{N}}^* \neq q_{\mathrm{UN}}^*$.*

In the following section, we consider how an increase in $(1-\alpha)L(q)$ is related to c/r, x^{i}, and q.

4 Model implications

This section numerically considers the important implications of our solution.

In the numerical calculation, the cost function for investment, $I(q) > 0$, is assumed to be

$$I(q) = F + q^2 > 0, \tag{26}$$

where $F > 0$ is a positive constant.[20] In addition, the cost function for costly reversibility, $g(k) \geq 0$, is assumed to be

$$g(k) = a\frac{k}{1-k} \geq 0, \tag{27}$$

where $a > 0$ is a positive constant.[21] The parameter $a > 0$ is interpreted as a measure of the "efficiency" of the cost function for costly reversibility. Because $g'(k) = a/(1-k)^2$, we have $g'(0) = a$ and $g'(1-s) = a/s^2$. Thus, we obtain

$$k(q) = \begin{cases} 0, & \text{if } I(q) \in \big[0, a\big), \\ 1 - \left(\dfrac{a}{I(q)}\right)^{1/2}, & \text{if } I(q) \in \big[a, \dfrac{a}{s^2}\big], \\ 1 - s, & \text{otherwise.} \end{cases} \tag{28}$$

From (28), whether $k(q) = 0$ or $k(q) > 0$ depends on F, a, and q. In particular, an increase in a enlarges the region of $k(q) = 0$. Additionally, whether $s + k(q) < 1$ or $s + k(q) = 1$ depends on F, a, s, and q. Specifically, an increase in a and s shrinks and enlarges, respectively, the region of $s + k(q) = 1$.

Figure 2 depicts the regions of $k = 0$ and $k > 0$ in the space (F, a). The two lines indicate the boundaries of $k = 0$ for $\sigma = 0.1$ and $\sigma = 0.2$. Thus, the regions from the upper-left to the boundaries are the regions of $k = 0$. This implies that the firm is more likely to adopt costly reversibility for a smaller a and larger F. In addition, when $F = 5$ and $a = 50$ (corresponding to the blue point), we obtain $k = 0$ for $\sigma = 0.1$, but $k > 0$ for $\sigma = 0.2$. This implies that an increase in σ enlarges the regions of $k > 0$. Moreover, as in (28), the magnitude of s is independent of whether $k = 0$ or $k > 0$, but depends on whether $k < 1 - s$ or $k = 1 - s$. Thus, the key parameters are s, σ, F, and a.

[20] We confirm that $I(q)$ satisfies $I(0) > 0$, $I'(q) > 0$, and $I''(q) > 0$ for any $q > 0$.

[21] Note that $g(k)$ satisfies $g(0) = 0$, $g'(k) > 0$, and $g''(k) > 0$ for any $k \geq 0$, and $\lim_{k \to 1} g(k) = +\infty$.

[insert Figure 2 about here]

Section 4.1 considers the effects of s (costless reversibility). Section 4.2 discusses the comparative statics with σ (volatility), F (fixed cost for investment), and a (cost efficiency for costly reversibility).

4.1 Effect of costless reversibility ratio

In this subsection, we consider the effects of s on various solutions and values.

[insert Figure 3 about here]

4.1.1 Financing and investment strategies with liquidation value

The top-left panel of Figure 3 depicts the optimal ratio of costly reversibility k^* with s. Recall that k^* and s are endogenous and exogenous, respectively. We know that k^* increases with $s \in [0, \hat{s}]$, where $\hat{s} = 0.74$, but decreases with $s \in (\hat{s}, 1]$. Here, $\hat{s}$ represents the minimum costless reversibility ratio that satisfies $k+s = 1$. In addition, $k(q) = 0.2607$ is maximized at $\hat{s} = 0.74$. Importantly, k^* depends on $s \in [0, \hat{s}]$ because q^* depends on s.

The other five panels of Figure 3 depict the effects of s on $(1-\alpha)L(q^*)$ (liquidation value), c^*/r (face value of debt), q^* (investment quantity), $x^{\mathrm{i}*}$ (investment trigger), and $x^{\mathrm{d}*}$ (default trigger).

As a benchmark, we assume that debt holders do not maximize the amount of debt collected at liquidation (i.e., the no costly reversibility case of $k = 0$). In the top-right panel, $(1-\alpha)L_{\mathrm{N}}(q_{\mathrm{N}}^*)$ increases monotonically with s. In the middle-left and the middle-right panels, c_{N}^*/r and $x_{\mathrm{N}}^{\mathrm{d}*}$ have a V-shaped curve with s. In the middle-right panel, $x_{\mathrm{N}}^{\mathrm{i}*}$ decreases monotonically with s. In the bottom-left panel, q_{N}^* is independent of s. In the bottom-right panel, $x_{\mathrm{N}}^{\mathrm{d}*}$ has a V-shaped curve with s and $x_{\mathrm{N}}^{\mathrm{s}*}$ increases monotonically with s. Thus, we conclude that there is no consistent relationship between these five panels.

By contrast, we assume that debt holders maximize the amount of debt collected at liquidation (i.e., the costly reversibility case of $k \geq 0$). In the top-right, middle-left, middle-right, bottom-left, and bottom-right panels, $(1-\alpha)L(q^*)$, c^*/r, q^*, $x^{\mathrm{i}*}$, and $x^{\mathrm{d}*}$ have a Λ-shaped curve with s. They increase monotonically with $s \in [0, \hat{s}]$, but decrease with $s \in (\hat{s}, 1]$. To clarify the relationship between the financing–investment strategies and the liquidation value, the three panels of Figure 4 display c/r, x^{i}, and q with respect to $(1-\alpha)L(q)$, showing that all three have a positive relationship with $(1-\alpha)L(q)$. In addition, to clarify whether these have a positive relationship, we calculate the correlation in Table 2. In particular, the correlation between $(1-\alpha)L(q)$ and c/r is 0.7228, the correlation between $(1-\alpha)L(q)$ and x^{i} is 0.3416, and the correlation between $(1-\alpha)L(q)$ and q is 0.7629.

[insert Figure 4 about here]

[insert Table 2 about here]

Thus, we summarize the results are follows.

Observation 1 *Suppose that debt holders do not have the option of maximizing the amount of debt collected at liquidation (i.e., the no costly reversibility case of $k = 0$). An increase in the liquidation value does not have a consistent relationship with the financing and investment decisions. In particular, if the liquidation value increases, the investment trigger is decreased, investment quantity is invariant, and debt issuance is either increased or decreased. By contrast, suppose that debt holders have the option of maximizing the amount of debt collected at liquidation (i.e., the costly reversibility case of $k \geq 0$). An increase in the liquidation value has a consistent relationship with the financing and investment decisions. In particular, an increase in the liquidation value leads to an increase in the investment trigger, investment quantity, and debt issuance.*

Observation 1 implies that if the liquidation value increases, the firm increases the amount of debt issuance and the investment quantity, leading to delayed investment. These results fit well with the empirical studies of Riordan and Williamson (1985), Choate (1997), and Vilasuso and Minkler (2001).

Returning to Figure 3, there are two additional interesting results. First, in the middle-right panel, $x^{\mathrm{i}} \leq x^{\mathrm{i}}_{\mathrm{U}}$ is not always obtained, but $x^{\mathrm{i}}_{\mathrm{N}} \leq x^{\mathrm{i}}_{\mathrm{UN}}$ is always obtained. To be more precise, we obtain $x^{\mathrm{i}} > x^{\mathrm{i}}_{\mathrm{U}}$ for $s \in (0.5023, 0.74)$. Without the option of costly reversibility, debt financing always enables the firm to hasten the investment.[22] By contrast, with the

[22] Recall that in our model, the firm exercises the investment once $X(t)$ reaches x^{i} from below. We define the notion as follows: if the investment trigger is larger (smaller) than the benchmark trigger, the investment will be exercised later (earlier).

option of costly reversibility, debt financing does not always enable the firm to hasten the investment. For $s \in [0.5023, 0.74]$, the investment is delayed under debt-equity financing rather than under all-equity financing. The reason is as follows. Because debt financing enables the firm to increase the investment quantity when the firm has the option of costly reversibility, the debt-equity-financed firm implies the delayed investment.[23]

We summarize these results as follows.

Observation 2 *Suppose that debt holders do not maximize the amount of debt collected at liquidation (i.e., the no costly reversibility case of $k = 0$). Debt-equity financing always hastens the investment, compared with all-equity financing. By contrast, suppose that debt holders maximize the amount of debt collected at liquidation (i.e., the costly reversibility case of $k \geq 0$). Then, debt-equity financing does not always hasten the investment.*

Second, we observe another interesting result in the bottom-right panel. On the one hand, under no costly reversibility (i.e., $k = 0$), the firm exercises the sequential bankruptcy (default and shutdown) strategy for $s \in [0, 0.84)$, but adopts the simultaneous bankruptcy (default and shutdown) strategy for $s \in [0.84, 1]$. On the other hand, under costly reversibility (i.e., $k \geq 0$), the firm exercises the sequential bankruptcy (default and shutdown) strategy for $s \in [0, 0.74)$, but adopts the simultaneous bankruptcy (default and shutdown) strategy for $s \in [0.74, 1]$. Thus, if we assume $s \in [0.74, 0.84)$, by the option of costly reversibility, the bankruptcy strategy is changed from sequential to simultaneous. We summarize these results as follows.

Observation 3 *Costly reversibility induces the firm to change the bankruptcy (default and shutdown) strategies from sequential to simultaneous.*

Observation 3 reflects the findings of Nishihara and Shibata (2018) and Shibata and Nishihara (2018).

4.1.2 Option value and total firm value

[insert Figure 5 about here]

[23]Without the option of costly reversibility, because debt financing is invariant to the investment quantity, the debt-equity-financed firm implies the hastened investment.

The top-left panel of Figure 5 depicts O (option value) with s. We see that $O \geq O_{\mathrm{N}}$ (and $O_{\mathrm{U}} \geq O_{\mathrm{UN}}$). The reason is intuitive. When the firm has the option of costly reversibility, the value with the option is increased. The top-right panel gives $V(x^{\mathrm{i}*}, q^*, c^*)$ (total firm value at the time of the investment). We see that V has a V-shaped curve with s.

4.1.3 Credit spread, leverage, and default probability

The middle-left panel shows that $c/r - D_2(x^{\mathrm{i}}, q, c) \geq 0$; this represents the difference between the face value of the debt and the market value. A positive difference is guaranteed because debt is risky. Interestingly, the difference is larger with the option of costly reversibility than without the option. Thus, costly reversibility increases the difference between the face value of the debt and the market value.

The middle-right and bottom-left panels depict the credit spread cs and leverage D/V, respectively. The credit spread (in basis points) is defined as

$$cs := cs(x^{\mathrm{i}}, q, c) = \left(\frac{c}{D_2(x^{\mathrm{i}}, q, c)} - r \right) \times 10^4 \geq 0.$$

Here, $cs \geq 0$ is always positive because debt is risky (i.e., $c/r > D_2(x^{\mathrm{i}}, q, c)$). The leverage (as a percentage) is defined as

$$\frac{D}{V} := \frac{D_2(x^{\mathrm{i}}, q, c)}{V_2(x^{\mathrm{i}}, q, c)} \times 10^2 \geq 0.$$

We see that cs and D/V are kinked at $s = 0.74$. The reason for these kinked curves is the change of bankruptcy strategies from sequential to simultaneous. We see that cs decreases with s, whereas D/V increases with s. Importantly, we find that $cs \leq cs_{\mathrm{N}}$ and $D/V \geq D_{\mathrm{N}}/V_{\mathrm{N}}$.[24] From these results, when the liquidation value is increased under the option of costly reversibility, the credit spread is decreased and the leverage is increased.

The bottom-right panel depicts p (default probability) with s. The default probability (as a percentage) is defined as

$$p := \left(\frac{x^{\mathrm{i}}}{x^{\mathrm{d}}(q, c)} \right)^{\gamma} \times 10^2 \geq 0.$$

[24]When the firm does not have the option of costly reversibility, the credit spread and leverage are defined as cs_{N} and $D_{\mathrm{N}}/V_{\mathrm{N}}$, respectively.

When the firm exercises the sequential bankruptcy (default and shutdown) strategy, the default probability is constant. When the firm exercises the simultaneous bankruptcy (default and shutdown) strategy, the default probability is not constant. We have $p \geq p_{\mathrm{N}}$.

These four panels provide the following observation.

Observation 4 *Costly reversibility increases the difference between the face value of the debt and the market value, leverage, and default probability, but decreases the credit spread.*

4.2 Comparative statics

[insert Figure 6 about here]

The three top panels of Figure 6 depict the effects of σ (volatility) on k (costly reversible ratio), c/r (face value of debt), and x^{i} (investment trigger), respectively. In the top-left panel, the firm does not exercise the costly reversibility option for $\sigma \in [0.15, 0.186]$, but does do so for $\sigma \in (0.186, 0.2]$. Thus, an increase in σ means that the firm is more likely to adopt the costly reversibility option. This result is consistent with those of the empirical studies of Leahy and Whited (1996), Guiso and Parigi (1999), and Ghosal and Loungani (2000).

The middle-left, middle-middle, and middle-right panels of Figure 4 depict the effects of F (fixed investment cost) on k, c/r, and x^{i}, respectively. In the lower-left panel, the firm does not exercise the option of costly reversibility for $F \in [2, 3.2]$, but does do so for $\sigma \in (3.2, 10]$. Thus, k^* increases with F. A larger fixed cost of investment (F) corresponds to a larger production facility $I(q)$, which results in greater production (q). A larger firm is assumed to have a larger production facility. Thus, our result fits well with that of the empirical study of Folta et al. (2006), where a large firm is better able to redeploy its assets. We see that c/r and x^{i} increase with F. An increase in F increases the amount of debt issuance and delays the investment.

The bottom-left, bottom-middle, and bottom-right panels of Figure 4 depict the effects of a (costly reversibility efficiency) on k, c/r, and x^{i}, respectively. In the lower-left panel, the firm exercises the option of costly reversibility for $a \in [40, 78]$, but does not do so for $a \in (78, 100]$. Thus, an increase in a decreases k^*. This is consistent with the empirical findings of Asplund (2000), where the salvage values for the high transaction costs of the second-hand market are reduced. We find that c/r and x^{i} decrease with a. An increase in a decreases the amount of debt issuance and accelerates the investment.

5 Concluding remarks

We examine the interaction between financing and investment decisions under the condition that debt holders have the option of maximizing the debt-collection amount if a firm is liquidated during financial distress. We add to the literature by incorporating the optimization considerations of the debt holders' debt-collection amount.

We provide a novel result on the interactions between financing and investment decisions by incorporating the debt holders' maximization considerations. As a benchmark, unless we do not consider the option of maximizing the debt-collection amount at liquidation, the relationship between the financing–investment decisions and the liquidation value is inconsistent with that identified by empirical studies in this literature. By contrast, suppose that debt holders have the option of maximizing the debt-collection amount at liquidation. Now, the relationship between the financing–investment decisions and the liquidation value fits well with existing empirical evidence (as previously cited). To be more precise, as the liquidation value increases, the amounts of debt issuance and investment quantity increase, leading to a delayed investment.

Appendix A

In this appendix, we provide the proofs of Lemmas 1–3 and Propositions 1–3.

Proof of Lemma 1

The Lagrangian can be formulated as

$$\max_{k,\lambda_1,\lambda_2} \ \mathcal{L} = (s+k)I(q) - g(k) + \lambda_1 k + \lambda_2(1-s-k), \tag{A.1}$$

where $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ denote the multipliers on the constraints. The Karush–Kuhn–Tucker (KKT) conditions are given by

$$\frac{\partial \mathcal{L}}{\partial k} = I(q) - g'(k) + \lambda_1 - \lambda_2 = 0, \tag{A.2}$$

and

$$\begin{aligned} &k \geq 0, & &\lambda_1 \geq 0, \quad \lambda_1 k = 0, \\ &k \leq 1-s, & &\lambda_2 \geq 0, \quad \lambda_2(1-s-k) = 0. \end{aligned} \tag{A.3}$$

We obtain the optimal costly reversible ratio $k(q)$ based on whether $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ are strictly positive. First, suppose $\lambda_1 > 0$ and $\lambda_2 > 0$. We then obtain $k = 0$ and $k = 1-s$, which is a contradiction. This implies that at least one of $\lambda_1 \geq 0$ and $\lambda_2 \geq 0$ must be zero. Second, suppose $\lambda_1 = 0$ and $\lambda_2 = 0$. We then obtain $k \in (0, 1-s)$ and $I(q) = g'(k)$. Using $g''(k) > 0$, $g'(0) \leq (q) \leq g'(1-s)$ must be satisfied. We thus obtain $k(q) = {g'}^{-1}(I(q))$. Third, suppose $\lambda_1 = 0$ and $\lambda_2 > 0$. We then have $k(q) = 1-s$ and $I(q) > g'(1-s)$. Finally, suppose $\lambda_1 > 0$ and $\lambda_2 = 0$. We then obtain $k(q) = 0$ and $I(q) < g'(0)$.

Proof of Lemma 2

It is straightforward to obtain

$$\begin{aligned} &x_1^{\mathrm{s}}(q, \theta_1(q)) = \lim\nolimits_{c \downarrow \theta_1(q)} x^{\mathrm{d}}(q, c), \\ &E_1(X(t), q, \theta_1(q)) = \lim\nolimits_{c \downarrow \theta_1(q)} E_2(X(t), q, c), \\ &D_1(X(t), q, \theta_1(q)) = \lim\nolimits_{c \downarrow \theta_1(q)} D_2(X(t), q, c). \end{aligned} \tag{A.4}$$

From (A.4), we show that the value functions are continuous at $c = \theta_1(q)$.

Proof of Lemma 3

First, we show that q_{N}^* is obtained by solving (18) if $k = 0$. This proof is easily obtained from the proof of Proposition 3. Recall that $g(k) = 0$ if $k = 0$. By substituting $k = 0$ into (A.17) and (A.18), q_{N}^* must be satisfied with $q_{\mathrm{N}} I'(q_{\mathrm{N}})/I(q_{\mathrm{N}}) = \beta/(\beta-1)$. This completes the proof. In addition, to ensure the existence and uniqueness of q_{N}^*, we need the condition that $(qI'(q)/I(q))' \geq 0$. We assume this condition, as in Wong (2010) and Cui and Shibata (2017).

Second, we show that $q_{\mathrm{N}}^* = q_{\mathrm{UN}}^*$. Similarly, this proof is easily obtained by substituting $k = 0$ and $c = 0$ into (A.17) and (A.18). q_{UN}^* must satisfy $q_{\mathrm{UN}} I'(q_{\mathrm{UN}})/I(q_{\mathrm{UN}}) = \beta/(\beta-1)$. This completes the proof.

Proof of Proposition 1

First, differentiating $V_1(X(t),q,c)$ with respect to c yields

$$\frac{\mathrm{d}V_1(X(t),q,c)}{\mathrm{d}c} = \frac{\partial V_1(X(t),q,c)}{\partial c} + \underbrace{\frac{\partial V_1(X(t),q,c)}{\partial x_1^{\mathrm{s}}(q,c)}}_{=0} \frac{\partial x_1^{\mathrm{s}}(q,c)}{\partial c}$$
$$= \frac{\tau}{r}\left(1 - \left(\frac{X(t)}{x_1^{\mathrm{s}}(q,c)}\right)^{\gamma}\right) \geq 0, \tag{A.5}$$

where we have the envelope theorem (i.e., $\partial V_1(X(t),q,c)/\partial x_1^{\mathrm{s}}(q,c) = 0$). The positive sign of Inequality (A.5) is obtained from $X(t) > x_1^{\mathrm{s}}(q,c)$ and $\gamma < 0$.

Next, we prove that there exists an optimal coupon payment $c(X(t),q)$ such that $c(X(t),q) > \theta_1(q)$ (i.e., the firm always issues risky debt) by using the properties of two functions, $V_1(X(t),q,c)$ and $V_2(X(t),q,c)$. Substituting $x^{\mathrm{d}}(q,c) = c/(\kappa q)$ into $V_2(X(t),q,c)$ yields

$$V_2(X(t),q,c) \tag{A.6}$$
$$= \begin{cases} vqX(t) + \tau\dfrac{c}{r} + \left((1-\alpha)L(q) - \left(\dfrac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\dfrac{c}{r}\right)\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma}, \\ \qquad\qquad c \in (\theta_1(q), \theta_2(q)], \\ vqX(t) + \tau\dfrac{c}{r} - \left(\alpha\dfrac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\dfrac{c}{r}\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma} + \dfrac{1-\alpha}{1-\gamma}L(q)\left(\dfrac{X(t)}{x_2^{\mathrm{s}}(q)}\right)^{\gamma}, \\ \qquad\qquad c \in (\theta_2(q), +\infty), \end{cases}$$

where $\kappa = r/((1-\tau)\varepsilon) > 0$. By differentiating $V_2(X(t),q,c)$ in (A.6) with respect to c, we have

$$\frac{\mathrm{d}V_2(X(t),q,c)}{\mathrm{d}c} = \begin{cases} \dfrac{\tau}{r}\left\{1 - \left(1 + \dfrac{\gamma}{\tau}\underbrace{\left(\dfrac{\theta_1(q)}{c} - 1\right)}_{<0}\right)\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma}\right\}, & c \in (\theta_1(q), \theta_2(q)], \\ \dfrac{\tau}{r}\left\{1 - \left(\underbrace{1 - \gamma\left(1 + \alpha\dfrac{1-\tau}{\tau}\right)}_{>0}\right)\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma}\right\}, & c \in (\theta_2(q), +\infty), \end{cases} \tag{A.7}$$

and

$$\frac{\mathrm{d}^2V_2(X(t),q,c)}{\mathrm{d}c^2} = \begin{cases} \dfrac{\tau\gamma}{rc}\left(1 + \dfrac{\gamma}{\tau}\underbrace{\left(\dfrac{\theta_1(q)}{c} - 1\right)}_{<0} + \dfrac{\theta_1(q)}{\tau c}\right)\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma} < 0, & c \in (\theta_1(q), \theta_2(q)], \\ \dfrac{\tau\gamma}{rc}\left(\underbrace{1 - \gamma\left(1 + \alpha\dfrac{1-\tau}{\tau}\right)}_{>0}\right)\left(\dfrac{\kappa q X(t)}{c}\right)^{\gamma} < 0, & c \in (\theta_2(q), +\infty). \end{cases} \tag{A.8}$$

From (A.8), $V_2(X(t), q, c)$ is a concave function of c. As $c \downarrow \theta_1(q)$, the upper equation of (A.7) is

$$\left.\frac{\mathrm{d}V_2(X(t), q, c)}{\mathrm{d}c}\right|_{c\downarrow\theta_1(q)} = \frac{\tau}{r}\left(1 - \left(\frac{\kappa q X(t)}{c}\right)^{\gamma}\right) > 0. \tag{A.9}$$

From (A.4) and (A.5), we have $V_1(X(t), q, \theta_1(q)) = \lim_{c\downarrow\theta_1(q)} V_2(X(t), q, c)$; also, $V_1(X(t), q, c)$ is maximized at $c = \theta_1(q)$. In addition to these two results, from (A.8) and (A.9), there exists $c(X(t), q)$ such that $c(X(t), q) > \theta_1(q)$, where $c(X(t), q)$ is given as

$$c(X(t), q) = \underset{c}{\operatorname{argmax}}\, V_2(X(t), q, c). \tag{A.10}$$

The result $c(X(t), q) > \theta_1(q)$ means that $V_1(X(t), q, \theta_1(q)) < V_2(X(t), q, c(X(t), q))$ (the firm issues risky debt). Therefore, from $X(t) = x^{\mathrm{i}*}$ and $q = q^*$, we obtain $c^* := c(x^{\mathrm{i}*}, q^*) > \theta_1(q^*)$, which completes the proof. The left-hand side panel of Figure 1 provides a graphical proof of $c(X(t), q) > \theta_1(q)$.

Proof of Proposition 2

We provide the solution to the problem (i.e., $x^{\mathrm{i}*}$, q^*, and c^*). Depending on the magnitude of c^*, there are two cases: $c^* \in (\theta_1(q^*), \theta_2(q^*)]$ and $c^* \in (\theta_2(q^*), +\infty)$.

Suppose that $c \in (\theta_1(q), \theta_2(q)]$ (i.e., $x^{\mathrm{d}}(q, c) \leq x_2^{\mathrm{s}}(q)$). The problem is defined as $\max_{x^{\mathrm{i}}, q, c} J_2(x^{\mathrm{i}}, q, c)$ in (16). By differentiating $J_2(x^{\mathrm{i}}, q, c)$ with respect to x^{i}, q, and c, we obtain $(\partial J_2/\partial x^{\mathrm{i}})x^{\mathrm{i}} = 0$, $(\partial J_2/\partial q)q = 0$, and $(\partial J_2/\partial c)c = 0$ as

$$-\beta\{V_2(x^{\mathrm{i}}, q, c) - I(q)\} + vqx^{\mathrm{i}} + \gamma\left\{(1-\alpha)L(q) - \left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c}{r}\right\}\left(\frac{\kappa q x^{\mathrm{i}}}{c}\right)^{\gamma} = 0, \tag{A.11}$$

$$vqx^{\mathrm{i}} + \left(\gamma\left\{(1-\alpha)L(q) - \left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c}{r}\right\} + (1-\alpha)qL'(q)\right)\left(\frac{\kappa q x^{\mathrm{i}}}{c}\right)^{\gamma} - qI'(q) = 0, \tag{A.12}$$

$$\tau\frac{c}{r} + \left((-\gamma)(1-\alpha)L(q) + (\gamma-1)\left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c}{r}\right)\left(\frac{\kappa q x^{\mathrm{i}}}{c}\right)^{\gamma} = 0. \tag{A.13}$$

By arranging (A.11)–(A.13), we obtain (20)–(22).

Suppose that $c^* \in (\theta_2(q^*), +\infty)$ (i.e., $x^{\mathrm{d}}(q^*, c^*) > x_2^{\mathrm{s}}(q^*)$). The problem is defined as $\max_{x^{\mathrm{i}}, q, c} J_2(x^{\mathrm{i}}, q, c)$ in (16). Here, by arranging the lower equation of (A.7), we obtain the optimal coupon payment as $c(x^{\mathrm{i}}, q) := \operatorname{argmax}_c = (\kappa/h)qx^{\mathrm{i}}$, where $h = (1 - \gamma(1 + \alpha(1 -$

$\tau)/\tau))^{-1/\gamma}$. Substituting $c = c(x^{\mathrm{i}}, q)$ into the three components in the lower equation of (A.6) gives

$$
\begin{aligned}
&V_2(x^{\mathrm{i}}, q, c(x^{\mathrm{i}}, q)) - \frac{1-\alpha}{1-\gamma} L(q) \left(\frac{q x^{\mathrm{i}}}{\varepsilon L(q)} \right)^{\gamma} \\
&= v q x^{\mathrm{i}} + \frac{\tau}{r} \underbrace{\frac{\kappa}{h} q x^{\mathrm{i}}}_{=c(x^{\mathrm{i}}, q)} - \big(\alpha \frac{\gamma}{\gamma - 1}(1-\tau) + \tau\big) h^{\gamma} \frac{1}{r} \underbrace{\frac{\kappa}{h} q x^{\mathrm{i}}}_{=c(x^{\mathrm{i}}, q)} \\
&= v q x^{\mathrm{i}} + \Bigg(\underbrace{\frac{\gamma - 1}{\gamma} - \big(\overbrace{\alpha \frac{1-\tau}{\tau} + \frac{\gamma-1}{\gamma}}^{=-h^{-\gamma}\gamma^{-1}}\big) h^{\gamma}}_{=1} \Bigg) \frac{\tau}{1-\tau} \frac{1}{h} v q x^{\mathrm{i}} \\
&= \underbrace{\big(1 + \frac{\tau}{1-\tau} \frac{1}{h}\big)}_{=:\psi^{-1}} v q x^{\mathrm{i}}.
\end{aligned}
$$

Thus, $J_2(x^{\mathrm{i}}, q, c(x^{\mathrm{i}}, q))$ is rewritten as

$$
J_2(x^{\mathrm{i}}, q, c(x^{\mathrm{i}}, q)) = (x^{\mathrm{i}})^{-\beta} \Bigg\{ \underbrace{\frac{v}{\psi} v q x^{\mathrm{i}} + \frac{1-\alpha}{1-\gamma} L(q) \left(\frac{q x^{\mathrm{i}}}{\varepsilon L(q)} \right)^{\gamma}}_{=V_2(x^{\mathrm{i}}, q, c(x^{\mathrm{i}}, q))} - I(q) \Bigg\}. \tag{A.14}
$$

Note that (A.14) has two control variables, x^{i} and q. By differentiating (A.14) with respect to x^{i} and q, $(\partial J_2/\partial x^{\mathrm{i}})x^{\mathrm{i}} = 0$ and $(\partial J_2/\partial q)q = 0$ become

$$
-\beta \big\{ V_2(x^{\mathrm{i}}, q, c(x^{\mathrm{i}}, q)) - I(q) \big\} + \frac{v}{\psi} q x^{\mathrm{i}} + \gamma \left(\frac{q x^{\mathrm{i}}}{\varepsilon L(q)} \right)^{\gamma} = 0, \tag{A.15}
$$

$$
\frac{v}{\psi} q x^{\mathrm{i}} + \frac{1-\alpha}{1-\gamma} \big(\gamma L(q) + (1-\gamma) q L'(q) \big) \left(\frac{q x^{\mathrm{i}}}{\varepsilon L(q)} \right)^{\gamma} - q I'(q) = 0, \tag{A.16}
$$

respectively. By arranging (A.15) and (A.16), we obtain (23) and (24).

Proof of Proposition 3

First, we show the proof of $q^* \geq q^*_{\mathrm{N}}$. For $c \in (\theta_1(q), \theta_2(q)]$, substituting (A.12) and (A.13) into (A.11) gives

$$
\begin{aligned}
&\underbrace{\left(1 - (1-\alpha)(s + k(q)) \left(\frac{\kappa q x^{\mathrm{i}}}{c} \right)^{\gamma} \right)}_{\geq 0} \left(q I'(q) - \frac{\beta}{\beta - 1} I(q) \right) \\
&= (1-\alpha) \frac{\beta}{\beta - 1} g(k(q)) \left(\frac{\kappa q x^{\mathrm{i}}}{c} \right)^{\gamma} \geq 0.
\end{aligned} \tag{A.17}
$$

For $c \in (\theta_2(q), +\infty)$, substituting (A.16) into (A.15) gives

$$\underbrace{\left(1-(1-\alpha)(s+k(q))\left(\frac{qx^{\mathrm{i}}}{\varepsilon L(q)}\right)^{\gamma}\right)}_{\geq 0}\left(qI'(q)-\frac{\beta}{\beta-1}I(q)\right)$$
$$=(1-\alpha)\frac{\beta}{\beta-1}g(k(q))\left(\frac{qx^{\mathrm{i}}}{\varepsilon L(q)}\right)^{\gamma} \geq 0. \tag{A.18}$$

In both (A.17) and (A.18), q^* satisfies the following condition:

$$\frac{qI'(q)}{I(q)} \geq \frac{\beta}{\beta-1}. \tag{A.19}$$

Recall that q^*_{N} satisfies (18), as shown in Lemma 3. Because $qI'(q)/I(q)$ is increasing with q, we obtain $q^* \geq q^*_{\mathrm{N}}$ for $c^* \in (\theta_1(q^*), +\infty)$.

Second, we show the proof that q^* is not independent of s. For $c^* \in (\theta_1(q^*), \theta_2(q^*)]$, we obtain q^* by solving (20)–(22), which include s. For $c^* \in (\theta_2(q^*), +\infty)$, we obtain q^* by solving (23) and (24), which include s. Thus, it is clear that q^* is not independent of s.

Third, we derive the implicit solution of q^*_{U}. The problem is formulated as

$$\max_{x^{\mathrm{i}}_{\mathrm{U}}, q_{\mathrm{U}}} J_1(x^{\mathrm{i}}_{\mathrm{U}}, q_{\mathrm{U}}, 0), \tag{A.20}$$

where

$$J_1(x^{\mathrm{i}}_{\mathrm{U}}, q_{\mathrm{U}}, 0) := (x^{\mathrm{i}}_{\mathrm{U}})^{-\beta}\Big\{\underbrace{vq_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}+\frac{1-\alpha}{1-\gamma}L(q)\left(\frac{x^{\mathrm{i}}_{\mathrm{U}}}{x^{\mathrm{s}}_1(q_{\mathrm{U}},0)}\right)^{\gamma}}_{=V_1(x^{\mathrm{i}}_{\mathrm{U}},q_{\mathrm{U}},0)}-I(q_{\mathrm{U}})\Big\}. \tag{A.21}$$

By differentiating (A.20) with $x^{\mathrm{i}}_{\mathrm{U}}$ and q_{U}, $(\partial J_1/\partial x^{\mathrm{i}}_{\mathrm{U}})x^{\mathrm{i}}_{\mathrm{U}}$ and $(\partial J_1/\partial q_{\mathrm{U}})q_{\mathrm{U}}$ become

$$(\beta-1)vq_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}+(\beta-\gamma)\frac{1-\alpha}{1-\gamma}L(q_{\mathrm{U}})\left(\frac{q_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}}{\varepsilon(1-\alpha)L(q_{\mathrm{U}})}\right)^{\gamma}-\beta I(q_{\mathrm{U}})=0, \tag{A.22}$$

$$vq_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}+\frac{1-\alpha}{1-\gamma}\big(\gamma L(q_{\mathrm{U}})+(1-\gamma)q_{\mathrm{U}}L'(q_{\mathrm{U}})\big)\left(\frac{q_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}}{\varepsilon(1-\alpha)L(q_{\mathrm{U}})}\right)^{\gamma}-q_{\mathrm{U}}I'(q_{\mathrm{U}})=0. \tag{A.23}$$

Substituting (A.23) into (A.22) gives

$$\underbrace{\left(1-(1-\alpha)(s+k(q_{\mathrm{U}}))\left(\frac{q_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}}{\varepsilon(1-\alpha)L(q_{\mathrm{U}})}\right)^{\gamma}\right)}_{\geq 0}\left(q_{\mathrm{U}}I'(q_{\mathrm{U}})-\frac{\beta}{\beta-1}I(q_{\mathrm{U}})\right)$$
$$=(1-\alpha)\frac{\beta}{\beta-1}g(k(q_{\mathrm{U}}))\left(\frac{q_{\mathrm{U}}x^{\mathrm{i}}_{\mathrm{U}}}{\varepsilon(1-\alpha)L(q_{\mathrm{U}})}\right)^{\gamma} \geq 0. \tag{A.24}$$

Because both (A.17) and (A.18) differ from (A.24), we obtain $q^* \neq q^*_{\mathrm{U}}$.

Appendix B

In this appendix, we provide the solution to the financing and investment decisions problem without costly reversibility. The proof is the same as that of Proposition 1.

First, in the problem without costly reversibility, we show that $c_{\mathrm{N}}(x_{\mathrm{N}}^{\mathrm{i}}, q_{\mathrm{N}}) > \theta_{1\mathrm{N}}(q_{\mathrm{N}})$, where $c_{\mathrm{N}}(x_{\mathrm{N}}^{\mathrm{i}}, q_{\mathrm{N}}) = \mathrm{argmax}_{c_{\mathrm{N}}} V_2(x_{\mathrm{N}}^{\mathrm{i}}, q_{\mathrm{N}}, c_{\mathrm{N}})$ subject to $L = L_{\mathrm{N}}(q_{\mathrm{N}}) := sI(q_{\mathrm{N}})$. This inequality implies that the firm always prefers risky debt. Here, $\theta_{1\mathrm{N}}(q_{\mathrm{N}})$ and $\theta_{2\mathrm{N}}(q_{\mathrm{N}})$ are defined as $\theta_{1\mathrm{N}}(q_{\mathrm{N}}) := r(1-\alpha)L_{\mathrm{N}}(q_{\mathrm{N}})$ and $\theta_{2\mathrm{N}}(q_{\mathrm{N}}) := r(1-\tau)^{-1}L_{\mathrm{N}}(q_{\mathrm{N}})$.

Next, recall that, for $c_{\mathrm{N}}^* \in (\theta_{1\mathrm{N}}(q_{\mathrm{N}}^*), +\infty)$, q_{N}^* is obtained by solving (18). The two other solutions, x_{N}^* and c_{N}^* are obtained as follows. For $c_{\mathrm{N}}^* \in (\theta_{1\mathrm{N}}(q_{\mathrm{N}}^*), \theta_{2\mathrm{N}}(q_{\mathrm{N}}^*)]$, we obtain x_{N}^* and c_{N}^* by solving

$$(\beta-1)vq_{\mathrm{N}}^*x_{\mathrm{N}}^{\mathrm{i}} + \beta\tau\frac{c_{\mathrm{N}}}{r} + (\beta-\gamma)\left\{(1-\alpha)L_{\mathrm{N}}(q_{\mathrm{N}}^*) - \left(\frac{\gamma}{\gamma-1}(1-\tau)+\tau\right)\frac{c_{\mathrm{N}}}{r}\right\}\left(\frac{\kappa q_{\mathrm{N}}^* x_{\mathrm{N}}^{\mathrm{i}}}{c_{\mathrm{N}}}\right)^{\gamma}$$

$$-\beta I(q_{\mathrm{N}}^*) = 0, \tag{B.1}$$

$$1 - \left(1 - \frac{\gamma}{\tau}\left(\frac{\theta_{1\mathrm{N}}(q_{\mathrm{N}}^*)}{c_{\mathrm{N}}} - 1\right)\right)\left(\frac{\kappa q_{\mathrm{N}}^* x_{\mathrm{N}}^{\mathrm{i}}}{c_{\mathrm{N}}}\right)^{\gamma} = 0. \tag{B.2}$$

We derive (B.1) and (B.2) by substituting $k=0$ into (20) and (22). For $c_{\mathrm{N}}^* \in (\theta_{2\mathrm{N}}(q_{\mathrm{N}}^*), +\infty)$, we obtain x_{N}^* by solving

$$(\beta-1)v\psi^{-1}q_{\mathrm{N}}^*x^{\mathrm{i}} + (\beta-\gamma)\frac{1-\alpha}{1-\gamma}L_{\mathrm{N}}(q_{\mathrm{N}}^*)\left(\frac{q_{\mathrm{N}}^*x^{\mathrm{i}}}{\varepsilon L_{\mathrm{N}}(q_{\mathrm{N}}^*)}\right)^{\gamma} - \beta I(q_{\mathrm{N}}^*) = 0. \tag{B.3}$$

We derive (B.3) by substituting $k=0$ into (23). The optimal coupon payment is obtained as $c_{\mathrm{N}}^* = c(x_{\mathrm{N}}^{\mathrm{i}*}, q_{\mathrm{N}}^*)$.

References

Abel, A. B., Eberly, J. C., 1994. A unified model of investment under uncertainty. American Economic Review, 84(5), 1369–1384.

Abel, A. B., Eberly, J. C., 1996. Optimal investment with costly reversibility. Review of Economic Studies, 63(4), 581–593.

Abel, A. B., Dixit, A. K., Eberly, J. C., Pindyck, R. S., 1996. Options, the value of capital, and investment. Quarterly Journal of Economics, 111(3), 753–777.

Acharya, V. V., Bharath, S. T., Srinivasan, A., 2007. Does industry-wide distress affect defaulted firms? Evidence from creditor recoveries. Journal of Financial Economics, 85(3), 787–821.

Asplund, M., 2000. What fraction of a capital investment is sunk cost? Journal of Industrial Economics, 48(3), 287–304.

Black, F., Cox, J. C., 1976. Valuing corporate securities: Some effects of bond indenture provisions. Journal of Finance, 31(2), 351–367.

Brennan, A., Schwartz, E., 1984. Optimal financial policy and firm valuation. Journal of Finance, 39(3), 593–607.

Chirinko, R. S., Schaller, H., 2009. The irreversibility premium. Journal of Monetary Economics, 56(3), 390–408.

Choate, G. M., 1997. The governance problem, asset specificity, and corporate financing decisions. Journal of Economic Behavior and Organization, 33(1), 75–90.

Cui, X., Shibata, T., 2017. Investment timing and quantity strategies under asymmetric information. Theory of Probability and its Applications, 61(1), 151–159.

Dixit, A. K., Pindyck, R. S., 1994. Investment under uncertainty. Princeton University Press, Princeton, NJ.

Fan, H., Sundaresan, S., 2000. Debt valuation, strategic debt service and optimal dividend policy. Review of Financial Studies, 13(4), 1057–1099.

Folta, T. B., Johnson, D. R., O'Brien, J., 2006. Uncertainty, irreversibility, and the likelihood of entry: An empirical assessment of the option to defer. Journal of Economic Behavior and Organization, 61(3), 432–452.

Ghosal, V., Loungani, P., 2000. The differential impact of uncertainty and investment in small and large businesses. Review of Economics and Statistics, 82(2), 338–343.

Goldstein, R., Ju, N., Leland, H. E., 2001. An EBIT-based model of dynamic capital structure. Journal of Business, 74(4), 483–512.

Guiso, L., Parigi, G., 1999. Investment and demand uncertainty. Quarterly Journal of Economics, 114(1), 185–228.

Leahy, J. V., Whited, T. M., 1996. The effect of uncertainty on investment: Some stylized facts. Journal of Money, Credit and Banking, 28(1), 64–83.

Leland, H. E., 1994. Corporate debt value, bond covenants, and optimal capital structure. Journal of Finance, 49(4), 1213–1252.

Mauer, D. C., Sarkar, S., 2005. Real options, agency conflicts, and optimal capital structure. Journal of Banking and Finance, 29(6), 1405–1428.

Mauer, D. C., Triantis, A. J., 1994. Interactions of corporate financing and investment decisions: A dynamic framework. Journal of Finance, 49(4), 1253–1277.

Mella-Barral, P., Perraudin, W., 1997. Strategic debt service. Journal of Finance, 52(2), 531–556.

Nishihara, M., Shibata, T., 2018. Dynamic bankruptcy procedure with asymmetric information between insiders and outsiders. Journal of Economic Dynamics and Control, 90(C), 118–137.

Ramey, V. A., Shapiro, M. D., 2001. Displaced capital: A study of aerospace plant closings. Journal of Political Economy, 109(5), 958–992.

Riordan, M. H., Williamson, O. E., 1985. Asset specificity and economic organization. International Journal of Industrial Organization, 3(4), 365–378.

Shibata, T., Nishihara, M., 2015. Investment-based financing constraints and debt renegotiation. Journal of Banking and Finance, 51(C), 79–92.

Shibata, T., Nishihara, M., 2018. Investment tming, reversibility, and financing constraints. Journal of Corporate Finance, 48(C), 771–796.

Shibata, T., Wong, K. P., 2019. Investment under uncertainty with varialbe costly reversibility. International Review of Economics and Finance, 59(C), 14–28.

Sibikov, V., 2009. Asset liquidity and capital structure. Journal of Financial and Quantitative Analysis, 44(5), 1173–1196.

Sundaresan, S., Wang, N., 2007. Investment under uncertainty with strategic debt service. American Economic Review Paper and Proceedings, 97(2), 256–261.

Vilasuso, J., Minkler, A., 2001. Agency costs, asset specificity, and capital structure of the firm. Journal of Economic Behavior and Organization, 44(1), 55–69.

Wong, K. P., 2010. The effects of irreversibility on the timing and intensity of lumpy investment. Economic Modelling, 27(1), 97–102.

Control variables	Decision makers
At the time of investment	
Investment trigger	Equity holders
Investment quantity	Equity holders
Coupon payment	Equity holders
Risk-free debt or risky debt	Equity holders
At the time of default after investment	
Default trigger	Equity holders
Continuation or shutdown	Debt holders
At the time of shutdown (liquidation) after default	
Shutdown trigger	Debt holders
Liquidation value	Debt holders

Table 1: Control variables and decision makers

There are eight control variables. Two of eight control variables are decided automatically by determining the other six control variables. In particular, the decisions about risk-free debt or risky debt and continuation or shutdown are automatically decided by determining the optimal coupon payment and shutdown trigger, respectively.

Costly reversibility			No costly reversibility		
$L(q)$	c/r	0.7228	$L_{\mathrm{N}}(q_{\mathrm{N}})$	c_{N}/r	−0.8206
	x^{i}	0.3416		$x^{\mathrm{i}}_{\mathrm{N}}$	−0.9491
	q	0.7629		q_{N}	0.1566

Table 2: Correlation of numerical solution

This table shows the correlation for numerical solution in Figure 4.

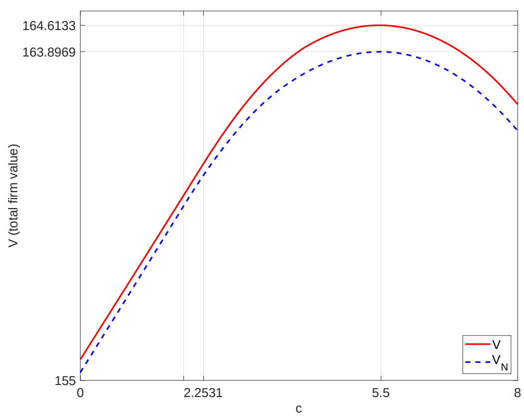

Figure 1: Total firm value with c

The parameters are $r = 0.06$, $\sigma = 0.2$, $\mu = 0.005$, $s = 0.5$, $\tau = 0.15$, $\alpha = 0.4$, $F = 5$, $a = 50$, $x^{\mathrm{i}} = 1$, and $q = 10$. We see $c(x^{\mathrm{i}}, q) = 5.5 > \theta_1(q) = 2.2531$.

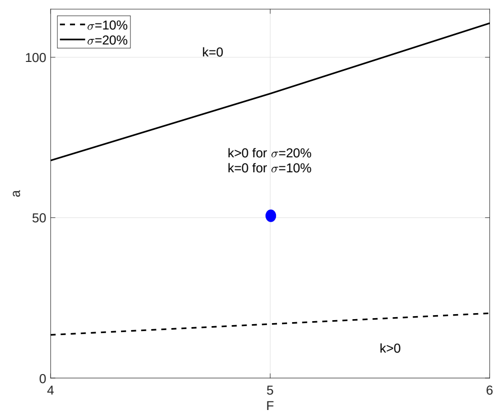

Figure 2: Regions of $k = 0$ and $k > 0$ in space (F, a)

Two lines indicate the boundaries of $k = 0$ for $\sigma = 10\%$ and $\sigma = 20\%$, respectively. The regions upper-left to the boundaries represent the regions of $k = 0$. For $F = 5$ and $a = 50$, we have $k = 0$ for $\sigma = 10\%$ but $k > 0$ for $\sigma = 20\%$.

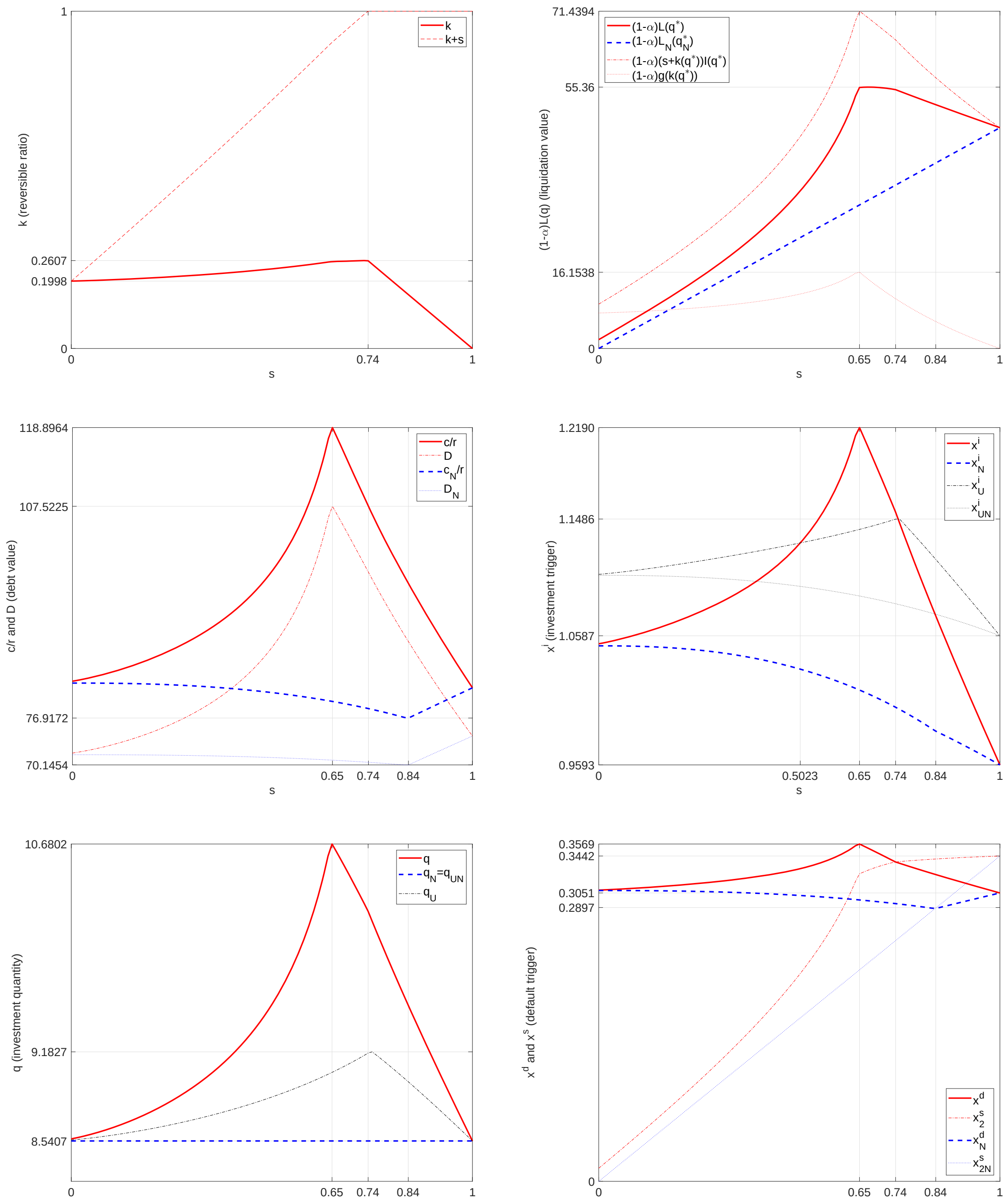

Figure 3: Effects of costly reversibility

The parameters are $r = 0.06$, $\sigma = 0.2$, $\mu = 0.005$, $F = 5$, $\tau = 0.15$, $\alpha = 0.4$, and $a = 50$.

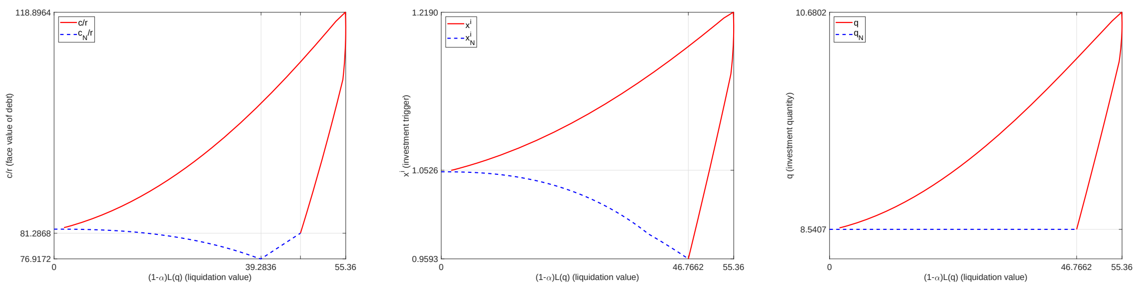

Figure 4: Effects with liquidation value $L(q)$

The basic parameters are $r = 0.06$, $\mu = 0.005$, $\tau = 0.15$, $\alpha = 0.4$, $\sigma = 0.2$, $F = 5$, and $a = 50$.

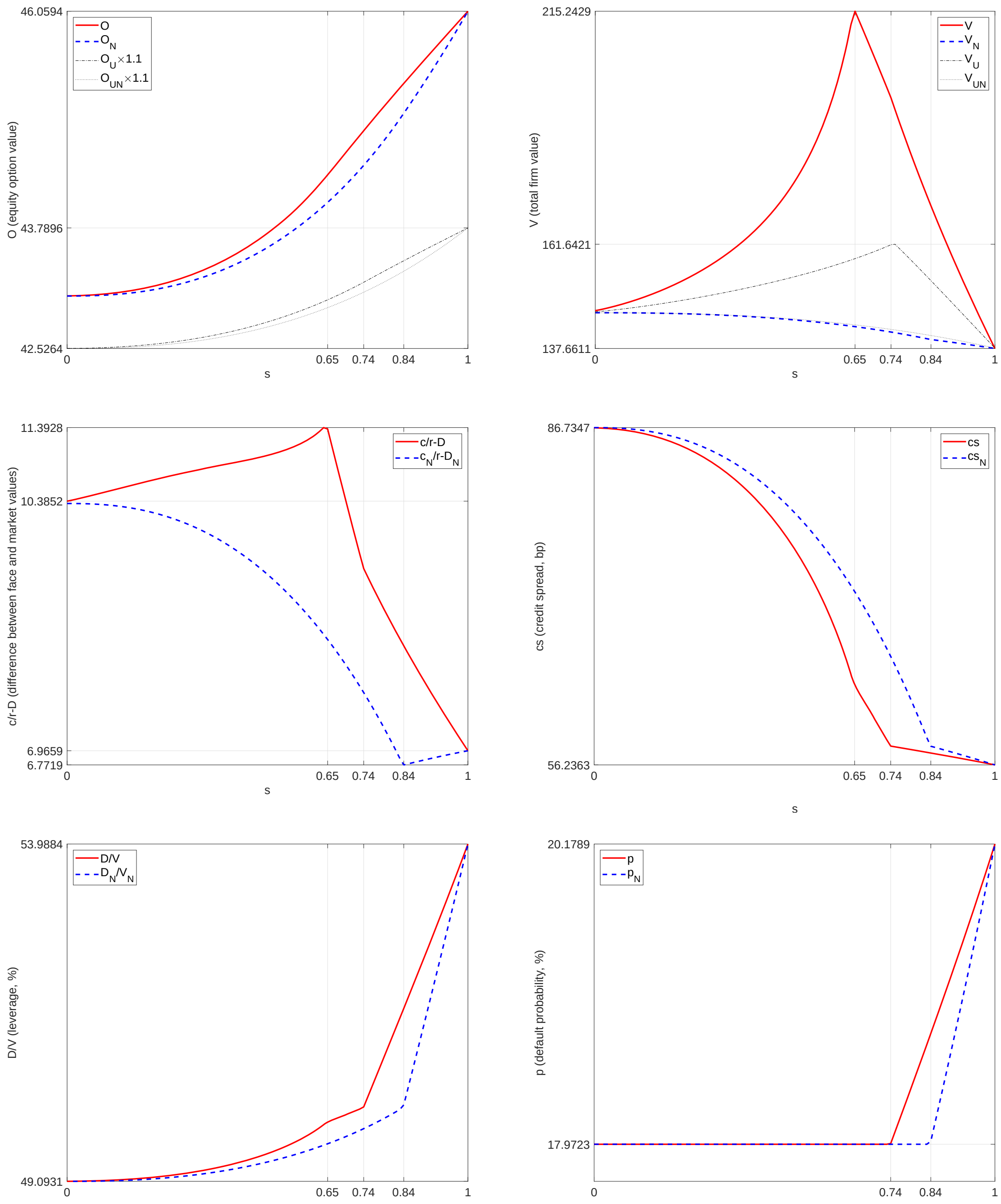

Figure 5: Leverage and credit spread

The parameters are $r = 0.06$, $\sigma = 0.2$, $\mu = 0.005$, $F = 5$, $\tau = 0.15$, $\alpha = 0.4$, and $a = 50$.

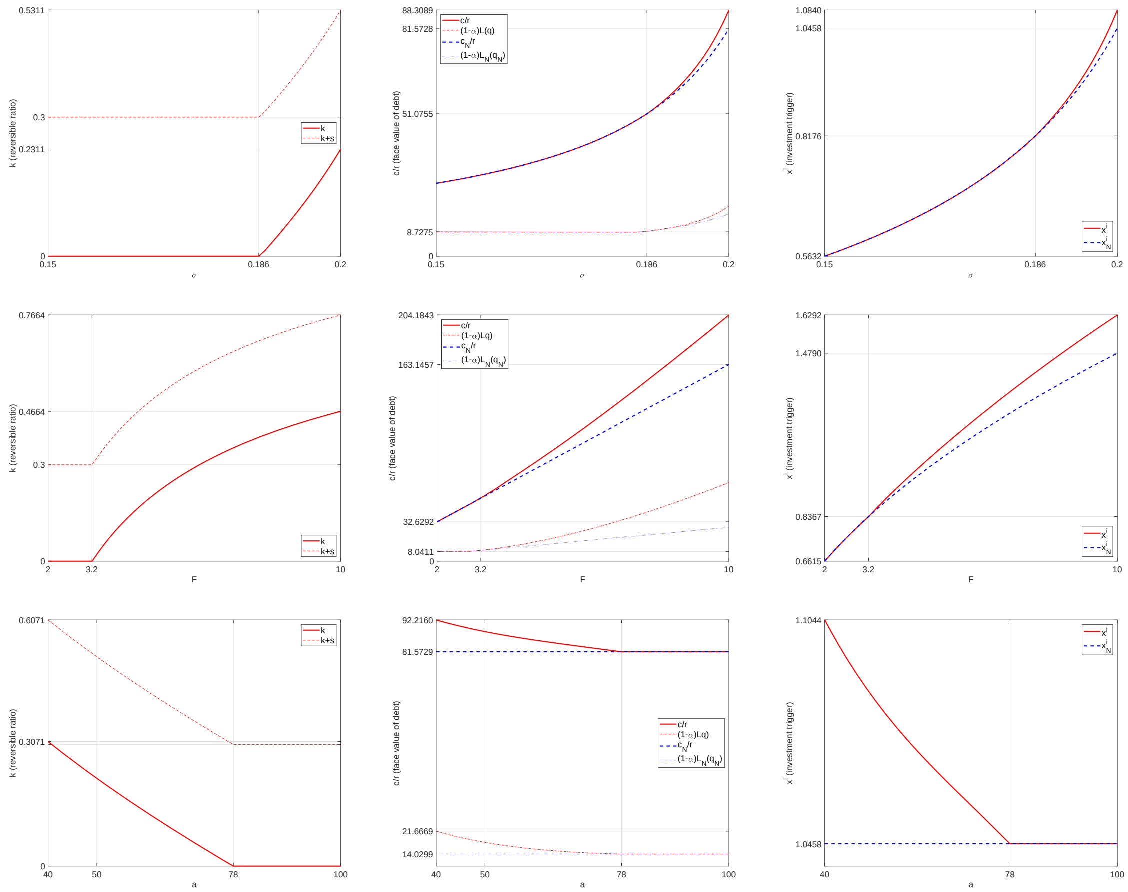

Figure 6: Comparative statics with σ, F, and a

The basic parameters are $r = 0.06$, $s = 0.3$, $\mu = 0.005$, $\tau = 0.15$, $\alpha = 0.4$, $\sigma = 0.2$, $F = 5$, and $a = 50$.