Real options and performance-sensitive debt*

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Abstract

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1 Introduction

We consider a situation where a firm can decide when and by how much to expand its production scale. Following the terminology in Trigeorgis (1996), this situation is analogous to the exercising of an option to expand capacity (which implies a choice of an investment time); yet, in contrast to the standard model, we allow for an endogenous choice of size. The firm's operating strategy consists of selecting a time of investment, namely a stopping time, as well as choosing the extra capacity lump added to its existing stock. Our benchmark model (see Bensoussan and Chevalier-Roignant, 2013) corresponds to a all equity financed firm with deep pockets.

We then allow for debt financing in the spirit of Leland (1994). Within the literature on debt financing, there are basically two streams depending on whether defaulting is part of the firm's strategy. Our intent is to investigate the impact of a debt-overhang on both the timing and the size of investment. While it is well known that debt will induce under-investment (Myers (1977), Henessy 2004), the combined effect on debt on the exercise strategy of an expansion option is understudied; here the strategy is twofold, including a choice of investment time and production scale.

Then we investigate more complex type of debt instruments, in particular the performance-sensitive debt (PSD) following the terminology introduced by Manso, Strulovici, and Tchistyi (2010). An interesting question is whether such instruments mitigate some of the welfare losses induced by debt (namely the debt overhang). Assuming managers's incentives are perfectly aligned with those of shareholders, will they pick an investment policy closer to first best (i.e. to the unlevered case)? Will the gain in terms of less under-investment compensate for the expected higher probability of bankruptcy due to PSD? In particular, will the model give an interior solution for the optimal sensitivity of debt (as well as the optimal leverage)?

2 Model

We consider a setting where a firm endowed with a capacity of size $\delta > 0$ faces a gross operating profit of $x^\gamma \delta^\epsilon$ with $\gamma > 0$. The state x can be interpreted as the commodity price of one unit of the output. We assume the firm finances itself via performance sensitive debt, whereby the coupon amount depends on the gross operating profit. For simplicity, we consider a coupon amount which is an affine function of the operating cashflows, with a negative slope, of the form

$$\beta_0 - \beta_\gamma \, x^\gamma \delta^\epsilon, \quad \beta_0, \beta_\gamma > 0.$$

This form for the coupon rate corresponds to *linear performance-sensitive debt* (PSD) following the terminology introduced in Manso et al. (2010). Let $\theta \in (0,1)$ denote the corporate tax rate. Given these specifications, we can define the firm's net operating profit as

$$\begin{aligned} \pi(x,\delta) &:= x^\gamma \delta^\epsilon - (1-\theta)\left(\beta_0 - \beta_\gamma x^\gamma \delta^\epsilon\right) \\ &= \alpha_0 + \alpha_\gamma x^\gamma \delta^\epsilon, \end{aligned} \tag{1}$$

where the parameters α_0 and α_γ are given by

$$\alpha_0 := -(1-\theta)\beta_0 \,[<0] \quad \text{and} \quad \alpha_\gamma := 1 + (1-\theta)\beta_\gamma \,[>0].$$

We introduce uncertainty in this environment and consider a Brownian motion $Z : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ defined on a probability space $(\Omega, \mathscr{F}, \mathbb{Q})$. This Brownian motion generates a filtration $\mathbb{F}$, i.e., a family of subtribes $(\mathscr{F}; t \geq 0)$ that is increasing in the sense that $\mathscr{F}_t \subseteq \mathscr{F}_s$ for $t < s$ and $\mathscr{F}_\infty = \mathscr{F}$. The filtration captures how uncertainty resolves as time passes, allowing for a perfect recall from the agents. The commodity price $X : \Omega \times \mathbb{R}_+ \to \mathbb{R}$ follows a geometric Brownian motion, i.e., it is adapted to the filtration $\mathbb{F}$ and solves a stochastic differential equation (SDE) of the form

$$\mathrm{d}X_t = \mu X_t \mathrm{d}t + \sigma X_t \, \mathrm{d}Z_t.$$

We denote the riskfree discount rate by $r > 0$, which we assume constant over time. We adopt a long-term perspective and are interested in the present value of the firm's net operating profit, which is given by

$$\Pi(x,\delta) := \mathbb{E}_{x,\delta} \int_0^\infty e^{-rt}\pi(X_t,\delta)\,\mathrm{d}t \tag{2}$$

provided the (improper) integral converges. Here, given the Markov setup, the expectation $\mathbb{E}_{x,\delta}$ is conditional on the initial values of the states (x,δ).

The firm may expand capacity once, choosing a $\mathbb{F}$-stopping time τ and a $\mathscr{F}_\tau$-measurable random variable ξ by which it increases its capacity stock. The cost incurred to increase the capacity from δ to $\delta+\xi$ is quadratic, given by $k_0+k_1\xi+k_2\xi^2$. The capacity stock $\Delta:\Omega\times\mathbb{R}_+\to\mathbb{R}_+$ is a stochastic process, adapted to the filtration $\mathbb{F}$; the capacity stock at time t is given by

$$\Delta_t^{\tau,\xi} = \delta + \xi\mathbf{1}_{[\tau,\infty)}(t),$$

where $\mathbf{1}_{\cdot}$ stands for the indicator function. In addition to deciding on the timing and scaling of capacity additions, management—who acts to the benefits of the shareholders–can decide on when to default on the debt in the spirit of Leland (1994); we note the ($\mathbb{F}$-stopping) time at which default takes place as T. We are interested in the value the firm achieves by selecting an optimal capacity expansion and defaulting strategy; we note the control or strategy as $\nu = \{\tau,\xi,T\}$. The value function corresponding to this problem is

$$\begin{aligned} F(x,\delta) := \sup_\nu \mathbb{E}_{x,\delta}\Bigg[&\int_0^T e^{-rt}\pi\left(X_t,\delta\right)\mathbf{1}_{\tau>T}\mathbf{1}_{T<\infty}\mathrm{d}t + \int_0^\tau e^{-rt}\pi\left(X_t,\delta\right)\mathbf{1}_{\tau\leq T}\mathbf{1}_{T<\infty}\mathrm{d}t \\ &+ \int_\tau^T e^{-rt}\pi\left(X_t,\delta+\xi\right)\mathbf{1}_{\tau\leq T}\mathbf{1}_{T<\infty}\mathrm{d}t - e^{-r\tau}(k_0+k_1\xi+k_2\xi^2)\mathbf{1}_{\tau\leq T}\mathbf{1}_{T<\infty}\Bigg], \end{aligned} \tag{3}$$

where the net operating profit function π is given in (1). The first integral on the RHS corresponds to the present value of the net operating profits should the firm not exercise the expansion option, but default on its debt at time T. The next two integral terms on the RHS of (3) correspond to the present value of the net profits should the firm expands capacity before defaulting, while the last term is the present value of the capacity expansion cost. We define another value function relating to the problem of defaulting on the debt commitment *only*:

$$\phi(x,\delta) := \sup_T \mathbb{E}_{x,\delta} \int_0^T e^{-rt}\pi\left(X_t,\delta\right)\mathrm{d}t. \tag{4}$$

Because ξ is $\mathscr{F}_\tau$-measurable, we can write (3) as

$$\begin{aligned} F(x,\delta) := \sup_{\tau,T} \mathbb{E}_{x,\delta}\Bigg[\int_0^T e^{-rt}\pi\left(X_t,\delta\right)\mathbf{1}_{t>T}\mathbf{1}_{T<\infty}\mathrm{d}t + \int_0^\tau e^{-rt}\pi\left(X_t,\delta\right)\mathbf{1}_{\tau\le T}\mathbf{1}_{T<\infty}\mathrm{d}t \\ + \mathbf{1}_{\tau\le T}\mathbf{1}_{T<\infty}e^{-r\tau}\Big\{\tilde{\phi}(X_\tau,\delta) - k_0\Big\}\Bigg]. \end{aligned} \tag{5}$$

where we define

$$\tilde{\phi}(x,\delta) := \sup_{\xi\ge 0}\Big\{\phi(x,\delta+\xi) - k_1\xi - k_2\xi^2\Big\}. \tag{6}$$

In this fashion, we transformed a relatively complex problem of stochastic control (3) into an optimal stopping problem (6) for which the "obstacle" is $x \mapsto \tilde{\phi}(x,\delta) - k_0$. The optimal choice of lump capacity investment amount is considered via the maximization program (6) that defines the obstacle, while the optimal choice of defaulting strategy is accounted for via the definition of the objective function $\xi \mapsto \phi(x,\delta+\xi) - k_1\xi - k_2\xi^2$ in (4).

3 Perpetuity of net operating profits

Throughout the manuscript, we solve and discuss the problems using techniques rooted in functional analysis. We denote by ∂_y the first-order differential operation in the variable y and introduce the operator $\mathcal{L} : C^2(\mathbb{R}) \to C^0(\mathbb{R})$ defined by

$$\mathcal{L} := r - \mu x\,\partial_x - \frac{1}{2}\sigma^2 x^2\,\partial_{xx}.$$

Using standard methods, it can easily be shown that the function $\Pi : \mathbb{R}_+^2 \to \mathbb{R}$ in (2) solves the second-order ordinary differential equation (ODE)

$$\mathcal{L}\Pi(x,\delta) = \pi(x,\delta), \qquad \text{a.e. } x \in \mathbb{R}_+, \tag{7a}$$

$$\Pi(\cdot,\delta) \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \text{ a.e.} \tag{7b}$$

which is parametrized in the capacity stock δ. To solve the nonhomogeneous ODE (7a)–(7b), we first look at the homogeneous ODE $\mathcal{L}g(x) = 0$. It readily obtains that the function $g : x \mapsto x^\gamma$ (which is C^2) is a solution of $\mathcal{L}g(x) = 0$ if γ is a root of the function $\mathcal{Q}(\cdot)$ given by

$$\mathcal{Q}(\gamma) = r - \gamma\mu - \frac{1}{2}\gamma\,(\gamma - 1)\,\sigma^2. \tag{8}$$

We study the variation of the function $\mathcal{Q}(\cdot)$ in (8). By differentiation, $\mathcal{Q}'(\gamma) \lesseqqgtr 0 \Leftrightarrow \gamma \gtreqqless \gamma^\star := -[\mu - \sigma^2/2]/\sigma^2$. Besides, as a polynomial function, $\mathcal{Q}(\cdot)$ is asymptotically equivalent to $\gamma \mapsto -\gamma^2\sigma^2/2$ at ∞ and $-\infty$; hence, $\mathcal{Q}(\infty) = \mathcal{Q}(-\infty) = -\infty$. It follows that $\mathcal{Q}(\cdot)$ is monotone increasing on $(-\infty, \gamma^\star)$ from $-\infty$ to $\mathcal{Q}(\gamma^\star)$ and monotone decreasing on $(\gamma^\star, \infty)$ from $\mathcal{Q}(\gamma^\star)$ to $-\infty$. The maximum, $\mathcal{Q}(\gamma^\star) = r + \frac{1}{2}[\mu - \sigma^2/2]^2/\sigma^2$, is strictly positive because we assume $r > 0$. It follows that $\mathcal{Q}(\cdot)$ has two distinct roots, γ_B and $\gamma_A > \gamma_B$, given by

$$\gamma_A, \gamma_B := -\frac{\mu - \sigma^2/2}{\sigma^2} \pm \sqrt{\left(\frac{\mu - \sigma^2/2}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{9}$$

Functions $x \mapsto x^{\gamma_B}$ and $x \mapsto x^{\gamma_A}$ are thus two independent solutions of $\mathcal{L}g(x) = 0$. We want to prove that $\gamma_B < 0 < \gamma_A$. We have $0 < \mathcal{Q}(0) = r \leq \mathcal{Q}(\gamma^\star)$. Because $\mathcal{Q}(\cdot)$ is monotone increasing on $(-\infty, \gamma^\star)$, we have $\gamma_B < 0$. Because $\mathcal{Q}(\cdot)$ is monotone decreasing on $(\gamma^\star, \infty) \subset (0, \infty)$, $\gamma_A > 0$.

We can easily verify that the function

$$\Pi(x, \delta) := \eta_0 + \eta_\gamma x^\gamma \delta^\epsilon, \tag{10}$$

whereby parameters η_0 and η_γ are respectively

$$\eta_0 := -\frac{(1-\theta)\beta_0}{r}\,[< 0] \quad \text{and} \quad \eta_\gamma := \frac{1 + (1+\theta)\beta_\gamma}{\mathcal{Q}(\gamma)}\,[> 0],$$

solves the ODE (7a)–(7b). This perpetuity value is positive if $\gamma \in \{\gamma \mid \mathcal{Q}(\gamma) > 0\}$, i.e., if $\gamma \in (\gamma_B, \gamma_A)$. We are thus focusing on parameter values for γ in the range $(0, \gamma_A)$.

4 Performance-sensitive debt (benchmark model)

4.1 Variational inequality

We first ignore the capacity expansion option to concentrate on the defaulting strategy; in other words, we consider the optimization program (4), which is essentially an optimal stopping problem. Following Bensoussan and Lions (1982), it is known that the dynamic programming equation corresponding to an optimal stopping problem is a *variational inequality* (VI). Using the notations (1) and (10), it can be shown that the value function ϕ solves the variational inequality (VI),

$$\phi(x,\delta) \geq 0, \qquad \forall x \in \mathbb{R}_+, \tag{11a}$$

$$\mathcal{L}\phi(x,\delta) \geq \pi(x,\delta), \qquad \text{a.e. } x \in \mathbb{R}_+, \tag{11b}$$

$$\phi(x,\delta)\left[\mathcal{L}\phi(x,\delta) - \pi(x,\delta)\right] = 0, \qquad \text{a.e. } x \in \mathbb{R}_+, \tag{11c}$$

$$\liminf_{x\uparrow\infty} \Big\{\phi(x,\delta) - \Pi(x,\delta)\Big\} = 0 \tag{11d}$$

$$\phi(\cdot,\delta) \in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \text{ a.e.} \tag{11e}$$

which is parametrized in the capacity stock $\delta > 0$. To prove that the solution of VI (11a)–(11e) coincides with the value function (4) requires a proof—known as a *verification theorem*—which we omit here/at this stage.

The variational inequality approach allows the define the continuation set implicitly as

$$\mathcal{C}_A = \Big\{(x,\delta) \in \mathbb{R}_+^2 \mid \phi(x,\delta) > 0\Big\}.$$

4.2 Free-boundary problem

We conjecture a certain structure for the continuation set $\mathcal{C}_A$ and assume the existence of a free boundary $\underline{x}_A(\delta)$ that delimits the continuation and stopping sets of the optimal stopping problem (4) which is parametrized in the capacity stock δ. Instead of the VI (11a)–(11e), we now consider a free-boundary problem (FBP), namely

$$\phi(x,\delta) = 0, \qquad \forall x < \underset{\sim}{x}_A(\delta), \tag{12a}$$

$$\mathcal{L}\phi(x,\delta) = \pi(x,\delta), \qquad \forall x \geq \underset{\sim}{x}_A(\delta) \tag{12b}$$

$$\lim_{x\uparrow\infty} \inf \left\{ \phi(x,\delta) - \Pi(x,\delta) \right\} = 0, \tag{12c}$$

$$\phi\left(\underset{\sim}{x}_A(\delta),\delta\right) = 0, \tag{12d}$$

$$\partial_x^+ \phi\left(\underset{\sim}{x}_A(\delta),\delta)\right) = 0, \tag{12e}$$

Note that a solution to the FBP (12a)–(12e) is not necessarily a solution to the VI (11a)–(11e). For this to be the case, we would need to verify that the solution of the FBP (12a)–(12e) satisfies the inequalities

$$\phi(x,\delta) \geq 0, \qquad \forall x \geq \underset{\sim}{x}_A(\delta), \tag{13a}$$

$$\mathcal{L}\phi(x,\delta) \geq \pi(x,\delta), \qquad \forall x < \underset{\sim}{x}_A(\delta). \tag{13b}$$

Having this in mind, we first (try to) solve the FBP (12a)–(12e) explicitly. The second-order ODE (12b) admits a solution of the form

$$\phi(x,\delta) = \Pi(x,\delta) + A(\delta)x^{\gamma_A} + B(\delta)x^{\gamma_B}, \quad \forall x \geq \underset{\sim}{x}_A(\delta), \tag{14}$$

where Π is the perpetuity operating profit given in (10) and the parameters γ_A and γ_B are provided in (9). So, now from (12a)–(12b), the function ϕ is defined piecewise as

$$\phi(x,\delta) = \begin{cases} 0, & x < \underset{\sim}{x}_A(\delta), \\ \Pi(x,\delta) + A(\delta)x^{\gamma_A} + B(\delta)x^{\gamma_B}, & x \geq \underset{\sim}{x}_A(\delta), \end{cases}$$

while the three unknown functions $A(\cdot)$, $B(\cdot)$ and $\underset{\sim}{x}_A(\cdot)$ are obtained from the boundary conditions (12c)–(12e). From (12c), it follows that

$$A(\cdot) \equiv 0.$$

For any $\delta > 0$, it obtains from (12d)–(12e) and the definition of Π in (10) that the threshold $\bar{x}(\delta)$ is the unique, positive root of the function $G(\cdot, \delta)$ given by

$$G(x, \delta) := \left(\frac{\gamma_B - \gamma}{\gamma_B}\right) \eta_\gamma x^\gamma \delta^\epsilon + \eta_0. \tag{15}$$

We study the variation of the function $G(\cdot, \delta)$ in (15) to determine whether such a root exists. We have

$$G(0, \delta) = \begin{cases} \eta_0 < 0, & \gamma_A > \gamma > 0 \\ \infty, & \gamma_B < \gamma < 0 \end{cases}$$

$$G(\infty, \delta) = \begin{cases} \infty, & \gamma_A > \gamma > 0 \\ \eta_0 < 0, & \gamma_B < \gamma < 0 \end{cases}$$

$$\partial_x G(x, \delta) = \eta_\gamma \gamma x^{\gamma - 1} \delta^\epsilon \left(\frac{\gamma_B - \gamma}{\gamma_B}\right)$$

$$\partial_x G(x, \delta) = \begin{cases} > 0, & \gamma_A > \gamma > 0 \\ < 0, & \gamma_B < \gamma < 0. \end{cases}$$

If $\gamma_A > \gamma > 0$, then $G(\cdot, \delta)$ is increasing on $\mathbb{R}_+$ from a negative number to ∞, while if $\gamma_B < \gamma < 0$, then this function decreases on $\mathbb{R}_+$ from ∞ to a negative number. In both cases, there is a unique positive root, given by

$$\underline{x}_A(\delta) := \left[-\frac{\eta_0}{\eta_\gamma} \frac{\gamma_B}{\gamma_B - \gamma}\right]^{1/\gamma} \delta^{-\epsilon/\gamma}. \tag{16}$$

By differentiation, it obtains that

$$\underline{x}'_A(\delta) = -\frac{\epsilon}{\gamma} \underline{x}_A(\delta) \delta^{-1}, \tag{17}$$

which implies that $\delta \mapsto \underline{x}_A(\delta)$ is monotone decreasing. In other words, when the firm's capacity stock δ becomes larger, the threshold level $\underline{x}_A(\delta)$ at which shareholders decide to default is decreased because the firm achieves a larger (gross) profit and is therefore more able to pay back its debt obligations. Because $\underline{x}_A(\cdot)$ is monotone decreasing, so is its inverse $\underline{x}_A^{-1}(\cdot)$ given by

$$\underline{x}_A^{-1}(x) := \left[\frac{\eta_\gamma}{\eta_0} \frac{\gamma - \gamma_B}{\gamma_B}\right]^\epsilon x^{-\gamma/\epsilon}.$$

We now want to derive $B(\delta)$. From (14), we have

$$\partial_x^+ \phi(x, \delta) = \gamma \eta_\gamma x^{\gamma - 1} \delta^\epsilon + \gamma_B B(\delta) x^{\gamma_B - 1}, \quad x \geq \underline{x}_A(\delta)$$

From (12e),

$$B(\delta) := -\frac{\gamma \eta_\gamma \delta^\epsilon}{\gamma_B} \underline{x}_A(\delta)^{\gamma - \gamma_B}, \tag{18}$$

with $\underline{x}_A(\delta)$ given in (16). The quantity $B(\delta)$ given in (18) can be expressed in terms of model primitives as

$$B(\delta) = \frac{\gamma \eta_0}{\gamma_B - \gamma} \left[-\frac{\eta_0}{\eta_\gamma} \frac{\gamma_B}{\gamma_B - \gamma} \right]^{-\gamma_B/\gamma} \delta^{\epsilon \gamma_B/\gamma}.$$

In summary, the function $\phi : \mathbb{R}_+^2 \to \mathbb{R}$ given by

$$\phi(x, \delta) = \left[\eta_\gamma x^\gamma \delta^\epsilon + \eta_0 - \frac{\gamma}{\gamma_B} \eta_\gamma x^{\gamma_B} \delta^\epsilon \underline{x}_A(\delta)^{\gamma - \gamma_B} \right] \mathbf{1}_{[\underline{x}_A(\delta), \infty)}(x), \tag{19}$$

with the boundary $\underline{x}_A(\cdot)$ in (16) solves the FBP (12a)–(12e).

4.3 Connection VI–FBP

It remains to verify that the function ϕ in (19) satisfies (13a)–(13b). We start with the inequality (13a). It follows from (19) that

$$\partial_x \phi(x, \delta) = \gamma \eta_\gamma \delta^\epsilon \frac{\underline{x}_A(\delta)^\gamma}{x} \left[\left(\frac{x}{\underline{x}_A(\delta)} \right)^\gamma - \left(\frac{x}{\underline{x}_A(\delta)} \right)^{\gamma_B} \right], \quad x > \underline{x}_A(\delta).$$

Because the function $y \mapsto a^y$ is monotone increasing whenever $a > 1$, it follows that

$$\Big(\gamma > \gamma_B \Big) \cap \Big(x > \underline{x}_A(\delta) \Big) \implies \left(\frac{x}{\underline{x}_A(\delta)} \right)^\gamma > \left(\frac{x}{\underline{x}_A(\delta)} \right)^{\gamma_B}.$$

It is now immediate that $x \mapsto \phi(x, \delta)$ is monotone increasing on $(\underline{x}_A(\delta), \infty)$ if $\gamma > 0 > \gamma_B$; because $\phi(\underline{x}_A(\delta), \delta) = 0$, the function $x \mapsto \phi(x, \delta)$ is positive valued on $(\underline{x}_A(\delta), \infty)$ and, hence, that the inequality (13a) is satisfied if $\gamma > 0$. For a given parameter $x > \underline{x}_A(\delta)$, the (partial) derivative of the function $\phi(x,\cdot)$ in (19) reads

$$\partial_\delta \phi(x,\delta) = \eta_\gamma \epsilon x^\gamma \delta^{\epsilon-1} - \eta_\gamma \epsilon x^{\gamma_B} \delta^{\epsilon-1} \frac{\gamma}{\gamma_B} \underline{x}_A(\delta)^{\gamma-\gamma_B} - \eta_\gamma \delta^\epsilon \frac{\gamma(\gamma-\gamma_B)}{\gamma_B} x^{\gamma_B} \underline{x}_A(\delta)^{\gamma-\gamma_B-1} \underline{x}'_A(\delta).$$

Using the derivative $\underline{x}'_A(\delta)$ in (17), it obtains after a few operations that

$$\partial_\delta \phi(x,\delta) = \eta_\gamma x^\gamma \epsilon \delta^{\epsilon-1} \left[1 - \left(\frac{\underline{x}_A(\delta)}{x} \right)^{\gamma-\gamma_B} \right]. \tag{20}$$

Because $\gamma > \gamma_B$ and $x > \underline{x}_A(\delta)$, it follows that $\partial_\delta \phi(x,\delta) > 0$ and hence that the function $\delta \mapsto \phi(x,\delta)$ is monotone increasing on $\left(\underline{x}_A^{-1}(x), \infty\right)$. It obtains from (12d) that

$$\phi\left(x, \underline{x}_A^{-1}(x)\right) = 0,$$

and thus that the inequality (13a) is satisfied.

We now consider the inequality (13b). It is immediate that, if

$$x \leq \bar{x}_\star(\delta) := \left[-\frac{\alpha_0}{\alpha_\gamma} \right]^{1/\gamma} \delta^{-\epsilon/\gamma}, \tag{21}$$

then the inequality (11b), which specializes here to

$$\alpha_\gamma x^\gamma \delta^\epsilon + \alpha_0 \leq 0 = \mathcal{L}\phi(x,\delta),$$

is satisfied. Now we want to prove that $\bar{x}_\star > \underline{x}_A(\delta)$, so that the inequality (11b) is satisfied for any $x \in (0, \underline{x}_A(\delta))$. From (16) and (21), we thus want to prove that

$$1 > \frac{\mathcal{Q}(\gamma)}{r} \frac{\gamma_B}{\gamma_B - \gamma} \iff \gamma_B[r - \mathcal{Q}(\gamma)] < \gamma r.$$

We define the function

$$g(x) := \alpha_\gamma x^\gamma \delta^\epsilon + \alpha_0.$$

Assuming $\gamma > 0$, this function is monotone increasing from $\alpha_0 < 0$ to ∞; it has a unique root given by

$$\bar{x}_\star(\delta) := \left[-\frac{\alpha_0}{\alpha_\gamma}\right]^{1/\gamma} \delta^{-\epsilon/\gamma}. \tag{22}$$

From (16), we have

$$\begin{aligned} g\left(\underline{x}_A(\delta)\right) &= -\alpha_\gamma \frac{\eta_0}{\eta_\gamma} \frac{\gamma_B}{\gamma_B - \gamma} + \alpha_0 \\ &= \alpha_0 \frac{\mathcal{Q}(\gamma)}{r} \left(\frac{r}{\mathcal{Q}(\gamma)} - \frac{\gamma_B}{\gamma_B - \gamma} \right) \end{aligned}$$

Because $\mathcal{L}\phi(x, \delta) = 0$ when $x < \underline{x}_A(\delta)$, we want to show that

$$x < \underline{x}_A(\delta) \implies \alpha_\gamma x^\gamma \delta^\epsilon + \alpha_0 \leq 0,$$

which is equivalent to show that . For this to be satisfied, it suffices that $\underline{x}_A(\delta) \geq$. Besides, for $x < \underline{x}_A(\delta)$,

$$\alpha_\gamma x^\gamma \delta^\epsilon \begin{cases} < \alpha_\gamma \underline{x}_A(\delta) \delta^\epsilon + \alpha_0, & \gamma \geq 1, \\ > \alpha_\gamma \underline{x}_A(\delta) \delta^\epsilon + \alpha_0, & \gamma < 1, \end{cases}$$

The LHS reads

$$\alpha_\gamma x - \frac{\eta_0}{\eta_\gamma} \frac{\gamma_B}{\gamma_B - \gamma} + \alpha_0 = \alpha_0 \left[1 - \frac{\mathcal{Q}(\gamma)}{r} \frac{\gamma_B}{\gamma_B - \gamma} \right].$$

We define

$$
\begin{aligned}
h(\gamma) &:= \gamma_B - \gamma - \frac{\mathcal{Q}(\gamma)}{r}\gamma_B \\
&= \gamma_B \frac{r - \mathcal{Q}(\gamma)}{r} - \gamma, \\
&= \gamma_B \frac{\mu\gamma + \frac{1}{2}\gamma(\gamma-1)\sigma^2}{r} - \gamma, \\
h(1) &= \gamma_B \frac{\mu}{r} - 1 < 0, \\
h'(\gamma) &= \frac{\gamma_B}{r}\left[\mu + \frac{1}{2}(\gamma-1)\sigma^2 + \frac{1}{2}\gamma\sigma^2\right] - 1, \\
&= \frac{\gamma_B}{r}\left[\mu + \sigma^2\mu - \frac{1}{2}\sigma^2\right] - 1 \\
h''(\gamma) &= \frac{\gamma_B}{r}\sigma^2 < 0;
\end{aligned}
$$

hence $h'(\cdot)$ is monotone decreasing, taking negative values of $\gamma > 1$. It follows that $h(\cdot)$ is decreasing on $(1, \infty)$ taking negative values. So,

$$
\begin{aligned}
h(\gamma) &< 0, && \forall \gamma \geq 1, \\
\frac{h(\gamma)}{\gamma_B - \gamma} &> 0, && \forall \gamma \geq 1, \\
\alpha_0 \frac{h(\gamma)}{\gamma_B - \gamma} &< 0, && \forall \gamma \geq 1.
\end{aligned}
$$

Hence,

$$
\alpha_1 x^\gamma \delta^\epsilon + \alpha_0 < 0 = \mathcal{L}\phi(x, \delta)
$$

on $(0, \underline{x}_A(\delta)$ if $\gamma \geq 1$.

5 Capacity expansion

Now that we found the explicit expression (19) for the solution of the VI (11a)–(11e), we can investigate the optimization problem (6) further. We rewrite the problem (6) as

$$\tilde{\phi}(x,\delta) := \sup_{\Delta \geq \delta} \left\{ \phi(x,\Delta) - k_1(\Delta - \delta) - k_2(\Delta - \delta)^2 \right\}. \tag{23}$$

Let $\hat{\delta}(x,\delta)$—or $\hat{\delta}$ in short—denote the maximum (if it exists). The maximum $\hat{\delta}$ must satisfy

$$\partial_\delta \phi\big(x,\hat{\delta}\big) - k_1 - 2k_2(\hat{\delta} - \delta) \leq 0, \tag{24a}$$

$$\hat{\delta} \geq \delta, \tag{24b}$$

$$\Big[\partial_\delta \phi(x,\hat{\delta}) - k_1 - 2k_2(\hat{\delta} - \delta)\Big]\,[\hat{\delta} - \delta] = 0. \tag{24c}$$

Let $\bar{\delta}(x)$—or $\bar{\delta}$ in short—be a critical point of the function $\Delta \mapsto \phi(x,\Delta) - k_1(\Delta - \delta) - k_2(\Delta - \delta)^2$. By definition, $\bar{\delta}$ solves the equation

$$\partial_\delta \phi\big(x,\bar{\delta}\big) - 2k_2\bar{\delta} = k_1 - 2k_2\delta. \tag{25}$$

We want to prove that equation (25) admits a unique solution $\bar{\delta}$ whenever $\delta > \underline{x}_A^{-1}(x)$. We recall the expression of $\partial_\delta \phi(x,\delta)$ in (20). We compute the derivatives

$$\partial_{x\delta}\phi(x,\delta) = \epsilon\gamma\eta_\gamma\delta^{\epsilon-1}x^{\gamma-1}\left[1 + (\gamma - 1 - \gamma_B)\left(\frac{\underline{x}_A(\delta)}{x}\right)^{\gamma-\gamma_B}\right],$$

$$\partial_{\delta\delta}\phi(x,\delta) = \eta_\gamma x^\gamma \epsilon\delta^{\epsilon-2}\left[\epsilon - 1 + \frac{\gamma - \gamma_B\epsilon}{\gamma}\left(\frac{\underline{x}_A(\delta)}{x}\right)^{\gamma-\gamma_B}\right].$$

...

6 Capacity expansion option and PSD

6.1 Variational inequality

To investigate the interplay between the capacity expansion option and the default strategy induced by the performance-sensitive debt, we come back to the original problem in (5). The function F differs from the function ϕ; in particular the function F solves a different VI (also parametrized in $\delta > 0$), namely

$$
\begin{aligned}
F(x,\delta) &\geq \tilde{\phi}(x,\delta) - k_0, && \forall x \in \mathbb{R}_+, \quad \text{(26a)}\\
\mathcal{L}F(x,\delta) &\geq \pi(x,\delta), && \text{a.e. } x \in \mathbb{R}_+, \quad \text{(26b)}\\
F(x,\delta) &\geq 0, && \forall x \in \mathbb{R}_+ \quad \text{(26c)}\\
\Big[F(x,\delta) - \tilde{\phi}(x,\delta) - k_0\Big]\Big[\mathcal{L}F(x,\delta) - \pi(x,\delta)\Big]F(x,\delta) &= 0, && \text{a.e. } x \in \mathbb{R}_+ \quad \text{(26d)}\\
\liminf_{x\uparrow\infty}\Big\{F(x,\delta) - \Pi(x,\delta)\Big\} &= 0, && \text{(26e)}\\
F(\cdot,\delta) &\in C^1(\mathbb{R}_+) \cap C^2(\mathbb{R}_+) \text{ a.e.} && \text{(26f)}
\end{aligned}
$$

Inequality (26a) relates to the "obstacle" corresponding to the capacity expansion option in (6). Inequality (26b) corresponds to the decision to defer the decision making for an extra (marginal) unit of time. Inequality (26c) relates to the shareholders' decision to default on their debt obligations. The equality (26a) prescribes that the decisions to defer, expand capacity or default are mutually exclusive; this is the complementary slackness condition.

6.2 Free-boundary problem

Again, we conjecture a structure of the continuation set

$$\mathcal{C} = \Big\{(x,\delta) \in \mathbb{R}^2_+ \mid F(x,\delta) > 0\Big\}$$

in the form of

$$\mathcal{C} = \Big\{(x,\delta) \in \mathbb{R}^2_+ \mid \underline{x}(\delta) < x < \bar{x}(\delta)\Big\}.$$

Given the economic setup, it is reasonable to further assume that $\bar{x}(\cdot) > \underline{x}_A(\cdot) > \underline{x}(\cdot)$. Because $\underline{x}_A(\cdot)$ is monotone decreasing, it seems natural to also assume that the functions $\bar{x}(\cdot)$ and $\underline{x}(\cdot)$ are monotone decreasing and invertible. On the basis of this conjecture, we consider the FBP

$$F(x,\delta) = \tilde{\phi}(x,\delta) - k_0, \qquad \forall x > \bar{x}(\delta), \tag{27a}$$

$$\mathcal{L}F(x,\delta) = \pi(x,\delta), \qquad \forall x \in (\bar{x}(\delta), \bar{x}(\delta)) \tag{27b}$$

$$F(x,\delta) = 0, \qquad \forall x < \underline{x}(\delta), \tag{27c}$$

$$F(\bar{x}(\delta),\delta) = \tilde{\phi}(\bar{x}(\delta),\delta), \tag{27d}$$

$$\partial_x F(\bar{x}(\delta),\delta) = \partial_x \tilde{\phi}(\bar{x}(\delta),\delta), \tag{27e}$$

$$F(\underline{x}(\delta),\delta) = 0, \tag{27f}$$

$$\partial_x F(\underline{x}(\delta),\delta) = 0, \tag{27g}$$

$$\liminf_{x\uparrow\infty} \Big\{ F(x,\delta) - \Pi(x,\delta) \Big\} = 0. \tag{27h}$$

Again, the solution of the FBP (27a)–(27h) does not necessarily coincide with the solution of the VI (26a)–(26f); for this to be true, the solution of the FBP (27a)–(27h) will need to satisfy

$$F(x,\delta) \geq \tilde{\phi}(x,\delta) - k_0, \qquad \forall x \leq \bar{x}(\delta) \tag{28a}$$

$$\mathcal{L}F(x,\delta) \geq \pi(x,\delta), \qquad \forall x \in (0, \underline{x}(\delta)] \cup [\bar{x}(\delta), \infty), \tag{28b}$$

$$F(x,\delta) \geq 0, \qquad \forall x \geq \underline{x}(\delta). \tag{28c}$$

To investigate the FBP (27a)–(27h), we first look at the equality (27a). We recall that the solution to the problem (6) is

$$\tilde{\phi}(x,\delta) = \ldots$$

...

7 Numerical results

We make the following parametric assumptions

Linear PSD and in example 4 of MST (2010)

Figure 1. Shareholder Value Function. Shareholder value function, exercise, and default thresholds. The parameter values are $r = 0.1$;$\alpha = 0.25$;$\sigma = 0.10$;$\mu = 0.01$;$\beta_0 = 0$;$\beta_1 = 0$;$c = 500$,$\kappa = 50$

Figure 2. Comparative Statics. Comparative Statics with respect to σ,μ,κ, and c. The parameter values are $r = 0.1$;$\alpha = 0.25$;$\sigma = 0.10$;$\mu = 0.01$;$\beta_0 = 0$,$\beta_1 = 0$;$c = 500$,$\kappa = 50$

The profit function is given by $\pi(X, \delta) = X \log \delta$

We assume the cost of adjusting capital is quadratic (in contrast with BC-R)

We also consider some comparative statics with respect to volatility σ, the growth rate of profitability μ, the fix cost of exercising the option κ, and the variable cost of investment c.

Results are intuitive (expected):

- Higher volatility increases the option value of waiting for both the default and the investment option. Therefore higher σ generates "hysterisis
- Higher growth rate μ delays default as it makes it more likely the firm will recover. It also makes investment more attractive, and investment takes place earlier.
- Higher cost of exercising the option κ or higher variable cost c makes investment less attractive: investment is delayed and default is hastened.

We next consider comparative statics with respect to the financing parameters β_0 and β_1. Our goal here is to study the impact of debt in the timing of default and investment. In the next section we explore the effect of debt on the scale of investment.

Increasing the baseline coupon β_0 delays the timing of investment. This result is in line with the intuition of the debt overhang problem first described by Myers (1977). Because some of the benefits of investment will not be internalized by shareholders (due to debt induced default), investment will be reduced. Moreover, higher β_0 increases per period cashflows for shareholders thus hastening default.

Increasing the sensitivity of debt β_1 incentivizes shareholders to increase their per period cashflows $\pi(X, \delta)$, since higher cashflows means their debt obligations will be reduced. Since profits are increasing in firm size δ, this provides an incentive for shareholders to invest earlier. This first

Figure 3. Comparative Statics. Comparative Statics with respect to β_0 and β_1. The parameter values are $r = 0.1$;$\alpha = 0.25$;$\sigma = 0.10$;$\mu = 0.01$;$\beta_0 = 0$, $\beta_1 = 0$;$c = 500$,$\kappa = 50$.

Figure 4. Comparative Statics. Comparative Statics with respect to β_0 and β_1 for the scale and timing of investment. The parameter values are $r = 0.1;\alpha = 0.25;\sigma = 0.10;\mu = 0.01;\beta_0 = 0;\beta_1 = 0;c = 500,\kappa = 50$

observation hints at the potential of PSD of mitigating the debt overhang problem.

Figure "Comparative Statics Scale" depicts the effect of debt on the scale of investment. Panel A shows the benchmark optimal investment policy in the absence of the distortionary effects of debt. It then depicts comparative statics with respect to the baseline coupon β_0. Higher β_0 renders investment less attractive: investment is only delayed, but also the scale of investment is distorted downwards. Panel B depicts comparative statics of the scale of investment with respect to the performance sensitivity of debt parameter β_1. Higher β_1 makes investment more attractive thereby increasing the scale of investment. Again, this hints at the potential of PSD to mitigate the underinvestment problem in both dimensions: time and scale. Interestingly, PSD can potentially lead to over-investment, thus encouraging shareholders to investment in projects that are larger than their first best size.

References

Bensoussan, A., and B. Chevalier-Roignant. 2013. Real Options with Time and Scale Flexibility. *Real Options, Ambiguity, Risk and Insurance: World Class University Program in Financial Engineering, Ajou University, Volume Two* 5:66.

Bensoussan, A., and J.-L. Lions. 1982. *Applications of Variational Inequalities in Stochastic Control*. Studies in Mathematics and Its Applications. New York: North-Holland.

Leland, H. E. 1994. Corporate debt value, bond covenants, and optimal capital structure. *Journal of Finance* 49:1213–1252.

Manso, G., B. Strulovici, and A. Tchistyi. 2010. Performance-sensitive debt. *Review of Financial Studies* 23:1819–1854.

Myers, S. C. 1977. Determinants of corporate borrowing. *Journal of Financial Economics* 5:147 – 175.

Trigeorgis, L. 1996. *Real options: Managerial flexibility and strategy in resource allocation*. MIT Press.