Analytical Solution for the General Two-Factor Investment Model: Option Value and Derivatives

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Abstract

We provide simplified solutions for determining the real option value (and exercise threshold) for a perpetual opportunity to invest when there are two stochastic factors and circumstances which do not allow dimensionality reduction. Our solution is easy to compute, amenable to interpretation, and enables analytical derivations for the partial derivatives. We compare the properties of our model with one-factor and two-factor homogeneity degree models. Analytical and numerical illustrations show that some of the typical real option value and thresholds assumptions, such as positive "vegas" do not necessarily hold.

1 Introduction

We provide a simplified analytical solution method for obtaining the value of the option to invest as well as derivatives with respect to certain key parameters for an investment-style real option model having two stochastic factors. Although the analytical solution for an investment opportunity formulated either as a one-factor model or a two-factor model assuming the homogeneity degree-1 property has been known for over 30 years, McDonald & Siegel (1986) and Sick (1989), an equivalent solution for a general two-factor model was proposed only recently, Adkins & Paxson (2006), Heydari (2010) and Støre, Fleten, Hagspiel, & Nunes (2017). Although their methods are computationally less onerous than numerical solutions based for example on finite-differences, their results are incomplete, possibly computationally unwieldy and their precision unexplored. Also, no analytical method is proposed for obtaining the key derivatives of the option value including the "vega". In this paper, we revise their methods to develop a conceivably more elegant but computationally less onerous solution with the facility to yield the option derivatives analytically. Further we examine the precision of the solution method.

The paper is organized in the following way. In section 2, we develop an analytical solution method based on Adkins & Paxson (2011), Heydari, Ovenden, & Siddiqui (2012), Støre et al. (2017) for a general two-factor investment opportunity model. The model is based on two stochastic factors and designed in a way that makes the homogeneity degree-1 property inadmissible. For this model, we develop an alternative analytical method that yields the threshold boundary and the option value for all possible values. The results we obtain are straightforward to compute and amenable to interpretation. They enable the option derivatives with respect to certain parameters to be derived analytically and the conditions governing the sign of each derivative to be determined. Accordingly, the properties of the general two-factor model can be compared with those corresponding to the one-factor model and two-factor homogeneity degree-1 model. In section 3, we apply our methods to the general two-factor abandonment opportunity model. Section 4 explores the precision of our methods. We demonstrate at least for the data-set employed that the precision in meeting the requirements for the smooth-pasting conditions is satisfactory We end with a brief conclusion. Much of the explanatory analysis is confined to Appendices.

2 General two-factor investment model

In the absence of competition and other forms of optionality, we consider a project investment, whose value is determined by a stochastic periodic cash-flow, denoted by $X \geq 0$, a known periodic fixed operating cost, $f > 0$, and a one-off stochastic investment cost, $K \geq 0$. The formulation represents a general two-factor model, since the two factors X, K are stochastic and the presence of a non-zero fixed operating cost makes the homogeneity degree-1 assumption underpinning the McDonald & Siegel (1986) model inadmissible. In short, our representation cannot be reframed as a one-factor model based on the ratio X/K. However, if $f = 0$, our general model simplifies to the McDonald & Siegel (1986) formulation, while if K is treated as known, then it simplifies to the standard one-factor model of Dixit & Pindyck (1994).

2.1 Model

The periodic cash flow and investment cost are assumed to be described by the following geometric Brownian motion processes:

$$\mathrm{d}X = \alpha_X X\,\mathrm{d}t + \sigma_X X\,\mathrm{d}W_X\,, \tag{1}$$

$$\mathrm{dK} = \alpha_K K\,\mathrm{d}t + \sigma_K K\,\mathrm{d}W_K\,, \tag{2}$$

where α denotes the respective known drift rate, σ the volatility, and $\mathrm{d}W$ an increment of the standard Wiener process. The covariance between the periodic cash flow and investment cost is specified by $\mathrm{Cov}\left[\mathrm{d}X, \mathrm{d}K\right] = \rho_{XK}\sigma_X\sigma_K\mathrm{d}t$ where ρ_{XK} denotes the correlation coefficient with $-1 \le \rho_{XK} \le 1$. The risk neutral valuation relationship based on Ito's lemma and contingent claims, Dixit & Pindyck (1994), is specified by:

$$\begin{aligned} &\tfrac{1}{2}\sigma_X^2 X^2 \frac{\partial^2 F}{\partial X^2} + \tfrac{1}{2}\sigma_K^2 K^2 \frac{\partial^2 F}{\partial K^2} + \rho_{XK}\sigma_X\sigma_K X K \frac{\partial^2 F}{\partial X \partial K} \\ &+ (r-\delta_X) X \frac{\partial F}{\partial X} + (r-\delta_K) K \frac{\partial F}{\partial K} - rF = 0, \end{aligned} \tag{3}$$

where F denotes the investment option value, $\delta > 0$ the respective convenience yield and $r > \alpha$ the risk-free rate. A valuation function, representing the investment option, satisfying (3), Adkins & Paxson (2011), takes the form:

$$F = F(X,K) = AX^{\beta}K^{\gamma}, \tag{4}$$

where $A > 0$ is a non-negative coefficient. The generic parameters have the properties $\beta \ge 0$ and $\gamma \le 0$, since the attractiveness of the option to invest is enhanced as the cash-flow increases but as the investment cost decreases, such that the option value tends to zero as X tends to zero or as K tends to infinity, $\underset{\lim X \to 0}{F} = 0$ and $\underset{\lim K \to \infty}{F} = 0$. The parameters β and γ are related through the characteristic equation:

$$Q(\beta,\gamma) = \tfrac{1}{2}\sigma_X^2\beta(\beta-1) + \tfrac{1}{2}\sigma_K^2\gamma(\gamma-1) + \rho\sigma_X\sigma_K\beta\gamma + (r-\delta_X)\beta + (r-\delta_K)\gamma - r = 0. \tag{5}$$

The function Q is an ellipse. This is illustrated in Figure 1 for the base case values exhibited in Table 1. The set of possible solutions for β and γ lie on the arc AA'.

**** Figure 1 and Table 1 about here ****

A two-dimensional space defined by non-negative values, $X \ge 0$ and $K \ge 0$, is said to be separated into two mutually-exclusive exhaustive decision regions. One region named "hold" is defined such that for any X, K belonging to "hold", the optimal policy is to retain the unexercised option and to wait until sufficiently more favourable values of X and K are obtained. The opportunity value for the "hold" region is $F(X,K)$, (4). The second region named "invest" is defined such that for any X, K belonging to "invest", the optimal policy is to exercise the option, commit the investment cost and be in receipt of the net cash-flow stream. The project value for the "invest" region is $X/\delta_X - f/r - K$, the net present value. We can conceive the two decision regions as being separated by a discriminatory boundary. This boundary is defined by $G\left(X = \hat{X}, K = \hat{K}\right) = 0$, where $\hat{X}, \hat{K}$ denote the respective thresholds that signal the option to be exercised, with $G(X,K) < 0$ and $G(X,K) \ge 0$ for the "hold" and "invest" regions, respectively. The boundary can be represented by a set of infinite points $\left\{\hat{X},\hat{K}\right\}$, that denote a trade-off between the cash-flow and investment cost as any $\hat{K}$ increase can be compensated by a commensurate $\hat{X}$ increase.

Identifying the discriminatory boundary normally involves solving the equation:

$$0=\max_{X,K}\left\{\frac{X}{\delta_X}-\frac{f}{r}-K-F\left(X,K\right)\right\}. \tag{ 6}$$

The thresholds $\left\{\hat{X},\hat{K}\right\}$ satisfy (6):

$$\left\{\hat{X},\hat{K}\right\}=\arg\max_{X,K}\left\{\frac{X}{\delta_X}-\frac{f}{r}-K-F\left(X,K\right)\right\}.$$

The necessary first-order (smooth-pasting) conditions for a maximum are:

$$\frac{1}{\delta_X}-\left.\frac{\partial F\left(X,K\right)}{\partial X}\right|_{X=\hat{X},K=\hat{K}}=0\,, \tag{ 7}$$

$$-1-\left.\frac{\partial F\left(X,K\right)}{\partial K}\right|_{X=\hat{X},K=\hat{K}}=0\,. \tag{ 8}$$

The sufficient second-order conditions require, Sydsæter & Hammond (2006):

$$\begin{aligned}
&-\left.\frac{\partial^2 F\left(X,K\right)}{\partial X^2}\right|_{X=\hat{X},K=\hat{K}}\leq 0,\\
&-\left.\frac{\partial^2 F\left(X,K\right)}{\partial K^2}\right|_{X=\hat{X},K=\hat{K}}\leq 0,\\
&\left(\left.\frac{\partial^2 F\left(X,K\right)}{\partial X^2}\right|_{X=\hat{X},K=\hat{K}}\right)\left(\left.\frac{\partial^2 F\left(X,K\right)}{\partial K^2}\right|_{X=\hat{X},K=\hat{K}}\right)-\left(\left.\frac{\partial^2 F\left(X,K\right)}{\partial X\,\partial K}\right|_{X=\hat{X},K=\hat{K}}\right)^2\geq 0.
\end{aligned} \tag{ 9}$$

Along the boundary where $X=\hat{X},K=\hat{K}$, the realisable values for the "hold" and "invest" policies are identical, since sacrificing the option and simultaneously being compensated by owning the project neither creates nor destroys value. This equality is represented by the value matching relationship:

$$A\hat{X}^{\beta}\hat{K}^{\gamma}=\frac{\hat{X}}{\delta_X}-\frac{f}{r}-\hat{K}\,, \tag{ 10}$$

where $\hat{\beta},\hat{\gamma}$ are the respective β,γ values corresponding to $\hat{X},\hat{K}$. The set $\{\hat{X},\hat{K},\hat{\beta},\hat{\gamma}\}$ forming the discriminatory boundary is obtained numerically from (10), (7), (8) and $Q(\hat{\beta},\hat{\gamma})=0$, (5). The sufficiency conditions (9) are satisfied provided $\hat{\beta}\geq 1,\hat{\gamma}\leq 0$.

We obtain the option value for any point X_0,K_0 belonging to the "hold" region by:

1. Maximizing the option value with respect to $\hat{X}$ and $\hat{K}$ to identify the policy boundary and the implied relationships amongst the set $\{\hat{X},\hat{K},\hat{\beta},\hat{\gamma}\}$;
2. For X_0,K_0, minimizing the option value with respect to $\hat{X}$ (or $\hat{K},\hat{\beta},\hat{\gamma}$) to identify the relevant value of $\hat{X}_0$ (or $\hat{K}_0,\hat{\beta}_0,\hat{\gamma}_0$) that belongs to the set $\{\hat{X},\hat{K},\hat{\beta},\hat{\gamma}\}$ and is to be used in evaluating the option value as well as its derivatives.

2.2 Policy Boundary

Instead of deriving the policy boundary $\{\hat{X},\hat{K},\hat{\beta},\hat{\gamma}\}$ from the value matching relationship (10), Adkins & Paxson (2011), we adopt an alternative derivation based on the option value. Using (10) to eliminate A, the option value, $AX_0^{\beta_0}K_0^{\gamma_0}$ (4), for any point X_0,K_0 belonging to the "hold" region can be written as:

$$F_0 = F\left(X_0,K_0;\hat{X},\hat{K},\hat{\beta},\hat{\gamma}\right)=\frac{X_0^{\beta_0}K_0^{\gamma_0}}{\hat{X}^{\hat{\beta}}\hat{K}^{\hat{\gamma}}}\left(\frac{\hat{X}}{\delta_X}-\frac{f}{r}-\hat{K}\right). \qquad (11)$$

The thresholds $\hat{X},\hat{K}$ that define the policy boundary are selected as those that maximize F_0, since the greatest viable value rendered by the option has to be attained before being relinquished in exchange for the net present value of an active project. The first order conditions for a maximum can be expressed as[1]:

$$\hat{X}=\frac{f}{r}\delta_X\frac{\hat{\beta}}{\hat{\beta}+\hat{\gamma}-1}\geq 0, \qquad (12)$$

$$\hat{K}=-\frac{f}{r}\frac{\hat{\gamma}}{\hat{\beta}+\hat{\gamma}-1}\geq 0. \qquad (13)$$

[1] Full derivation is supplied in Appendix A, (A6) and (A7), including proof of sufficiency condition.

Since $\hat{X} \geq 0, \hat{K} \geq 0$ and

$$\hat{\beta} + \hat{\gamma} - 1 = \frac{f/r}{\hat{X}/\delta_X - f/r - \hat{K}} \geq 0,$$

from (12) and (13), then $\hat{\beta} \geq 1$ and $\hat{\gamma} \leq 0$. This corroborates our conjecture on their sign conditions and satisfies the sufficiency requirements. Moreover, (12) and (13) are identical to the solutions derived from the value matching relationship and smooth-pasting conditions[2].

The policy boundary is a relationship linking $\hat{X}$ and $\hat{K}$. It is obtained numerically by evaluating viable pairs of $\hat{X} \geq 0, \hat{K} \geq 0$ satisfying (12), (13) and $Q\left(\hat{\beta}, \hat{\gamma}\right) = 0$, (5). Starting with say a pre-specified $\hat{K}$ value, the corresponding $\hat{X}, \hat{\beta}, \hat{\gamma}$ values are evaluated, a process which is repeated for other $\hat{K}$ values until the boundary is formed[3]. Along the boundary the values $\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}$ vary, and the complete set of values $\left\{\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right\}$ constitutes the policy boundary. Figure 2 illustrates the boundary for Table 1 values. It shows the boundary CC' separating the X, K space into two distinct decision regions, "hold" and "invest". For any X_0, K_0 lying above the boundary, satisfying $\hat{X} = X_0$ and $K_0 > \hat{K}$, the optimal decision is "hold", otherwise "invest". The boundary slope is positive, so the trade-off is also positive indicating that any $\hat{K}$ increase has to be compensated by a commensurate $\hat{X}$ increase. In Figure 1, the set of possible solutions for $\hat{\beta}, \hat{\gamma}$ lie on the arc AA", so $\hat{\gamma}$ can range between 0 and $1 - \hat{\beta}$. When $\hat{\gamma} = 0$, K is being treated as absent from the formulation[4], which yields the one-factor solution of Dixit & Pindyck (1994); when $\hat{\gamma} = 1 - \hat{\beta}$, f is being treated as absent from the formulation, which yields the two-factor solution of McDonald & Siegel (1986).

**** Figure 2 about here ****

2.3 Option Value

For X_0, K_0 belonging to the "hold" region, its option value is determined from a to-be-selected point $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ belonging to the policy boundary:

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[2] This is shown in Appendix A, Appendix B offers a proof.

[3] A simple Excel procedure avoiding the solution of three simultaneous non-linear equations is provided in Appendix C.

[4] Alternatively, the investment cost is being treated as known instead of stochastic and its value is absorbed within the operating cost.

$$\hat{F}_0 = F\left(X_0, K_0; \hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0\right) = \frac{X_0^{\hat{\beta}_0} K_0^{\hat{\gamma}_0}}{\hat{X}_0^{\hat{\beta}_0} \hat{K}_0^{\hat{\gamma}_0}} \left(\frac{\hat{X}_0}{\delta_X} - \frac{f}{r} - \hat{K}_0\right), \quad (14)$$

where $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0 \in \left\{\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right\}$. A justifiable rule is required to identify the single point $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ on the boundary from the set of infinite points $\left\{\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right\}$. Støre et al. (2017) develop such a rule based on the envelope theorem that states:

$$\hat{X}_0 = \arg\min_{\hat{X}} F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right), \quad (15)$$

where $\hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ are the values on the boundary corresponding to $\hat{X}_0$, and $F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right)$ is the option reduced form function derived from $F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ by using (12), (13) and $Q\left(\hat{\beta}, \hat{\gamma}\right) = 0$ (5) to replace $\hat{K}, \hat{\beta}, \hat{\gamma}$. Also, $F_{S\hat{X}}\left(X_0, K_0; \hat{X}_0\right) = \hat{F}_0$. Intuitively, the minimization rule can be justified because any prospective option buyer is knowledgeable regarding the project properties, can ascertain the policy boundary, determine the option values for all possible boundary values, and select the cheapest because it is not known where the trajectory of stochastic variables will hit the boundary. We can illustrate its validity by considering $X_0 = 16.98593, K_0 = 75.0$ in Figure 2, which lies on the boundary so its option value equals the net present value for the project $X_0/\delta_X - f/r - K_0 = 249.6529$. The profile:

$$\hat{F} = F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right) = F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right)$$

versus $\hat{X}$ is reproduced in Figure 3; it is U-shaped and exhibits a minimum at $\hat{X}_0 = X_0$ with $\hat{F}_0 = 249.6529$.

**** Figure 3 about here ****

The first order condition for $F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right)$ yields $\hat{X}_0$, from which $\hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ can be evaluated. Since there is no closed form solution, Støre et al. (2017), it has to be generated numerically and because of this, it is not amenable to further analysis. We now develop an equivalent alternative, which yields the identical result but has the secondary merit of facilitating the derivatives of the option value to be obtained in an analytical form.

Since $\hat{X},\hat{K},\hat{\beta},\hat{\gamma}$ are intrinsically related through the boundary specification, then minimizing the option reduced form function with respect to any one of the $\hat{X},\hat{K},\hat{\beta},\hat{\gamma}$ produces the identical result:

$$\hat{F}_0 = \min_{\hat{X}} F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right) = \min_{\hat{K}} F_{S\hat{K}}\left(X_0, K_0; \hat{K}\right) = \min_{\hat{\beta}} F_{S\hat{\beta}}\left(X_0, K_0; \hat{\beta}\right) = \min_{\hat{\gamma}} F_{S\hat{\gamma}}\left(X_0, K_0; \hat{\gamma}\right),$$

where $F_{S\hat{K}}, F_{S\hat{\beta}}, F_{S\hat{\gamma}}$ are the respective reduced form functions derived in a similar way as $F_{S\hat{X}}$. This is illustrated in Figure 4 a-d for $X_0 = 15.0, K_0 = 75.0$ belonging to the "hold" region, where each figure exhibits the option value profile versus $\hat{X},\hat{K},\hat{\beta},\hat{\gamma}$. These reveal that by minimizing $F_{S\hat{X}}, F_{S\hat{K}}, F_{S\hat{\beta}}, F_{S\hat{\gamma}}$ individually with respect to $\hat{X},\hat{K},\hat{\beta},\hat{\gamma}$, the identical minimum option value $\hat{F}_0 = 201.8942$ results, with

$$\hat{X}_0 = 16.96064, \hat{K}_0 = 75.73068, \hat{\beta}_0 = 1.70777, \hat{\gamma}_0 = -0.30501.$$

**** Figure 4 a-d about here ****

It follows that (15) can be reformulated as a Lagrange minimization problem:

$$\begin{aligned}\left\{\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_1, \hat{\lambda}_2, \hat{\lambda}_3\right\} = \underset{\hat{X},\hat{K},\hat{\beta},\hat{\gamma}}{\arg\min} \Big[& F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right) \\ & + \lambda_1 \left(\hat{X} - \frac{f}{r} \delta \frac{\hat{\beta}}{\hat{\beta} + \hat{\gamma} - 1} \right) \\ & + \lambda_2 \left(\hat{K} + \frac{f}{r} \frac{\hat{\gamma}}{\hat{\beta} + \hat{\gamma} - 1} \right) + \lambda_3 Q\left(\hat{\beta}, \hat{\gamma}\right) \Big].\end{aligned} \tag{16}$$

where $\lambda_1, \lambda_2, \lambda_3$ denote the Lagrangian multipliers. Instead, it is more effective to perform the Lagrangian minimization with respect to $\hat{\beta},\hat{\gamma}$ after eliminating $\hat{X},\hat{K}$ from (16), since it both reduces the number of variables without increasing the complexity[5] and facilitates the analytical production of the derivatives. We can reformulate (16) as:

$$\left\{\hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_0\right\} = \underset{\hat{\beta},\hat{\gamma}}{\arg\min} \left[F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right) + \lambda_0 Q\left(\hat{\beta}, \hat{\gamma}\right) \right], \tag{17}$$

where λ_0 denotes the Lagrangian multiplier, $F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right)$ is the reduced form function of $F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ by using (12) and (13) to replace $\hat{X},\hat{K}$ and given by:

[5] The identical numerical result is obtainable by minimizing with respect to $\hat{X},\hat{K}$ after eliminating $\hat{\beta},\hat{\gamma}$, but the derivation is less straightforward.

$$F_R\left(X_0,K_0;\hat{\beta},\hat{\gamma}\right)=X_0^{\hat{\beta}}K_0^{\hat{\gamma}}\hat{\beta}^{-\hat{\beta}}\left(-\hat{\gamma}\right)^{-\hat{\gamma}}\left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right)^{\hat{\beta}+\hat{\gamma}-1}\delta_X^{-\hat{\beta}}\geq 0\,, \qquad (18)$$

and $\hat{F}_R=F_R\left(X_0,K_0;\hat{\beta}_0,\hat{\gamma}_0\right)$. The first order conditions for (17) with respect to $\hat{\beta},\hat{\gamma}$ are, respectively (see Appendix D for details):

$$\left[\ln\left(\frac{X_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{\hat{\beta}_0\,\delta_X\,f/r}\right)\right]\hat{F}_R+\hat{\lambda}_0\left[\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\tfrac{1}{2}\sigma_X^2\right]=0, \qquad (19)$$

$$\left[\ln\left(\frac{K_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{-\hat{\gamma}_0\,f/r}\right)\right]\hat{F}_R+\hat{\lambda}_0\left[\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\tfrac{1}{2}\sigma_K^2\right]=0, \qquad (20)$$

The numerical solutions for $\hat{\beta}_0,\hat{\gamma}_0,\hat{\lambda}_0$ are evaluated from (19), (20) and $Q\left(\hat{\beta}_0,\hat{\gamma}_0\right)=0$, (5). It is possible to combine (19) and (20) to eliminate $\hat{\lambda}_0$ to yield:

$$\frac{\ln\left(\dfrac{X_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{\hat{\beta}_0\,\delta_X\,f/r}\right)}{\ln\left(\dfrac{K_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{-\hat{\gamma}_0\,f/r}\right)}=\frac{\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\frac{1}{2}\sigma_X^2}{\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\frac{1}{2}\sigma_K^2}. \qquad (21)$$

with the solutions for $\hat{\beta}_0,\hat{\gamma}_0$ obtainable from (21) and $Q\left(\hat{\beta}_0,\hat{\gamma}_0\right)=0$, (5). Since (21) can be expressed as:

$$-\frac{\partial\hat{\gamma}_0}{\partial\hat{\beta}_0}=\frac{\dfrac{\partial F_R\left(X_0,K_0;\hat{\beta}_0,\hat{\gamma}_0\right)}{\partial\hat{\beta}_0}}{\dfrac{\partial F_R\left(X_0,K_0;\hat{\beta}_0,\hat{\gamma}_0\right)}{\partial\hat{\gamma}_0}}=\frac{\dfrac{\partial Q\left(\hat{\beta}_0,\hat{\gamma}_0\right)}{\partial\hat{\beta}_0}}{\dfrac{\partial Q\left(\hat{\beta}_0,\hat{\gamma}_0\right)}{\partial\hat{\gamma}_0}}=-\frac{\partial\hat{\gamma}_0}{\partial\hat{\beta}_0}\,, \qquad (22)$$

then for any X_0,K_0 belonging to the "hold" region, the optimal solution $\hat{\beta}_0,\hat{\gamma}_0$ occurs when the change in $\hat{\gamma}_0$ due to $\hat{\beta}_0$ as measured by the slope along $\hat{F}_R$ equals the corresponding change as measured by the slope along $Q\left(\hat{\beta}_0,\hat{\gamma}_0\right)$.

The investment option value for any X_0,K_0 belonging to the "hold" region is given by $\hat{F}_0=F_R\left(X_0,K_0;\hat{\beta}_0,\hat{\gamma}_0\right)$. A numerical rule for discriminating whether or not X_0,K_0 belongs to the “hold” region is obtained as follows. From (A8) and (A9), respectively, the values β_0, γ_0 corresponding to X_0, K_0 are:

$$\beta_0 = \frac{X_0/\delta_X}{X_0/\delta_X - f/r - K_0}, \gamma_0 = -\frac{K_0}{X_0/\delta_X - f/r - K_0}.$$

We introduce the indicator IND, where $IND = 1$ means the prevailing optimal decision is “invest” and $IND = -1$ means “hold”. If [6]:

$$\beta_0 \geq 1 \ \&\& \ \gamma_0 \leq 0 \ \&\& \ \beta_0 + \gamma_0 \geq 1 \ \&\& \ Q(\beta_0, \gamma_0) \leq 0$$

is $TRUE$, then $IND = 1$, otherwise $IND = -1$. The value of the project at X_0, K_0 is given by:

$$\begin{cases} = \hat{F}_0 & \text{if } IND = -1, \\ = X_0/\delta_X - f/r - K_0 & \text{if } IND = +1. \end{cases} \qquad (23)$$

Based on Table 1 values, the value function (23) is illustrated graphically in Figure 5 and numerically in Table 2, over domains covering both the “hold” and “invest” regions. As expected, both figure and table reveal that the project value increases for increases in X_0 but decreases in K_0. Further, both $\hat{X}_0$ and $\hat{K}_0$ decrease if either X_0 increases or K_0 decreases, so a favourable movement in the prevailing X, K values leads to a favourable change in the thresholds used in determining the option value.

**** Figure 5 and Table 2 about here ****

Table 2 values indicate $\hat{\lambda}_0$ to be non-negative, since the Q function constrains $\hat{\beta}_0, \hat{\gamma}_0$ to lie on the ellipse. If $\hat{\lambda}_0 = 0$, then the resulting $\hat{\beta}_0, \hat{\gamma}_0$ values imply that $\hat{X}_0 = X_0$, $\hat{K}_0 = K_0$, so $X_0, K_0 \in \{\hat{X}, \hat{K}\}$ belongs to the policy boundary. This can be shown by expressing (19) and (20), respectively, in terms of $\hat{X}_0$, (12) and $\hat{K}_0$, (13):

$$\hat{F}_R \ln\left(X_0/\hat{X}_0\right) + \hat{\lambda}_0\left[\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2\right] = 0, \qquad (24)$$

$$\hat{F}_R \ln\left(K_0/\hat{K}_0\right) + \hat{\lambda}_0\left[\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2\right] = 0, \qquad (25)$$

since for $\hat{\lambda}_0 = 0$, then $\ln\left(X_0/\hat{X}_0\right) = 0$ and $\ln\left(K_0/\hat{K}_0\right) = 0$. For $\hat{\lambda}_0 > 0$, $\hat{X}_0 \neq X_0$ and $\hat{K}_0 \neq K_0$. While we can surmise that $\hat{X}_0 > X_0$ because intuitively an unexercised option is most likely due to X_0 not being sufficiently high enough to trigger an exercise, no similar

[6] $\&\&$ is the truth functional operator of logical conjunction.

assumption can be made regarding the value of $\hat{K}_0$ relative to K_0. If $r-\delta_X-\frac{1}{2}\sigma_X^2$ in (24) can be treated as small, then $\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K$ is most likely to be positive since $\hat{\beta}_0>0, \hat{\gamma}_0<0$, so $\hat{X}_0>X_0$. In contrast, $\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K$ in (25) may adopt either sign suggesting that $\hat{K}_0$ may be greater or less than K_0. This effect is revealed in Table 2, where $Test_1$ and $Test_2$ are specified by:

$$Test_1=\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\tfrac{1}{2}\sigma_X^2, \qquad (26)$$

$$Test_2=\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\tfrac{1}{2}\sigma_K^2. \qquad (27)$$

For the illustrated values, $Test_1>0$ and $\hat{X}_0\geq X_0$, which suggests:

$$\ln\left(X_0/\hat{X}_0\right)\div\left[\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\tfrac{1}{2}\sigma_X^2\right]\leq 0, \qquad (28)$$

because $F_R\geq 0, \hat{\lambda}_0\geq 0$. In contrast, $Test_2$ can be positive or negative, but Table 2 reveals that if $Test_2>0$ then $\hat{K}_0\geq K_0$ and if $Test_2<0$ then $\hat{K}_0\leq K_0$, which suggests:

$$\ln\left(K_0/\hat{K}_0\right)\div\left[\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\tfrac{1}{2}\sigma_K^2\right]\leq 0. \qquad (29)$$

2.4 Option value derivatives

In this section, we investigate the sensitivity of the investment option value to changes in the various parameters of the Q function, (5). The derivatives, obtained from the chain rule and conditions (28) and (29), are examined for the extent that the findings of the 1-factor model, Dixit & Pindyck (1994), and the 2-factor homogeneity degree-1 model, McDonald & Siegel (1986), translate to the general 2-factor model. The derivations for the general model are presented in Appendix E; the key findings are presented in Table 3. We find that the disparity is greatest for the two volatility parameters, while that for the remaining parameters is either absent or small.

**** Table 3 about here ****

2.4.1 Cash-flow volatility

Using the chain rule, the sensitivity of the option value to changes in the cash-flow volatility σ_X , $\partial\hat{F}_0/\partial\sigma_X$ is given by (E6):

$$\begin{aligned}\frac{\partial\hat{F}_0}{\partial\sigma_X} &= -\hat{F}_R \ln\left(X_0/\hat{X}_0\right)\frac{\sigma_X\hat{\beta}_0\left(\hat{\beta}_0-1\right)+\hat{\beta}_0\hat{\gamma}_0\rho\sigma_K}{\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\frac{1}{2}\sigma_X^2} \\ &\quad -\hat{F}_R \ln\left(K_0/\hat{K}_0\right)\frac{\sigma_X\hat{\beta}_0\left(\hat{\beta}_0-1\right)+\hat{\beta}_0\hat{\gamma}_0\rho\sigma_K}{\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\frac{1}{2}\sigma_K^2}.\end{aligned} \tag{30}$$

From (30), the sign of $\partial\hat{F}_0/\partial\sigma_X$ depends on the sign $\sigma_X\hat{\beta}_0\left(\hat{\beta}_0-1\right)+\hat{\beta}_0\hat{\gamma}_0\rho\sigma_K$ because of (28) and (29). This result contrasts with the standard finding of the 1-factor model where volatility increases lead to increases in the option value, but is similar to that for the 2-factor homogeneity degree-1 model, see Table 3. The option value $\hat{F}_0$ is an increasing function of σ_X if:

$$\frac{\hat{\beta}_0-1}{-\hat{\gamma}_0} > \frac{\rho\sigma_X\sigma_K}{\sigma_X^2}. \tag{31}$$

Condition (31) always holds provided $\rho \le 0$. For $\rho = 0$, the two assets are uncorrelated and the circumstances influencing their respective evolutions are unrelated, which may arise when the cash-flow is subject to purely economic uncertainties while the investment cost is subject to unrelated technological uncertainties. For $\rho < 0$, they are negatively correlated, which may arise if an oil price rise is associated with an investment cost fall due to economies of scale. Realistically, however, it is more plausible to treat ρ as positive, since asset prices in an economy tend to move together where (un)favourable economic prospects are likely to be accompanied with both higher (lower) cash-flows and higher (lower) investment costs. Even if $\rho > 0$, breaking (31) requires that the investment cost volatility dominates that for the cash-flow, $\sigma_K \gg \sigma_X$. This arises for those capital intensive projects like nuclear powered electricity generation, which are subject to considerable technological uncertainty that is significantly greater than the cash-flow uncertainty, particularly if the output price is underpinned by a fixed tariff policy.

It is interesting to compare (31) with the similar condition for the 2-factor homogeneity degree-1 model derived from (F7), which can be expressed as $\rho\sigma_X\sigma_K/\sigma_X^2 < 1$. Because $\hat{\beta}_0 + \hat{\gamma}_0 > 1$, then (31) is the less restrictive of the two conditions. For $\rho > 0$, the relationship between the option value for some X_0, K_0 belonging to the "hold" region and the cash-flow volatility is U-shaped and exhibits a minimum at

$$\sigma_X = \frac{-\hat{\gamma}_0}{\hat{\beta}_0 - 1}\rho\sigma_K < \rho\sigma_K$$

for the general 2-factor model, and at $\sigma_X = \rho\sigma_K$ for the 2-factor homogeneity degree-1 model. If the option value exhibits a minimum, then this occurs at a lower σ_X value for the general 2-factor model than for the 2-factor homogeneity degree-1 model. Because of this, the 2-factor homogeneity degree-1 model is more likely than the general 2-factor model with identical parametric values to produce negative vegas in contrast to the 1-factor model.

2.4.2 Investment cost volatility

We obtain similar findings for vega with reference to the investment cost volatility σ_K. The sensitivity $\partial\hat{F}_0/\partial\sigma_K$ is given by (E10):

$$\begin{aligned}\frac{\partial\hat{F}_0}{\partial\sigma_K} = &-\ln\left(\frac{X_0}{\hat{X}_0}\right)\frac{\sigma_K\hat{\gamma}_0(\hat{\gamma}_0-1)+\rho\sigma_X\hat{\beta}_0\hat{\gamma}_0}{\hat{\beta}_0\sigma_X^2+\hat{\gamma}_0\rho\sigma_X\sigma_K+r-\delta_X-\tfrac{1}{2}\sigma_X^2} \\ &-\ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R\frac{\sigma_K\hat{\gamma}_0(\hat{\gamma}_0-1)+\rho\sigma_X\hat{\beta}_0\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2+\hat{\beta}_0\rho\sigma_X\sigma_K+r-\delta_K-\tfrac{1}{2}\sigma_K^2}.\end{aligned} \qquad (32)$$

In (32), the sign of $\partial\hat{F}_0/\partial\sigma_K$ depends on the sign $\sigma_K\hat{\gamma}_0(\hat{\gamma}_0-1)+\hat{\beta}_0\hat{\gamma}_0\rho\sigma_X$. The option value $\hat{F}_0$ to be an increasing function of σ_K if:

$$\frac{1-\hat{\gamma}_0}{\hat{\beta}_0} > \frac{\rho\sigma_X\sigma_K}{\sigma_K^2}, \qquad (33)$$

which is always true provided $\rho \le 0$. However, (33) may not hold for $\rho > 0$ particularly if $\sigma_K \ll \sigma_X$, as illustrated by a project having a high cash-flow volatility due to market uncertainty but a low investment cost volatility due to technological certainty.

2.4.3 Cash-flow investment cost correlation

The impact of changes in the correlation coefficient on the option value is given by, see Appendix E (E14):

$$\frac{\partial \hat{F}_0}{\partial \rho} = -\hat{F}_R \sigma_X \sigma_K \hat{\beta}_0 \hat{\gamma}_0 \times \left\{ \frac{\ln\left(X_0/\hat{X}_0\right)}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} + \frac{\ln\left(K_0/\hat{K}_0\right)}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2} \right\}. \quad (34)$$

In (34), $\partial \hat{F}_0 / \partial \rho < 0$ since $\hat{\gamma}_0 < 0$. A fall in the correlation coefficient causes a rise in the option value and makes the investment opportunity more attractive. This is partly due to any movement towards to an increasingly negative correlation coefficient between the cash-flow and investment cost being interpreted as beneficial, since a random event producing an increased cash-flow level becomes more likely to lead to a fall in the investment cost. But as we have shown above, it is also because a negative correlation enables *vega* to be positive. The finding $\partial \hat{F}_0 / \partial \rho < 0$ matches the corresponding result for the homogenous degree-1 model, (F8).

2.4.4 Investment cost drift

The effect of an investment cost drift δ_K change on the option value is:

$$\frac{\partial \hat{F}_0}{\partial \delta_K} = \hat{F}_R \hat{\gamma}_0 \times \left\{ \frac{\ln\left(X_0/\hat{X}_0\right)}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} + \frac{\ln\left(K_0/\hat{K}_0\right)}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2} \right\} > 0. \quad (35)$$

This finding matches the corresponding result for the homogenous degree-1 model, (F10).

2.4.5 Cash-flow drift

Since the cash-flow drift δ_X occurs in both the $\hat{F}_R$ and Q functions, the effect of changes on the option value is:

$$\frac{\partial \hat{F}_0}{\partial \delta_X} = \frac{\hat{\beta}_0}{\delta_X}\hat{F}_R + \hat{F}_R\hat{\beta}_0$$
$$\times\left\{\frac{\ln\left(X_0/\hat{X}_0\right)}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} + \frac{\ln\left(K_0/\hat{K}_0\right)}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}\right\} \quad (36)$$
$$< 0.$$

This finding matches the corresponding result for the homogenous degree-1 model, (F11).

2.4.6 Risk-free rate

Since the risk-free rate r occurs in both the $\hat{F}_R$ and Q functions, the effect of changes on the option value is:

$$\frac{\partial \hat{F}_0}{\partial r} = \frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{r}\hat{F}_R - \hat{F}_R\left(\hat{\beta}_0 + \hat{\gamma}_0 - 1\right)$$
$$\times\left\{\frac{\ln\left(X_0/\hat{X}_0\right)}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} + \frac{\ln\left(K_0/\hat{K}_0\right)}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}\right\} \quad (37)$$
$$\geq 0,$$

since $\hat{\beta}_0 + \hat{\gamma}_0 \geq 1$. The equality condition governing (37) holds only if $\hat{\beta}_0 + \hat{\gamma}_0 = 1$, in which case the general two-factor simplifies to the homogenous degree-1 model and $\partial \hat{F}_0 / \partial r = 0$.

3 Divestment

We consider an operational asset having an embedded divestment option, but in the absence of competition and other forms of optionality. The project value is determined not only by a stochastic periodic cash-flow, $Y \geq 0$, and a known periodic fixed operating cost, $f > 0$, as before, but also a divestment option that confers the right to sacrifice the operational project for a one-off stochastic value, denoted by $Z \geq 0$. Since the analysis and findings are similar to those presented in section 2, only the results in brief are reproduced.

3.1 The Model

Similar to cash-flow process defined by (1), the value acquired by abandoning the project is described by a geometric Brownian motion process:

$$\mathrm{d}Y = \alpha_Y Y \,\mathrm{d}t + \sigma_Y Y \,\mathrm{d}W_Y\,, \tag{38}$$

$$\mathrm{d}Z = \alpha_Z Z \,\mathrm{d}t + \sigma_Z Z \,\mathrm{d}W_Z\,, \tag{39}$$

where α denotes the respective known drift rates, σ the volatilities, and $\mathrm{d}W$ an increment of the standard Wiener process. The covariance between Y, Z is $\mathrm{Cov}\left[\mathrm{d}Y, \mathrm{d}Z\right] = \rho_{YZ}\sigma_Y\sigma_Z \mathrm{d}t$ where ρ_{YZ} denotes the correlation coefficient with $-1 \le \rho_{XY} \le 1$. The risk neutral valuation relationship is:

$$\begin{aligned} &\tfrac{1}{2}\sigma_Y^2 Y^2 \frac{\partial^2 H}{\partial Y^2} + \tfrac{1}{2}\sigma_Z^2 Z^2 \frac{\partial^2 H}{\partial Z^2} + \rho_{YZ}\sigma_Y\sigma_Z YZ \frac{\partial^2 H}{\partial Y \partial Z} \\ &+ \left(r - \delta_Y\right) Y \frac{\partial H}{\partial Y} + \left(r - \delta_Z\right) Z \frac{\partial H}{\partial Z} - rH = 0, \end{aligned} \tag{40}$$

where H denotes the divestment option value and $\delta > 0$ the respective convenience yields. A valuation function satisfying (40) adopts the form:

$$H = H\left(Y, Z\right) = BY^{\phi} Z^{\theta} \tag{41}$$

where $A > 0$. The parameters have the properties $\phi \le 0$ and $\theta \ge 0$, since the attractiveness of the divestment option increases as the cash-flow decreases and as the divestment value increases, such that $\underset{\lim Y \to \infty}{H} = 0$ and $\underset{\lim Z \to 0}{H} = 0$. ϕ and θ are related through:

$$Q_{YZ}\left(\phi, \theta\right) = \tfrac{1}{2}\sigma_Y^2 \phi\left(\phi - 1\right) + \tfrac{1}{2}\sigma_Z^2 \theta\left(\theta - 1\right) + \rho\sigma_Y\sigma_Z\phi\theta + \left(r - \delta_Y\right)\phi + \left(r - \delta_Z\right)\theta - r = 0\,. \tag{42}$$

A two-dimensional region defined by $Y \ge 0, Z \ge 0$ is divided into two mutually-exclusive exhaustive regions, named “retain” and “divest”. For any $Y = Y_0, Z = Z_0$ belonging to the former, the best policy is to retain the operational asset, with value $Y/\delta_Y - f/r$, together with its divestment option value $BY^{\phi} Z^{\theta}$, and to divest the asset and obtain the divestment value Z if otherwise. Along the boundary separating “retain” and “divest”, since the sacrificing the asset and its option in exchange for the divestment value neither creates nor destroys value at the optimal thresholds, denoted by $\hat{Y}$ and $\hat{Z}$, respectively, the value matching relationship is:

$$\frac{\hat{Y}}{\delta_Y} - \frac{f}{r} + B\hat{Y}^{\phi} \hat{Z}^{\theta} = \hat{Z}\,. \tag{43}$$

By using (43) to eliminate B from

4 Testing the Solution Precision

The precision of the quasi-analytical solution is assessed in three ways: (i) from evaluating the value matching relationship along the policy boundary, (ii) evaluating the two smooth pasting conditions, and (iii) evaluating the option value from the valuation relationship.

4.1 Value Matching Relationship

The value matching relationship requires that for any point on the policy boundary, $\{X_0, K_0\} \in \partial\Omega_P$, the value of the investment option $F(X_0, K_0)$ is equal to the immediately post exercise net value $X_0/\delta_X - f/r - K_0$ for the project. We show this by first considering any $\{X_0, K_0\}$ as if it belongs to the "hold" region. Since $\{X_0, K_0\} \in \Omega$, then both (24) and (25) apply. By setting $\hat{\lambda}_0 = 0$, a measure of the distance between $\{X_0, K_0\}$ and the policy boundary, it follows that $\hat{X}_0 = X_0$ and $\hat{K}_0 = K_0$. Since $\{\hat{X}_0, \hat{K}_0\} \in \partial\Omega_P$, by definition, it follows that $\{X_0, K_0\} \in \partial\Omega_P$. Along the policy boundary, the value matching relationship holds so the option value $F(X_0, K_0)$ and the net value $X_0/\delta_X - f/r - K_0$ are equal.

4.2 Smooth Pasting Conditions

The two smooth pasting conditions require that for any $\{X_0, K_0\} \in \partial\Omega_P$, the two first order conditions with respect to X and K have to hold:

$$\left.\frac{\partial F(X,K)}{\partial X}\right|_{X=X_0, K=K_0} = \frac{1}{\delta_X}; \left.\frac{\partial F(X,K)}{\partial K}\right|_{X=X_0, K=K_0} = -1.$$

The extent the solution satisfies these two conditions is assessed by comparing the option gradient as measured by the finite difference approximation with the predicted value.

For any positive cash flow, $X_0 > 0$, we identify the corresponding K_o for $\{X_0, K_0\} \in \partial\Omega_P$ by solving the quadratic equation (5):

$$Q\left(\frac{X_0/\delta_X}{X_0/\delta_X - f/r - K_0}, -\frac{K_0}{X_0/\delta_X - f/r - K_0}\right) = 0 \qquad (44)$$

and selecting the positive root such that $K_0 < X_0/\delta_X - f/r$. For some small increment ΔX_0, the finite difference representation is applied to obtain an approximation for the option gradient with respect to cash flow:

$$\frac{F\left(X_0,K_0\right)-F\left(X_0-\Delta X_0,K_0\right)}{\Delta X_0}\approx\left.\frac{\partial F\left(X,K\right)}{\partial X}\right|_{X=X_0,K=K_0}=\frac{1}{\delta_X}$$

where the option value $F(\bullet)$ is determined from (14) using the method explained in §2.3. The percentage error in satisfying the cash flow smooth pasting condition, err_{X_0}, is obtained from:

$$err_{X_0}=\frac{F\left(X_0,K_0\right)-F\left(X_0-\Delta X_0,K_0\right)}{\Delta X_0}\times\frac{1}{\delta_X}-1. \tag{45}$$

The percentage error in satisfying the investment cost smooth pasting condition is obtained in a similar way. For any positive investment cost, $K_0>0$, we identify the corresponding X_o for $\{X_0,K_0\}\in\partial\Omega_P$ by solving (44) and selecting the positive root such that $X_0/\delta_X>f/r+K_0$. For some small increment ΔK_0 the pertaining finite difference approximation is given by:

$$-\frac{F\left(X_0,K_0\right)-F\left(X_0,K_0+\Delta K_0\right)}{\Delta K_0}\approx\left.\frac{\partial F\left(X,K\right)}{\partial K}\right|_{X=X_0,K=K_0}=-1,$$

and the percentage error in satisfying the smooth pasting condition, err_{K_0}, by:

$$err_{K_0}=-\frac{F\left(X_0,K_0\right)-F\left(X_0,K_0+\Delta K_0\right)}{\Delta K_0}-1. \tag{46}$$

The absolute percentage errors relating to the two smooth pasting conditions, cash flow and investment cost, are exhibited respectively in Figure 6 (a,b). The error profile for the cash flow condition is based on $\Delta X_0=0.01$ and for the investment cost condition on $\Delta K_0=0.04$. The two profiles reveal that the error magnitudes in each case are very small so we can conclude that the analytical solution to identifying the discriminatory boundary is determined with a satisfactorily high level of precision.

**** Figure 6 (a, b) about here ****

5 Conclusion

Our contribution of our paper lies in developing and extending the quasi-analytical for the general two-factor real-option model and determining the properties of its discriminatory boundary and option value for all feasible factor values. Although it builds on the works by

Adkins & Paxson (2011), Heydari et al. (2012), Støre et al. (2017), the solution method it adopts is distinctive and insightful. Particularly, it turns the focus of attention to the option value, which is a more fundamental element than the discriminatory boundary. By performing the process of maximization on the option value after eliminating the coefficient A instead of on the value matching relationship, we obtain a solution to the option value, which is not only identical in magnitude to that proposed by Støre et al. (2017), but also potentially more straightforward, with the secondary merit that the derivatives of the option value with respect to each of the various Q function parameters can be obtained analytically. This demonstrates that in contrast to the standard result for a one-factor model that an increase in the underlying volatility is associated with an option value increase, an increase in volatility may or may not produce an option value increase for the two-factor model, but only if the correlation between the two factors is positive. Even in a world with multiple sources of uncertainty, asset prices tend to rise and fall in tandem, to a greater or lesser extent, so we can conclude that a strict positive relationship between volatility and option value is uniquely specific to only one-factor models. Although two-factor general models and homogeneity degree-1 models share the simultaneous property of a potentially positive and negative volatility option value relationship, their turning points identifying the change from negative (positive) to positive (negative) slopes are distinctly determined and subtle.

The quasi-analytical method described here for determining the option value is designed for general two-factor formulations that obey the maximization requirement. It is essential that the power function proposed to solve the risk-neutral valuation relationship has characteristics that strictly obey the sufficiency condition for a maximum. In our case, that required the power parameter associated with cash flow to exceed one and that associated with investment cost to be negative. However, this sufficiency maximization requirement may not be met for representations of all other phenomena, in which case the solution method described here is inappropriate. The conventional approach to solving general two-factor models is to apply numerical techniques such as finite-differences. We demonstrate that the quasi-analytical method is favoured by having a high degree of precision when the smooth pasting conditions are evaluated using finite differences. This means that the method has the merits of being both less computationally onerous and sufficiently accurate. Finally, our method focuses on the option value. It is not absolutely necessary to determine the discriminatory boundary before evaluating the option value for any feasible point in the “hold” region; the option value is obtainable without knowing the discriminatory boundary. If required, the boundary can be formed by evaluating the option value for every point in the “hold” region.

Appendix A: Sufficient conditions for a maximum

From (11), the thresholds are selected to maximize the option value:

$$F = F\left(X_0, K_0; \hat{X}, \hat{K}\right) = \left(\frac{X_0}{\hat{X}}\right)^{\beta} \left(\frac{K_0}{\hat{K}}\right)^{\gamma} \left(\frac{\hat{X}}{\delta} - \frac{f}{r} - \hat{K}\right). \tag{A1}$$

At the optimal thresholds where $X = \hat{X}, K = \hat{K}$, the corresponding values of β, γ are $\beta = \hat{\beta}, \gamma = \hat{\gamma}$.

Differentiating F (A1) with respect to $\hat{X}$ yields:

$$\frac{\partial F}{\partial \hat{X}} = \left(\frac{X_0}{\hat{X}}\right)^{\beta} \left(\frac{K_0}{\hat{K}}\right)^{\gamma} \frac{1}{\hat{X}} \left(\frac{\hat{X}}{\delta} - \frac{\beta \hat{X}}{\delta} + \frac{f}{r} + \hat{K}\right). \tag{A2}$$

From (A2), the first order condition for a maximum requires:

$$\hat{X} = \frac{\hat{\beta}}{\hat{\beta} - 1} \delta_X \left(\frac{f}{r} + \hat{K}\right), \tag{A3}$$

Which demonstrates $\hat{\beta} > 1$ since $\hat{X} \geq 0$ and $\hat{\beta} \geq 0$. If $\hat{K} = 0$, then (A3) simplifies to the standard one-factor result, Dixit & Pindyck (1994). Differentiating F (A1) with respect to $\hat{K}$ yields:

$$\frac{\partial F}{\partial \hat{K}} = \left(\frac{X_0}{\hat{X}}\right)^{\beta} \left(\frac{K_0}{\hat{K}}\right)^{\gamma} \frac{1}{\hat{K}} \left(-\frac{\gamma \hat{X}}{\delta} + \frac{\gamma f}{r} + \gamma \hat{K} - \hat{K}\right). \tag{A4}$$

From (A4), the first order condition for a maximum requires:

$$\hat{K} = \frac{\hat{\gamma}}{\hat{\gamma} - 1} \left(\frac{\hat{X}}{\delta_X} - \frac{f}{r}\right), \tag{A5}$$

for $\hat{X}/\delta > f/r$. If $f = 0$, then (A3) and (A5) entail $\hat{\beta} + \hat{\gamma} = 1$ as well as $\hat{\gamma} < 0$, the result of McDonald & Siegel (1986). For $f > 0$, the respective thresholds from (A3) and (A5) are given by:

$$\hat{X} = \frac{f}{r} \delta_X \frac{\hat{\beta}}{\hat{\beta} + \hat{\gamma} - 1} > 0, \tag{A6}$$

provided $\hat{\beta} > 0$ and $\hat{\beta} + \hat{\gamma} - 1 > 0$, and:

$$\hat{K} = -\frac{f}{r} \frac{\hat{\gamma}}{\hat{\beta} + \hat{\gamma} - 1} > 0, \tag{A7}$$

provided $\gamma < 0$ and $\beta + \gamma - 1 > 0$. From (A6) and (A7):

$$\hat{\beta} = \frac{\hat{X}/\delta_X}{\hat{X}/\delta_X - f/r - \hat{K}}, \tag{A8}$$

$$\hat{\gamma} = -\frac{\hat{K}}{\hat{X}/\delta_X - f/r - \hat{K}}. \tag{A9}$$

The second order conditions validate that the obtained thresholds represent maximum values. In the following, the first line specifies the relevant second derivative, the second line expresses it at the obtained thresholds, and the third line indicates the sign. From (A2):

$$\begin{aligned}\frac{\partial^2 F}{\partial \hat{X}^2} &= \left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{X}^2}\left(-\frac{\beta\hat{X}}{\delta}+\frac{\beta^2\hat{X}}{\delta}-\frac{\beta f}{r}-\frac{\beta^2 f}{r}-\beta\hat{K}-\beta^2\hat{K}\right)\\ &= -\left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{X}^2}\frac{f}{r}\frac{\beta(\beta-1)}{\beta+\gamma-1}\\ &< 0,\end{aligned} \tag{A10}$$

and:

$$\begin{aligned}\frac{\partial^2 F}{\partial \hat{X}\partial \hat{K}} &= \left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{X}\hat{K}}\left(-\frac{\gamma\hat{X}}{\delta}+\frac{\beta\gamma\hat{X}}{\delta}-\frac{\beta\gamma f}{r}+\gamma\hat{K}-\beta\gamma\hat{K}\right)\\ &= -\left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{X}\hat{K}}\frac{f}{r}\frac{\beta\gamma}{\beta+\gamma-1}\\ &> 0.\end{aligned} \tag{A11}$$

From (A4):

$$\begin{aligned}\frac{\partial^2 F}{\partial \hat{K}^2} &= \left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{K}^2}\left(\frac{\gamma\hat{X}}{\delta}+\frac{\gamma^2\hat{X}}{\delta}-\frac{\gamma f}{r}-\frac{\gamma^2 f}{r}+\gamma\hat{K}-\gamma^2\hat{K}\right)\\ &= -\left(\frac{X_0}{\hat{X}}\right)^{\beta}\left(\frac{K_0}{\hat{K}}\right)^{\gamma}\frac{1}{\hat{K}^2}\frac{f}{r}\frac{\gamma(\gamma-1)}{\beta+\gamma-1}\\ &< 0,\end{aligned} \tag{A12}$$

and:

$$\frac{\partial^2 F}{\partial \hat{X}\partial \hat{K}} = \frac{\partial^2 F}{\partial \hat{K}\partial \hat{X}}. \tag{A13}$$

The sufficient condition for a maximum requires, Sydsæter & Hammond (2006):

$$\frac{\partial^2 F}{\partial \hat{X}^2} \le 0, \frac{\partial^2 F}{\partial \hat{K}^2} \le 0, \frac{\partial^2 F}{\partial \hat{X}^2}\frac{\partial^2 F}{\partial \hat{K}^2} - \frac{\partial^2 F}{\partial \hat{X}\partial \hat{K}}\frac{\partial^2 F}{\partial \hat{K}\partial \hat{X}} \ge 0. \tag{A14}$$

Since:

$$\frac{\partial^2 F}{\partial \hat{X}^2}\frac{\partial^2 F}{\partial \hat{K}^2}-\frac{\partial^2 F}{\partial \hat{X}\partial \hat{K}}\frac{\partial^2 F}{\partial \hat{K}\partial \hat{X}}$$

$$=\left(\frac{X_0}{\hat{X}}\right)^{2\beta}\left(\frac{K_0}{\hat{K}}\right)^{2\gamma}\frac{1}{\hat{X}^2\hat{K}^2}\left(\frac{f}{r}\right)^2\frac{\{\beta(\beta-1)\gamma(\gamma-1)-\beta^2\gamma^2\}}{(\beta+\gamma-1)^2}$$

$$=-\left(\frac{X_0}{\hat{X}}\right)^{2\beta}\left(\frac{K_0}{\hat{K}}\right)^{2\gamma}\frac{1}{\hat{X}^2\hat{K}^2}\left(\frac{f}{r}\right)^2\frac{\beta\gamma}{\beta+\gamma-1} \tag{A15}$$

$$\geq 0,$$

all the sufficient conditions for a maximum are satisfied.

Appendix B: Equivalence between two methods

Consider a model having m variables $z_i, i=1,\ldots,m$. Denote the net present of an exercised project, defined as the present value of the net cash-flow stream less the investment cost, by $U(z_i)$ and the option to invest by $V(z_i)$. The optimal investment threshold is denoted by $\hat{z}_i, i=1,\ldots,m$. Then, by the value-matching smooth-pasting method, we set the thresholds according to the first order conditions for maximizing the value gain in exercising the option $U(z_i)-V(z_i)$ in order to derive the smooth pasting condition:

$$U'(\hat{z}_i)-V'(\hat{z}_i)=0 \tag{B1}$$

Next, the value gain in exercising the option is set equal to zero at the threshold in order to derive the value-matching relationship:

$$U(\hat{z}_i)-V(\hat{z}_i)=0. \tag{B2}$$

The alternative method is based on maximizing the option value, which after ignoring any constants, such as X_0, K_0 in (11), can be expressed as:

$$\max_{z_i}\left(\frac{U(z_i)}{V(z_i)}\right) \tag{B3}$$

The first order conditions for (B3) are:

$$\frac{U'(\hat{z}_i)}{V(\hat{z}_i)}-\frac{U(\hat{z}_i)V'(\hat{z}_i)}{V^2(\hat{z}_i)}=0. \tag{B4}$$

Now since the value gain in exercising the option equals zero at the threshold, $U(\hat{z}_i)-V(\hat{z}_i)=0$, (B4) and (B1) are identical.

Appendix C: Simple Excel procedure

The boundary $G\left(\hat{X},\hat{K}\right)=0$ can be obtained by applying a procedure that does not involve solving multiple non-linear equations, see also Heydari et al. (2012), Støre et al. (2017). Every point along the boundary entails β,γ values that lie on the arc AA" of Figure 1, between the end points $\left(\beta=1.64981,\gamma=0\right)$ and $\left(\beta=1.68055,\gamma=-0.68055\right)$. Starting from a pre-specified $\hat{\gamma}=\gamma$ value with $-0.68055\leq\gamma\leq0$, the corresponding value of $\hat{\beta}$ is obtained as the positive root of $Q\left(\hat{\beta},\hat{\gamma}\right)=0$, (5). From knowing $\hat{\beta},\hat{\gamma}$, the corresponding values of $\hat{X},\hat{K}$ are obtainable from (12) and (13), respectively. This is repeated until the boundary is formed.

An alternative approach is to use (12) and (13) to eliminate $\hat{\beta},\hat{\gamma}$ in order to express (5) in terms of $\hat{X},\hat{K}$. From (12) and (13):

$$
\begin{aligned}
\hat{\beta}&=\frac{r\hat{X}}{r\hat{X}-f\delta_X-r\delta_X\hat{K}}\geq0,\\
\hat{\gamma}&=\frac{-r\delta_X\hat{K}}{r\hat{X}-f\delta_X-r\delta_X\hat{K}}\leq0.
\end{aligned}
\tag{C1}
$$

Substituting (C1) in $Q\left(\hat{\beta},\hat{\gamma}\right)=0$ yields:

$$
\begin{aligned}
Q\left(\hat{\beta},\hat{\gamma}\right)=\tfrac{1}{2}\sigma_X^2\frac{r\hat{X}\left(f\delta_X+r\delta_X\hat{K}\right)}{\left(r\hat{X}-f\delta_X-r\delta_X\hat{K}\right)^2}&+\tfrac{1}{2}\sigma_K^2\frac{r\delta_X\hat{K}\left(r\hat{X}-f\delta_X\right)}{\left(r\hat{X}-f\delta_X-r\delta_X\hat{K}\right)^2}\\
&-\rho\sigma_X\sigma_K\frac{r^2\delta_X\hat{X}\hat{K}}{\left(r\hat{X}-f\delta_X-r\delta_X\hat{K}\right)^2}\\
&+\frac{\left(r-\delta_X\right)r\hat{X}}{r\hat{X}-f\delta_X-r\delta_X\hat{K}}-\frac{\left(r-\delta_Y\right)r\delta_X\hat{K}}{r\hat{X}-f\delta_X-r\delta_X\hat{K}}-r\\
=0.
\end{aligned}
\tag{C2}
$$

Simplifying, (C2) can be expressed as:

$$
\begin{aligned}
QQ\left(\hat{X},\hat{K}\right)&=c_{\hat{X}^2}\hat{X}^2+c_{\hat{K}^2}\hat{K}^2+c_{\hat{X}\hat{K}}\hat{X}\hat{K}+c_{\hat{X}}\hat{X}+c_{\hat{K}}\hat{K}+c_0\\
&=0.
\end{aligned}
\tag{C3}
$$

where the respective coefficients are given by:

$$c_{\hat{X}^2} = r^2\delta_X,$$
$$c_{\hat{K}^2} = r^2\delta_X^2\delta_K,$$
$$c_{\hat{X}\hat{K}} = -r^2\delta_X\left(\delta_X + \delta_K + \tfrac{1}{2}\sigma_X^2 - \rho\sigma_X\sigma_K + \tfrac{1}{2}\sigma_K^2\right),$$
$$c_{\hat{X}} = -fr\delta_X\left(r + \delta_X + \tfrac{1}{2}\sigma_X^2\right),$$
$$c_{\hat{K}} = fr\delta_X^2\left(r + \delta_Y + \tfrac{1}{2}\sigma_K^2\right),$$
$$c_0 = f^2r\delta_X^2.$$

By repeatedly selecting different pre-specified $\hat{K} \geq f/r$ values, (C3) a quadratic function in $\hat{X}$ can be solved by adopting the positive root where $\hat{X}/\delta_X > \hat{K} + f/r$.

Appendix D: Option value minimization

For any X_0, K_0 belonging to the "hold" region, its option value is defined by (14):

$$\hat{F}_0 = F\left(X_0, K_0; \hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0\right) = \frac{X_0^{\hat{\beta}_0} K_0^{\hat{\gamma}_0}}{\hat{X}_0^{\hat{\beta}_0} \hat{K}_0^{\hat{\gamma}_0}}\left(\frac{\hat{X}_0}{\delta_X} - \frac{f}{r} - \hat{K}_0\right), \tag{D1}$$

where $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ belongs to the policy boundary, $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0 \in \left\{\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right\}$. Then based on the envelope theorem, Støre et al. (2017) show $\hat{X}_0, \hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ is obtained from:

$$\hat{X}_0 = \arg\min_{\hat{X}} F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right),$$

where $\hat{K}_0, \hat{\beta}_0, \hat{\gamma}_0$ are the boundary values corresponding to $\hat{X}_0$ and $F_{S\hat{X}}\left(X_0, K_0; \hat{X}\right)$ is derived from $F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ through using (12), (13) and $Q\left(\hat{\beta}, \hat{\gamma}\right) = 0$ to replace $\hat{K}, \hat{\beta}, \hat{\gamma}$.

An alternative solution is by Lagrangian minimization. We define the Lagrangian function £:

$$£\left(\hat{\beta}, \hat{\gamma}\right) = F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right) + \lambda_0 Q\left(\hat{\beta}, \hat{\gamma}\right), \tag{D2}$$

where $F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right)$ is derived from $F\left(X_0, K_0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ through using (12) and (13) to replace $\hat{X}, \hat{K}$. The option value is given by:

$$\hat{F}_0 = F_R\left(X_0, K_0; \hat{\beta}_0, \hat{\gamma}_0\right) \tag{D3}$$

where:

$$\left(\hat{\beta}_0, \hat{\gamma}_0\right) = \arg\min_{\hat{\beta}, \hat{\gamma}}\left\{F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right) - \lambda_0 Q\left(\hat{\beta}, \hat{\gamma}\right)\right\}, \tag{D4}$$

$$F_R\left(X_0, K_0; \hat{\beta}, \hat{\gamma}\right) = \frac{-1}{\hat{\gamma}} X_0^{\hat{\beta}} K_0^{\hat{\gamma}} \left(\frac{-\hat{\gamma} f}{r\left(\hat{\beta}+\hat{\gamma}-1\right)}\right)^{1-\hat{\gamma}} \left(\frac{\hat{\beta}\delta_X f}{r\left(\hat{\beta}+\hat{\gamma}-1\right)}\right)^{-\hat{\beta}}$$
$$= X_0^{\hat{\beta}} K_0^{\hat{\gamma}} \hat{\beta}^{-\hat{\beta}} \left(-\hat{\gamma}\right)^{-\hat{\gamma}} \left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right)^{\hat{\beta}+\hat{\gamma}-1} \delta_X^{-\hat{\beta}}. \tag{D5}$$

The solution values for $\hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_0$ are obtained from the two first order conditions of $£\left(\hat{\beta}, \hat{\gamma}\right)$ and $Q\left(\hat{\beta}_0, \hat{\gamma}_0\right) = 0$. Differentiating the elements of F_R with respect to $\hat{\beta}$ yields:

$$\frac{\partial X_0^{\hat{\beta}}}{F_R \partial \hat{\beta}} = \ln\left(X_0\right),$$
$$\frac{\partial \hat{\beta}^{-\hat{\beta}}}{F_R \partial \hat{\beta}} = -\left(1 + \ln\left(\hat{\beta}\right)\right),$$
$$\frac{\partial \left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right)^{\hat{\beta}+\hat{\gamma}-1}}{F_R \partial \hat{\beta}} = 1 + \ln\left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right),$$
$$\frac{\partial \delta_X^{-\hat{\beta}}}{F_R \partial \hat{\beta}} = -\ln\left(\delta_X\right),$$

so:

$$\frac{\partial F_R}{\partial \hat{\beta}} = \left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\,\delta_X f/r}\right)\right] F_R. \tag{D6}$$

Differentiating the elements of F_R with respect to $\hat{\gamma}$ yields:

$$\frac{\partial K_0^{\hat{\gamma}}}{F_R \partial \hat{\gamma}} = \ln\left(K_0\right),$$
$$\frac{\partial \left(-\hat{\gamma}\right)^{-\hat{\gamma}}}{F_R \partial \hat{\gamma}} = -\left(1 + \ln\left(-\hat{\gamma}\right)\right),$$
$$\frac{\partial \left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right)^{\hat{\beta}+\hat{\gamma}-1}}{F_R \partial \hat{\beta}} = 1 + \ln\left(\frac{\hat{\beta}+\hat{\gamma}-1}{f/r}\right),$$

so:

$$\frac{\partial F_R}{\partial \hat{\gamma}} = \left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}\, f/r}\right)\right] F_R. \tag{D7}$$

Also:

$$\frac{\partial Q\left(\hat{\beta},\hat{\gamma}\right)}{\partial\hat{\beta}} = \hat{\beta}\sigma_X^2 + \hat{\gamma}\rho\sigma_X\sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2, \tag{D8}$$

$$\frac{\partial Q\left(\hat{\beta},\hat{\gamma}\right)}{\partial\hat{\gamma}} = \hat{\gamma}\sigma_K^2 + \hat{\beta}\rho\sigma_X\sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2. \tag{D9}$$

So differentiating $£\left(\hat{\beta},\hat{\gamma}\right)$ with respect to $\hat{\beta},\hat{\gamma}$ yields:

$$£_{\hat{\beta}}\left(\hat{\beta},\hat{\gamma}\right) = \left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\,\delta_X\, f/r}\right)\right]F_R + \lambda_0\left[\hat{\beta}\sigma_X^2 + \hat{\gamma}\rho\sigma_X\sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2\right], \tag{D10}$$

$$£_{\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right) = \left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}\, f/r}\right)\right]F_R + \lambda_0\left[\hat{\gamma}\sigma_K^2 + \hat{\beta}\rho\sigma_X\sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2\right], \tag{D11}$$

and the two respective first order conditions are given by:

$$\left[\ln\left(\frac{X_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{\hat{\beta}_0\,\delta_X\, f/r}\right)\right]\hat{F}_R + \hat{\lambda}_0\left[\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2\right] = 0, \tag{D12}$$

$$\left[\ln\left(\frac{K_0\left(\hat{\beta}_0+\hat{\gamma}_0-1\right)}{-\hat{\gamma}_0\, f/r}\right)\right]\hat{F}_R + \hat{\lambda}_0\left[\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2\right] = 0, \tag{D13}$$

where $\hat{F}_R = F_R\left(X_0, K_0; \hat{\beta}_0, \hat{\gamma}_0\right)$. From (D5), the option value for any X_0, K_0 belonging to the "hold" region is:

$$\hat{F}_0 = \hat{F}_R = X_0^{\hat{\beta}_0} K_0^{\hat{\gamma}_0} \hat{\beta}_0^{-\hat{\beta}_0}\left(-\hat{\gamma}_0\right)^{-\hat{\gamma}_0}\left(\frac{\hat{\beta}_0+\hat{\gamma}_0-1}{f/r}\right)^{\hat{\beta}_0+\hat{\gamma}_0-1}\delta_X^{-\hat{\beta}_0}. \tag{D14}$$

The sufficiency condition is inspected from the second derivatives:

$$£_{\hat{\beta},\hat{\beta}}\left(\hat{\beta},\hat{\gamma}\right) = \left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\,\delta_X\, f/r}\right)\right]^2 F_R + \frac{\left(1-\hat{\gamma}\right)F_R}{\hat{\beta}\left(\hat{\beta}+\hat{\gamma}-1\right)} + \lambda_0\sigma_X^2 \geq 0, \tag{D15}$$

$$£_{\hat{\gamma},\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right) = \left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}\, f/r}\right)\right]^2 F_R + \frac{\left(\hat{\beta}-1\right)F_R}{-\hat{\gamma}\left(\hat{\beta}+\hat{\gamma}-1\right)} + \lambda_0\sigma_K^2 \geq 0, \tag{D16}$$

$$£_{\hat{\beta},\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right)=\left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\delta_X f/r}\right)\right]\left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}f/r}\right)\right]F_R$$
$$+\frac{F_R}{\left(\hat{\beta}+\hat{\gamma}-1\right)}+\lambda_0\rho\sigma_X\sigma_K, \tag{D17}$$

and:

$$£_{\hat{\beta},\hat{\beta}}\left(\hat{\beta},\hat{\gamma}\right)£_{\hat{\gamma},\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right)-£_{\hat{\beta},\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right)£_{\hat{\beta},\hat{\gamma}}\left(\hat{\beta},\hat{\gamma}\right)=$$
$$\lambda_0^2\sigma_X^2\sigma_K^2\left(1-\rho^2\right)$$
$$+\lambda_0F_R\left\{\left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\delta_X f/r}\right)\right]^2\sigma_K^2+\left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}f/r}\right)\right]^2\sigma_X^2+\right.$$
$$\left.-2\times\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\delta_X f/r}\right)\times\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}f/r}\right)\times\rho\sigma_X\sigma_K\right\}$$
$$+\frac{F_R^2}{-\hat{\gamma}\hat{\beta}\left(\hat{\beta}+\hat{\gamma}-1\right)}$$
$$+\frac{\lambda_0F_R}{\hat{\beta}+\hat{\gamma}-1}\left\{\frac{\hat{\beta}-1}{-\hat{\gamma}}\sigma_X^2+\frac{1-\hat{\gamma}}{\hat{\beta}}\sigma_K^2-2\times\rho\sigma_X\sigma_K\right\}$$
$$+\frac{F_R^2}{\hat{\beta}+\hat{\gamma}-1}\left\{\frac{\hat{\beta}-1}{-\hat{\gamma}}\left[\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\delta_X f/r}\right)\right]^2+\frac{1-\hat{\gamma}}{\hat{\beta}}\left[\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}f/r}\right)\right]^2\right.$$
$$\left.-2\times\ln\left(\frac{X_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{\hat{\beta}\delta_X f/r}\right)\times\ln\left(\frac{K_0\left(\hat{\beta}+\hat{\gamma}-1\right)}{-\hat{\gamma}f/r}\right)\right\} \tag{D18}$$
$$\geq 0,$$

since each term on the right hand side of (D18) is non-negative. Therefore, the sufficiency condition for a minimum is fulfilled.

Appendix E: Option value derivatives

The option value derivatives with respect to the various parameters are essentially obtained by using the chain rule and their sign condition by using (28) and (29).

The option value derivative with respect to σ_X is obtained from:

$$\frac{\partial \hat{F}_0}{\partial \sigma_X} = \frac{\partial \hat{F}_R}{\partial \hat{\beta}_0}\frac{\partial \hat{\beta}_0}{\partial \sigma_X} + \frac{\partial \hat{F}_R}{\partial \hat{\gamma}_0}\frac{\partial \hat{\gamma}_0}{\partial \sigma_X}. \tag{E1}$$

Then from (D6) and (D7), respectively:

$$\frac{\partial \hat{F}_R}{\partial \hat{\beta}_0} = \left[\ln\left(\frac{X_0\left(\hat{\beta}_0 + \hat{\gamma}_0 - 1\right)}{\hat{\beta}_0\, \delta_X\, f/r} \right) \right] \hat{F}_R = \ln\left(\frac{X_0}{\hat{X}_0} \right) \hat{F}_R, \tag{E2}$$

$$\frac{\partial \hat{F}_R}{\partial \hat{\gamma}_0} = \left[\ln\left(\frac{K_0\left(\hat{\beta}_0 + \hat{\gamma}_0 - 1\right)}{-\hat{\gamma}_0\, f/r} \right) \right] \hat{F}_R = \ln\left(\frac{K_0}{\hat{K}_0} \right) \hat{F}_R. \tag{E3}$$

Also:

$$\frac{\partial \hat{\beta}_0}{\partial \sigma_X} = -\frac{\dfrac{\partial Q}{\partial \sigma_X}}{\dfrac{\partial Q}{\partial \hat{\beta}_0}} = -\frac{\sigma_X \hat{\beta}_0\left(\hat{\beta}_0 - 1\right) + \rho\sigma_K \hat{\beta}_0 \hat{\gamma}_0}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2}, \tag{E4}$$

$$\frac{\partial \hat{\gamma}_0}{\partial \sigma_X} = -\frac{\dfrac{\partial Q}{\partial \sigma_X}}{\dfrac{\partial Q}{\partial \hat{\gamma}_0}} = -\frac{\sigma_X \hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right) + \rho\sigma_X \hat{\beta}_0 \hat{\gamma}_0}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2}. \tag{E5}$$

It follows from (E2)-(E5) that (E1) becomes:

$$\begin{aligned} \frac{\partial \hat{F}_0}{\partial \sigma_X} = &-\hat{F}_R \ln\left(X_0/\hat{X}_0\right) \frac{\sigma_X \hat{\beta}_0\left(\hat{\beta}_0 - 1\right) + \hat{\beta}_0 \hat{\gamma}_0 \rho \sigma_K}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2} \\ &-\hat{F}_R \ln\left(K_0/\hat{K}_0\right) \frac{\sigma_X \hat{\beta}_0\left(\hat{\beta}_0 - 1\right) + \hat{\beta}_0 \hat{\gamma}_0 \rho \sigma_K}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \tfrac{1}{2}\sigma_K^2}. \end{aligned} \tag{E6}$$

The sign of $\partial \hat{F}_0 / \partial \sigma_X$ depends on the sign of $\sigma_X \hat{\beta}_0\left(\hat{\beta}_0 - 1\right) + \hat{\beta}_0 \hat{\gamma}_0 \rho \sigma_K$.

The option value derivative with respect to σ_K is obtained from:

$$\frac{\partial \hat{F}_0}{\partial \sigma_K} = \frac{\partial \hat{F}_R}{\partial \hat{\beta}_0}\frac{\partial \hat{\beta}_0}{\partial \sigma_K} + \frac{\partial \hat{F}_R}{\partial \hat{\gamma}_0}\frac{\partial \hat{\gamma}_0}{\partial \sigma_K}. \tag{E7}$$

Similarly:

$$\frac{\partial \hat{\beta}_0}{\partial \sigma_K} = -\frac{\dfrac{\partial Q}{\partial \sigma_K}}{\dfrac{\partial Q}{\partial \hat{\beta}_0}} = -\frac{\sigma_K \hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right) + \rho\sigma_X \hat{\beta}_0 \hat{\gamma}_0}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \tfrac{1}{2}\sigma_X^2}, \tag{E8}$$

$$\frac{\partial\hat{\gamma}_0}{\partial\sigma_K} = -\frac{\dfrac{\partial Q}{\partial\sigma_K}}{\dfrac{\partial Q}{\partial\hat{\gamma}_0}} = -\frac{\sigma_K\hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right) + \rho\sigma_X\hat{\beta}_0\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}, \tag{E9}$$

so:

$$\begin{aligned}\frac{\partial\hat{F}_0}{\partial\sigma_K} = &-\ln\left(\frac{X_0}{\hat{X}_0}\right)\frac{\sigma_K\hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right) + \rho\sigma_X\hat{\beta}_0\hat{\gamma}_0}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} \\ &-\ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R\frac{\sigma_K\hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right) + \rho\sigma_X\hat{\beta}_0\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}.\end{aligned} \tag{E10}$$

The sign of $\partial\hat{F}_0 / \partial\sigma_K$ depends on the sign of $\hat{\gamma}_0\left(\hat{\gamma}_0 - 1\right)\sigma_K + \hat{\beta}_0\hat{\gamma}_0\rho\sigma_X$.

The option value derivative with respect to ρ is obtained from:

$$\frac{\partial\hat{F}_0}{\partial\rho} = \frac{\partial\hat{F}_R}{\partial\hat{\beta}_0}\frac{\partial\hat{\beta}_0}{\partial\rho} + \frac{\partial\hat{F}_R}{\partial\hat{\gamma}_0}\frac{\partial\hat{\gamma}_0}{\partial\rho}. \tag{E11}$$

Similarly:

$$\frac{\partial\hat{\beta}_0}{\partial\rho} = -\frac{\dfrac{\partial Q}{\partial\rho}}{\dfrac{\partial Q}{\partial\hat{\beta}_0}} = -\frac{\sigma_X\sigma_K\hat{\beta}_0\hat{\gamma}_0}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2}, \tag{E12}$$

$$\frac{\partial\hat{\gamma}_0}{\partial\rho} = -\frac{\dfrac{\partial Q}{\partial\rho}}{\dfrac{\partial Q}{\partial\hat{\gamma}_0}} = -\frac{\sigma_X\sigma_K\hat{\beta}_0\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}, \tag{E13}$$

so:

$$\begin{aligned}\frac{\partial\hat{F}_0}{\partial\rho} = &-\ln\left(\frac{X_0}{\hat{X}_0}\right)\hat{F}_R\frac{\sigma_X\sigma_K\hat{\beta}_0\hat{\gamma}_0}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} \\ &-\ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R\frac{\sigma_X\sigma_K\hat{\beta}_0\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}.\end{aligned} \tag{E14}$$

The sign of $\partial\hat{F}_0 / \partial\rho$ depends on the sign of $\hat{\beta}_0\hat{\gamma}_0\sigma_X\sigma_K < 0$.

The option value derivative with respect to δ_X is obtained from:

$$\frac{\partial\hat{F}_0}{\partial\delta_X} = \frac{\partial\hat{F}_R}{\partial\delta_X} + \frac{\partial\hat{F}_R}{\partial\hat{\beta}_0}\frac{\partial\hat{\beta}_0}{\partial\delta_X} + \frac{\partial\hat{F}_R}{\partial\hat{\gamma}_0}\frac{\partial\hat{\gamma}_0}{\partial\delta_X}, \tag{E15}$$

since $\hat{F}_R$ (18) is a function of δ_X . Now:

$$\frac{\partial \hat{F}_R}{\partial \delta_X} = -\frac{\hat{\beta}_0}{\delta_X}\hat{F}_R . \tag{E16}$$

Similarly:

$$\frac{\partial \hat{\beta}_0}{\partial \delta_X} = -\frac{\dfrac{\partial Q}{\partial \delta_X}}{\dfrac{\partial Q}{\partial \hat{\beta}_0}} = -\frac{-\hat{\beta}_0}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2}, \tag{E17}$$

$$\frac{\partial \hat{\gamma}_0}{\partial \delta_X} = -\frac{\dfrac{\partial Q}{\partial \delta_X}}{\dfrac{\partial Q}{\partial \hat{\gamma}_0}} = -\frac{-\hat{\beta}_0}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}, \tag{E18}$$

so:

$$\begin{aligned}\frac{\partial \hat{F}_0}{\partial \delta_X} = &-\frac{\hat{\beta}_0}{\delta_X}\hat{F}_R - \ln\left(\frac{X_0}{\hat{X}_0}\right)\hat{F}_R \frac{-\hat{\beta}_0}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} \\ &-\ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R \frac{-\hat{\beta}_0}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}.\end{aligned} \tag{E19}$$

The sign of $\partial \hat{F}_0 / \partial \delta_X$ depends on the sign of $-\hat{\beta}_0 < 0$.

The option value derivative with respect to δ_K is obtained from:

$$\frac{\partial \hat{F}_0}{\partial \delta_K} = \frac{\partial \hat{F}_R}{\partial \hat{\beta}_0}\frac{\partial \hat{\beta}_0}{\partial \delta_K} + \frac{\partial \hat{F}_R}{\partial \hat{\gamma}_0}\frac{\partial \hat{\gamma}_0}{\partial \delta_K} . \tag{E20}$$

Similarly:

$$\frac{\partial \hat{\beta}_0}{\partial \delta_K} = -\frac{\dfrac{\partial Q}{\partial \delta_K}}{\dfrac{\partial Q}{\partial \hat{\beta}_0}} = -\frac{-\hat{\gamma}_0}{\hat{\beta}_0 \sigma_X^2 + \hat{\gamma}_0 \rho \sigma_X \sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2}, \tag{E21}$$

$$\frac{\partial \hat{\gamma}_0}{\partial \delta_K} = -\frac{\dfrac{\partial Q}{\partial \delta_K}}{\dfrac{\partial Q}{\partial \hat{\gamma}_0}} = -\frac{-\hat{\gamma}_0}{\hat{\gamma}_0 \sigma_K^2 + \hat{\beta}_0 \rho \sigma_X \sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}, \tag{E22}$$

so:

$$\frac{\partial \hat{F}_0}{\partial \delta_K} = -\ln\left(\frac{X_0}{\hat{X}_0}\right)\hat{F}_R \frac{-\hat{\gamma}_0}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} - \ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R \frac{-\hat{\gamma}_0}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}. \tag{E23}$$

The sign of $\partial \hat{F}_0 / \partial \delta_K$ depends on the sign of $-\hat{\gamma}_0 > 0$.

The option value derivative with respect to r is obtained from:

$$\frac{\partial \hat{F}_0}{\partial r} = \frac{\partial \hat{F}_R}{\partial r} + \frac{\partial \hat{F}_R}{\partial \hat{\beta}_0}\frac{\partial \hat{\beta}_0}{\partial r} + \frac{\partial \hat{F}_R}{\partial \hat{\gamma}_0}\frac{\partial \hat{\gamma}_0}{\partial r} \tag{E24}$$

since $\hat{F}_R$ (18) is a function of r. Now:

$$\frac{\partial \hat{F}_R}{\partial r} = \frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{r}\hat{F}_R. \tag{E25}$$

Similarly:

$$\frac{\partial \hat{\beta}_0}{\partial r} = -\frac{\frac{\partial Q}{\partial r}}{\frac{\partial Q}{\partial \hat{\beta}_0}} = -\frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2}, \tag{E26}$$

$$\frac{\partial \hat{\gamma}_0}{\partial r} = -\frac{\frac{\partial Q}{\partial r}}{\frac{\partial Q}{\partial \hat{\gamma}_0}} = -\frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}, \tag{E27}$$

so:

$$\frac{\partial \hat{F}_0}{\partial r} = \frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{r}\hat{F}_R - \ln\left(\frac{X_0}{\hat{X}_0}\right)\hat{F}_R \frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{\hat{\beta}_0\sigma_X^2 + \hat{\gamma}_0\rho\sigma_X\sigma_K + r - \delta_X - \frac{1}{2}\sigma_X^2} - \ln\left(\frac{K_0}{\hat{K}_0}\right)\hat{F}_R \frac{\hat{\beta}_0 + \hat{\gamma}_0 - 1}{\hat{\gamma}_0\sigma_K^2 + \hat{\beta}_0\rho\sigma_X\sigma_K + r - \delta_K - \frac{1}{2}\sigma_K^2}. \tag{E28}$$

The sign of $\partial \hat{F}_0 / \partial r$ depends on the sign of $\hat{\beta}_0 + \hat{\gamma}_0 - 1 > 0$.

Appendix F: Homogeneity degree-1 model

From (10), $f = 0$ yields the homogeneity degree-1 model of McDonald & Siegel (1986), with its option function power parameters that sum to 1:

$$A_1 \hat{X}_1^{\beta_1} \hat{K}_1^{\gamma_1} = \frac{\hat{X}_1}{\delta_X} - \hat{K}_1, \tag{F1}$$

where the subscript 1 indicates the homogeneity degree-1 model and $\beta_1 + \gamma_1 = 1$. From (5):

$$\begin{aligned} Q_1(\beta_1) &= Q(\beta = \beta_1, \gamma = \gamma_1; \beta_1 + \gamma_1 = 1) \\ &= \tfrac{1}{2}\sigma_{XK}^2 \beta_1(\beta_1 - 1) + \beta_1(\delta_K - \delta_X) - \delta_K = 0. \end{aligned} \tag{F2}$$

where $\sigma_{XK}^2 = \sigma_X^2 - 2\rho\sigma_X\sigma_K + \sigma_K^2$. $\hat{\beta}_1$ is evaluated as the positive root of (F2). The optimal thresholds $\hat{X}_1, \hat{K}_1$ are related by:

$$\frac{\hat{X}_1}{\hat{K}_1} = \frac{\delta_X \hat{\beta}_1}{\hat{\beta}_1 - 1} \tag{F3}$$

and $A_1 = (\beta_1 - 1)^{\beta_1 - 1} \beta_1^{-\beta_1} \delta_X^{-\beta_1}$. The option value is $F_1 = A_1 X_0^{\beta_1} K_0^{1-\beta_1} \geq 0$ for any X_0, K_0 belonging to the "hold" region so $X_0/Y_0 < \hat{X}_1/\hat{K}_1$. The option value derivative is obtained in a similar way as in Appendices D and E:

$$\frac{\partial F_1}{\partial \sigma_X} = \frac{\partial F_1}{\partial \hat{\beta}_1} \frac{\partial \hat{\beta}_1}{\partial \sigma_X}, \tag{F4}$$

where:

$$\frac{\partial F_1}{\partial \hat{\beta}_1} = \ln\left(\frac{X_0}{K_0} \frac{\beta_1 - 1}{\delta_X \beta_1}\right) F_1 = \ln\left(\frac{X_0}{K_0} \frac{\hat{K}_1}{\hat{X}_1}\right) F_1 < 0, \tag{F5}$$

$$\frac{\partial \hat{\beta}_1}{\partial \sigma_X} = -\frac{\frac{\partial Q}{\partial \sigma_X}}{\frac{\partial Q}{\partial \hat{\beta}_1}} = -\frac{\hat{\beta}_1(\hat{\beta}_1 - 1)(\sigma_X - \rho\sigma_K)}{\hat{\beta}_1\sigma_{XK}^2 + (\delta_K - \delta_X - \frac{1}{2}\sigma_{XK}^2)}. \tag{F6}$$

Treating the denominator of (F6) as positive, the sign of $\partial F_1/\partial \sigma_X$ depends on the sign of $\sigma_X - \rho\sigma_K$. The remaining derivatives are obtained in a similar way:

$$\frac{\partial F_1}{\partial \sigma_K} = -\ln\left(\frac{X_0}{K_0} \frac{\hat{K}_1}{\hat{X}_1}\right) F_1 \frac{\hat{\beta}_1(\hat{\beta}_1 - 1)(\sigma_K - \rho\sigma_X)}{\hat{\beta}_1\sigma_{XK}^2 + (\delta_K - \delta_X - \frac{1}{2}\sigma_{XK}^2)}, \tag{F7}$$

the sign of $\partial F_1/\partial \sigma_K$ depends on the sign of $\sigma_K - \rho\sigma_X$.

$$\frac{\partial F_1}{\partial \rho} = -\ln\left(\frac{X_0}{K_0} \frac{\hat{K}_1}{\hat{X}_1}\right) F_1 \frac{-\hat{\beta}_1(\hat{\beta}_1 - 1)\sigma_X\sigma_K}{\hat{\beta}_1\sigma_{XK}^2 + (\delta_K - \delta_X - \frac{1}{2}\sigma_{XK}^2)} < 0. \tag{F8}$$

$$\frac{\partial F_1}{\partial \delta_X} = -\frac{\beta_1}{\delta_X} F_1 - \ln\left(\frac{X_0}{K_0} \frac{\hat{K}_1}{\hat{X}_1}\right) F_1 \frac{-\hat{\beta}_1}{\hat{\beta}_1\sigma_{XK}^2 + (\delta_K - \delta_X - \frac{1}{2}\sigma_{XK}^2)} < 0. \tag{F9}$$

$$\frac{\partial F_1}{\partial \delta_K} = -\ln\left(\frac{X_0}{K_0}\frac{\hat{K}_1}{\hat{X}_1}\right) F_1 \frac{\hat{\beta}_1 - 1}{\hat{\beta}_1 \sigma_{XK}^2 + \left(\delta_K - \delta_X - \frac{1}{2}\sigma_{XK}^2\right)} > 0 \tag{F10}$$

$$\frac{\partial F_1}{\partial r} = 0\,. \tag{F11}$$

Table 1

Base Case Values

Parameter	**Symbol**	**Value**
Periodic fixed cost	f	5.0
Risk-free rate	r	5.0%
Convenience yield for X	δ_X	4.0%
Volatility for X	σ_X	25.0%
Convenience yield for K	δ_K	2.0%
Volatility for K	σ_K	25.0%
Correlation between X and K	ρ	0.25

The variables X and K denote the periodic cash-flow and investment cost, respectively.

Table 2
Illustrative Results

K_0		$X_0 = 5.0$	$X_0 = 10.0$	$X_0 = 15.0$	$X_0 = 20.0$	$X_0 = 25.0$
25.0	$\hat{\beta}_0$	1.68819	1.68509			
	$\hat{\gamma}_0$	-0.15785	-0.14176			
	$\hat{\lambda}_0$	457.13	339.12			
	$\hat{X}_0$	12.733	12.406			
	$\hat{K}_0$	29.764	26.091			
	$Test_1$	0.0818	0.0819			
	$Test_2$	0.0153	0.0162			
	Value	40.001	128.768	250.000	375.000	625.000
50.0	$\hat{\beta}_0$	1.70289	1.70127			
	$\hat{\gamma}_0$	-0.25488	-0.24151			
	$\hat{\lambda}_0$	475.49	544.81			
	$\hat{X}_0$	15.204	14.802			
	$\hat{K}_0$	56.892	52.531			
	$Test_1$	0.0812	0.0813			
	$Test_2$	0.0094	0.0102			
	Value	34.716	112.958	225.000	350.000	475.000
75.0	$\hat{\beta}_0$	1.70879	1.70820	1.70778		
	$\hat{\gamma}_0$	-0.31878	-0.31055	-0.305		
	$\hat{\lambda}_0$	481.13	677.91	307.26		
	$\hat{X}_0$	17.526	17.183	16.961		
	$\hat{K}_0$	81.737	78.098	75.731		
	$Test_1$	0.0806	0.0807	0.0807		
	$Test_2$	0.0055	0.0060	0.0064		
	Value	30.906	101.009	201.894	325.000	450.000
100.0	$\hat{\beta}_0$	1.71112	1.71097	1.71087		
	$\hat{\gamma}_0$	-0.36391	-0.35992	-0.3572		
	$\hat{\lambda}_0$	480.35	764.78	583.62		
	$\hat{X}_0$	19.713	19.495	19.351		
	$\hat{K}_0$	104.813	102.524	101.009		
	$Test_1$	0.0800	0.0801	0.0801		

	$Test_2$	0.0027	0.0030	0.0032		
	Value	28.015	91.720	183.545	300.000	425.000
125.0	$\hat{\beta}_0$	1.71183	1.71182	1.71182	1.71181	
	$\hat{\gamma}_0$	-0.39757	-0.39667	-0.39606	-0.39559	
	$\hat{\lambda}_0$	476.27	822.30	781.83	275.62	
	$\hat{X}_0$	21.789	21.727	21.685	21.653	
	$\hat{K}_0$	126.511	125.863	125.431	125.096	
	$Test_1$	0.0795	0.0795	0.0796	0.0796	
	$Test_2$	0.0006	0.0007	0.0007	0.0008	
	Value	25.732	84.292	168.742	276.119	400.000
150.0	$\hat{\beta}_0$	1.71178	1.71176	1.71175	1.71174	
	$\hat{\gamma}_0$	-0.42374	-0.42504	-0.42591	-0.42658	
	$\hat{\lambda}_0$	470.49	860.73	926.71	592.23	
	$\hat{X}_0$	23.771	23.881	23.954	24.011	
	$\hat{K}_0$	147.111	148.246	149.006	149.596	
	$Test_1$	0.0791	0.0791	0.0791	0.0791	
	$Test_2$	-0.0010	-0.0011	-0.0011	-0.0012	
	Value	23.875	78.207	156.556	256.172	375.000
175.0	$\hat{\beta}_0$	1.71134	1.71125	1.71119	1.71114	1.71110
	$\hat{\gamma}_0$	-0.44473	-0.44763	-0.44953	-0.45099	-0.45219
	$\hat{\lambda}_0$	463.86	886.40	1034.46	834.76	249.11
	$\hat{X}_0$	25.676	25.965	26.159	26.310	26.436
	$\hat{K}_0$	166.814	169.800	171.803	173.360	174.655
	$Test_1$	0.0788	0.0787	0.0787	0.0786	0.0786
	$Test_2$	-0.0023	-0.0025	-0.0026	-0.0027	-0.0028
	Value	22.329	73.118	146.338	239.413	350.729
200.0	$\hat{\beta}_0$	1.71071	1.71053	1.71040	1.71030	1.71022
	$\hat{\gamma}_0$	-0.46201	-0.46606	-0.46869	-0.47070	-0.47234
	$\hat{\lambda}_0$	456.85	903.29	1115.77	1023.63	589.91
	$\hat{X}_0$	27.514	27.987	28.305	28.552	28.757
	$\hat{K}_0$	185.769	190.637	193.904	196.444	198.557
	$Test_1$	0.0785	0.0784	0.0783	0.0783	0.0783
	$Test_2$	-0.0034	-0.0037	-0.0038	-0.0039	-0.0040
	Value	21.017	68.790	137.634	225.121	329.729

The computations for this table are processed in the following way. For a given pair $\{X_0, K_0\}$, we first test whether the pair belongs to the "hold" or the "invest" region.

If $\{X_0, K_0\}$ belongs to the "hold" region, then:

1. $\hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_0$ are obtained by solving (19), (20) and $Q\left(\hat{\beta}_0, \hat{\gamma}_0\right) = 0$, (5);
2. $\hat{X}_0$ and $\hat{K}_0$ are obtained from (12) and (13), respectively:

$$\hat{X}_0 = \frac{f}{r}\delta_X \frac{\hat{\beta}_0}{\hat{\beta}_0 + \hat{\gamma}_0 - 1}, \hat{K}_0 = -\frac{f}{r}\frac{\hat{\gamma}_0}{\hat{\beta}_0 + \hat{\gamma}_0 - 1};$$

3. $Test_1$ and $Test_2$ are obtained from (26) and (27);
4. $Value$ is determined from $\hat{F}_R = F_R\left(X_0, K_0; \hat{\beta}_0, \hat{\gamma}_0\right)$, (18).

If $\{X_0, K_0\}$ belongs to the "invest" region, then:

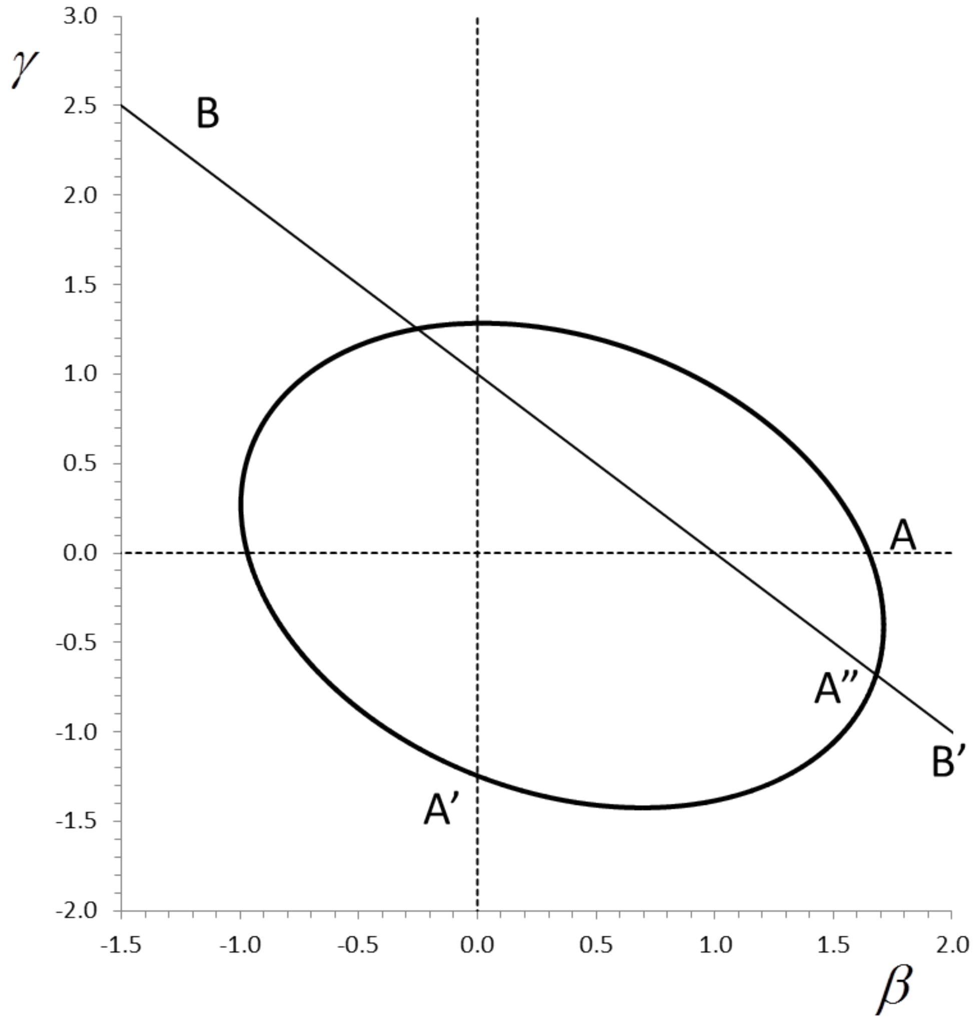
1. The cells relating to $\hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_0, \hat{X}_0, \hat{K}_0, Test_1, Test_2$ are left blank;
2. $Value$ is determined from the net present value $\left(X_0/\delta_X - f/r - K_0\right)$.

Table 3
Option Value Derivative Sign Conditions for General 2-Factor Model,
2-Factor Homogeneity Degree-1 Model and 1-Factor Model

Derivative with respect to	General 2-Factor Model	2-Factor Homogeneity Degree-1 Model	1-Factor Model
σ_X	$\sigma_X \hat{\beta}_0 \left(\hat{\beta}_0 - 1\right) + \hat{\beta}_0 \hat{\gamma}_0 \rho \sigma_K > 0$	$\sigma_X - \rho\sigma_K > 0$	+
σ_K	$\sigma_K \hat{\gamma}_0 \left(\hat{\gamma}_0 - 1\right) + \hat{\beta}_0 \hat{\gamma}_0 \rho \sigma_X > 0$	$\sigma_K - \rho\sigma_X > 0$	
ρ	–	–	
δ_X	–	–	–
δ_K	+	+	
r	+	0	+

The items in this table identify the option value derivative sign condition for the general 2-factor model, the 2-factor homogeneity degree-1 model, and the 1-factor model for the parameters defined in the respective Q function. Positive (negative) dependence between the option value and the parameter, indicating an increasing (decreasing) option function, is denoted by $+(-)$, while 0 denotes independence. If the item is stated as a condition, then this condition has to be met for the option function to be increasing. Blank items indicate that the derivative is not applicable. Results are drawn from Appendices E and F for the general 2-factor model and the 2-factor homogeneity degree-1 model, respectively, and from Dixit & Pindyck (1994) for the 1-factor model.

Figure 1
Characteristic Function $Q(\beta,\gamma)=0$



The elliptical function $Q(\beta,\gamma)=0$ is drawn based on Table 1 values. The maximum and minimum values for β and γ are:

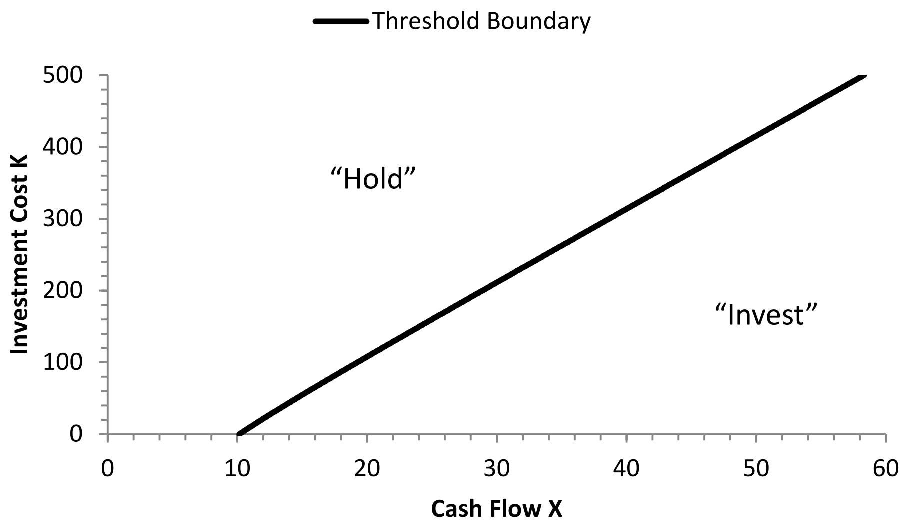
	β	γ
Maximum β	1.71187	-0.40797
Minimum β	-0.99721	0.26930
Maximum γ	0.01870	1.28521
Minimum γ	0.69597	-1.42387

The ellipse traverses the axes at $\beta=-0.96981$ and $\beta=1.64981$ (point A) when $\gamma=0$, and at $\gamma=-1.24507$ (point A') and $\gamma=1.28507$ when $\beta=0$. The set of values conforming to $\beta\geq 0, \gamma\leq 0$ and $Q(\beta,\gamma)=0$ is represented by points on the arc AA'.

The line BB' represents $\beta+\gamma=1$. It crosses the ellipse at $\beta=-0.25388$ and $\gamma=1.25388$, and at $\beta=1.68055$ and $\gamma=-0.68055$ (point A"). Points on the arc AA" represents the values satisfying $\beta \geq 0, \gamma \leq 0, \beta+\gamma \geq 1$ and $Q(\beta,\gamma)=0$.

Figure 2

Threshold Boundary $G\left(\hat{X},\hat{Y},\hat{A},\hat{\beta},\hat{\gamma}\right)=0$



This figure is determined from (5), (12) and (13) using the Table 1 values, by evaluating the values of $\hat{X},\hat{\beta},\hat{\gamma}$ repeatedly for various pre-specified $\hat{K}>0$ values. Typical values for G are given in the following table, with $\hat{A}=\left(\hat{\beta}\delta_X\hat{X}^{\hat{\beta}-1}\right)^{-1}$ from (10):

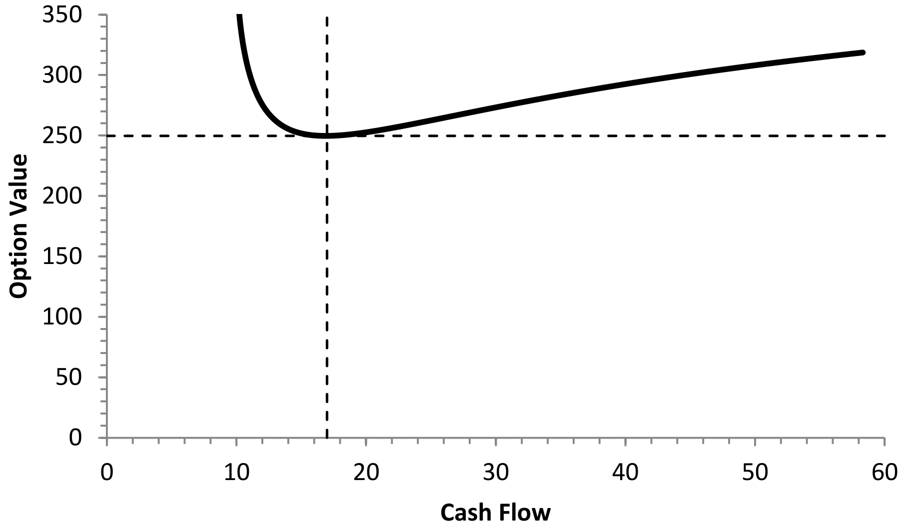
$\hat{K}$	$\hat{X}$	$\hat{A}$	$\hat{\beta}$	$\hat{\gamma}$
0.0	10.15565	3.35999	1.64981	0.00000
50.0	14.56870	5.61478	1.70022	-0.23341
75.0	16.89206	7.33576	1.70764	-0.30327
100.0	19.25498	9.17254	1.71080	-0.35540
150.0	24.05027	12.90607	1.71173	-0.42704
200.0	28.89753	16.47622	1.71016	-0.47344
250.0	33.77315	19.76486	1.70803	-0.50574
300.0	38.66583	22.74794	1.70591	-0.52943
350.0	43.56951	25.43907	1.70396	-0.54753
400.0	48.48070	27.86508	1.70223	-0.56178
450.0	53.39724	30.05567	1.70070	-0.57330
500.0	58.31770	32.03918	1.69935	-0.58279

The values of $\hat{\beta},\hat{\gamma}$ lie on the arc AA" of Figure 1.

Figure 3

Option Value Profile of $F_0 = F\left(X_0 = 16.98593, K_0 = 75.0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$

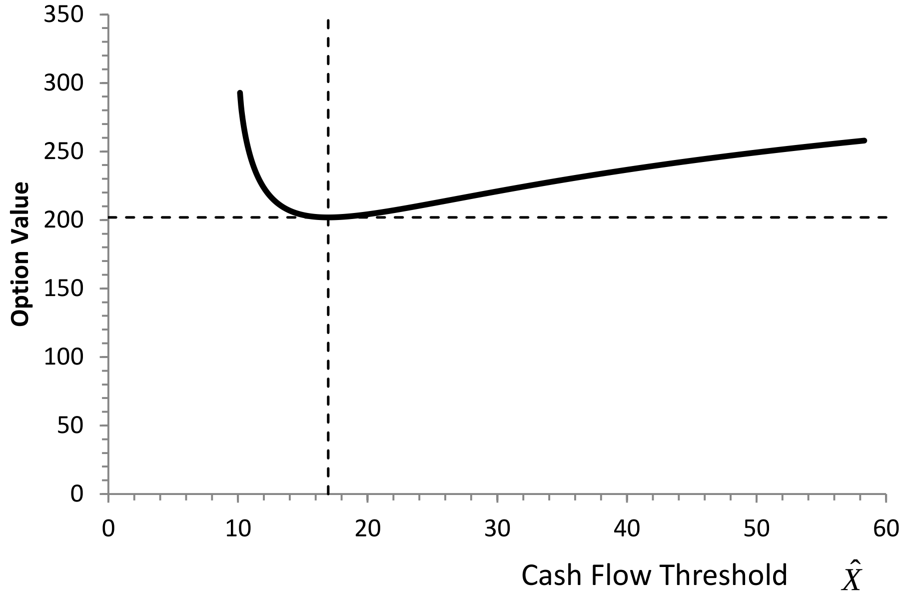
versus Cash Flow Threshold $10.15565 \le \hat{X} \le 60$



The option values are evaluated for the various $\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}$ values on the boundary as provided by Figure 2 for $X_0 = 16.98593, K_0 = 75.0$, which because they lie on the boundary, has a known option value. The minimum option value and corresponding cash flow threshold are identified.

Figure 4a

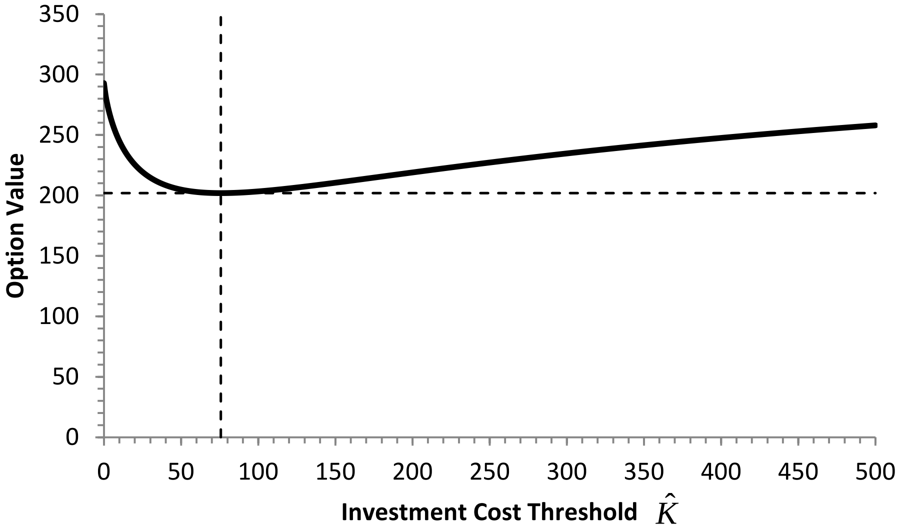
Option Value Profiles of $F\left(X_0 = 15.0, K_0 = 75.0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ versus $\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}$



The option value profile is constructed according to Figure 3, for $10.15565 \le \hat{X} \le 60$. The minimum option value occurs at $\hat{F}_0 = 201.8942$ for $\hat{X}_0 = 16.96064$.

Figure 4b

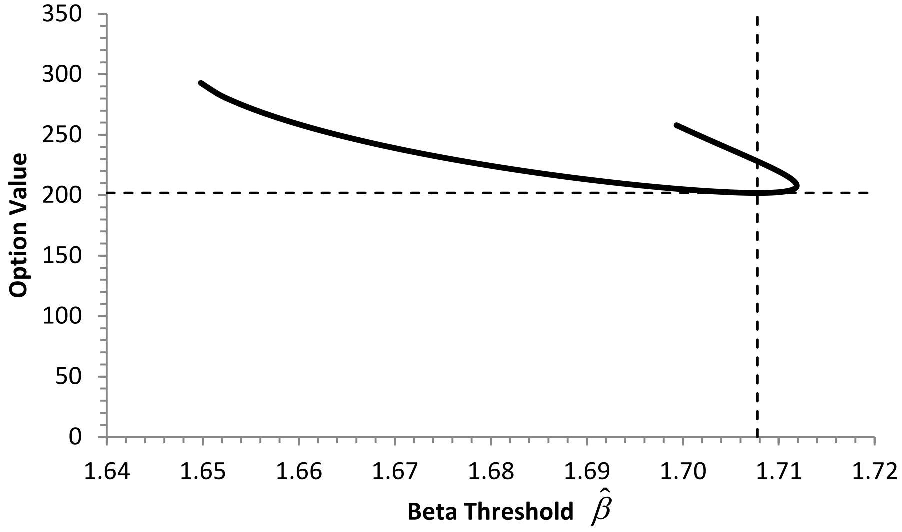
Option Value Profiles of $\hat{F}_0 = F\left(X_0 = 15.0, K_0 = 75.0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ versus $\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}$



The option value profile is constructed according to Figure 3, for $0 \le \hat{K} \le 500$. The minimum option value occurs at $\hat{F}_0 = 201.8942$ for $\hat{K}_0 = 75.73068$.

Figure 4c

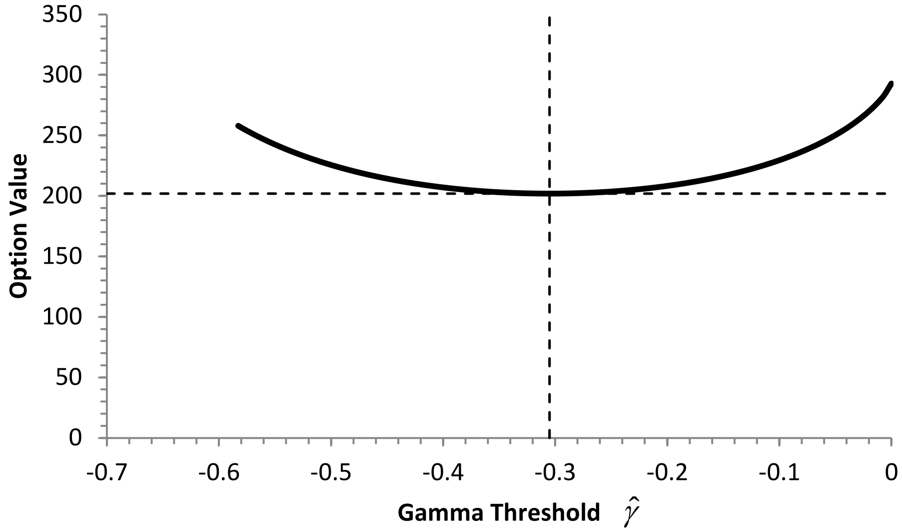
Option Value Profiles of $F\left(X_0 = 15.0, K_0 = 75.0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ versus $\hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}$



The option value profile is constructed according to Figure 3, for $1.64981 \leq \hat{\beta} \leq 1.71187$, where $\hat{\beta} = 1.64981$ and $\hat{\beta} = 1.71187$ represent in Figure 1 point A and the maximum β value, respectively. The minimum option value occurs at $\hat{F}_0 = 201.8942$ for $\hat{\beta}_0 = 1.70777$.

Figure 4d

Option Value Profiles of $F\left(X_0 = 15.0, K_0 = 75.0; \hat{X}, \hat{K}, \hat{\beta}, \hat{\gamma}\right)$ versus $\hat{\gamma}$



The option value profile is constructed according to Figure 3, for $-0.68055 \le \hat{\gamma} \le 0.0$, where $\hat{\gamma} = 0.0$ and $\hat{\gamma} = -0.68055$ represent in Figure 1 points A and A", respectively. The minimum option value occurs at $\hat{F}_0 = 201.8942$ for $\hat{\gamma} = -0.30501$.

Figure 5
Project value for the two stochastic general investment model

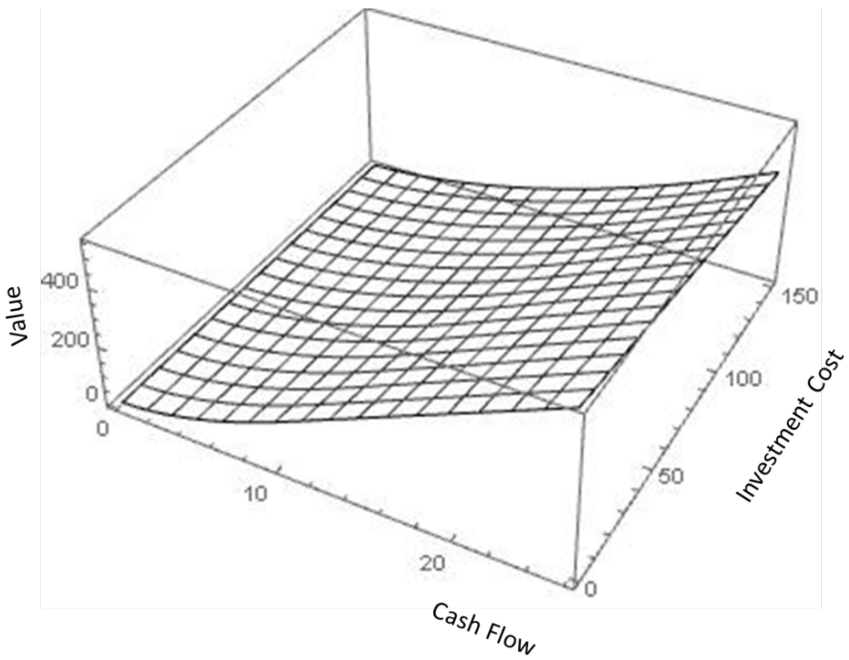
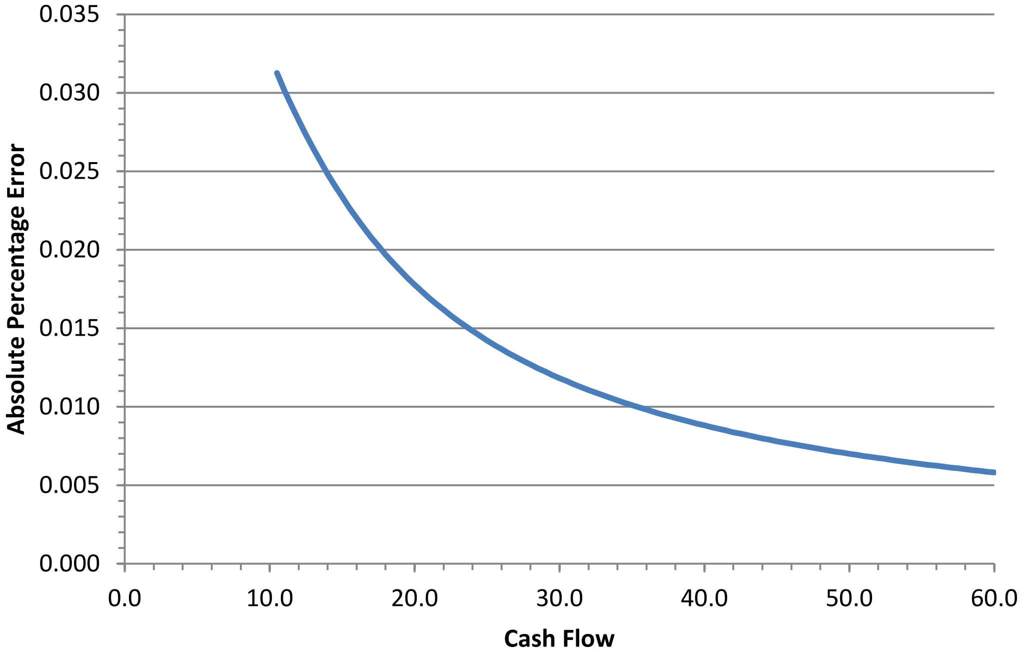


Figure 5 illustrates the project value for $0 \leq X_0 \leq 26$ and $0 \leq K_0 \leq 150$. Project values and the corresponding $\hat{\beta}_0, \hat{\gamma}_0, \hat{\lambda}_0, \hat{X}_0, \hat{K}_0$ values for $X_0 = \{5, 10, \ldots, 25\}$ and $K_0 = \{25, 50, \ldots, 200\}$ are presented in Table 2, together with the computational explanation.

Figure 6a

Absolute percentage error relating to the cash flow smooth-pasting condition based on a finite-difference approximation with $\Delta X_0 = 0.01$

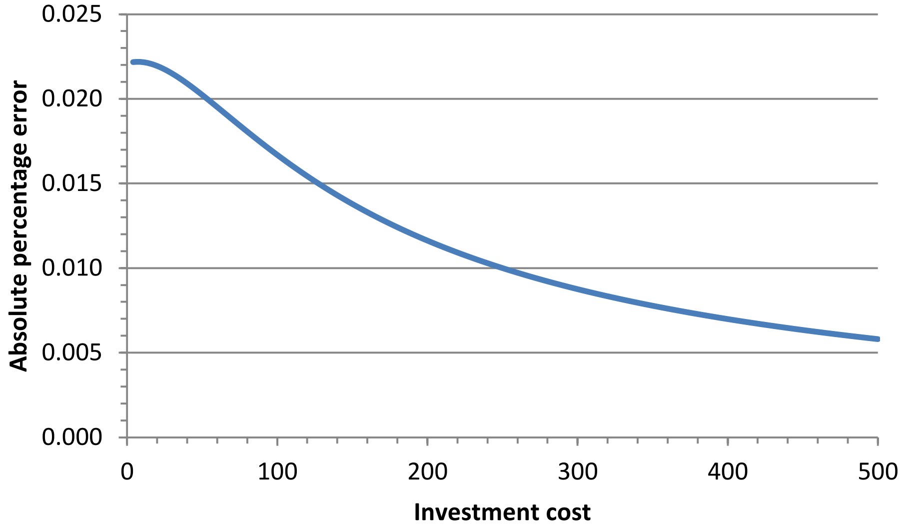


The calculations supporting this figure are processed in the following way. Using an arbitrarily feasible X_0 value, we compute the corresponding K_0 value belonging to the discriminatory boundary. Then the option value $\hat{F}_0$ at $\{X_0, K_0\}$ is determined from (24), (25) and $Q(\hat{\beta}_0, \hat{\gamma}_0) = 0$, (5); since $\{X_0, K_0\} \in \partial\Omega_P$, this can be alternatively obtained from $(\hat{X}_0/\delta_X - f/r - \hat{K}_0)$. The X_0 value is reset to $X_0 - \Delta X_0$ where $\Delta X_0 = 0.01$, while the K_0 value remains unchanged. The option value $\hat{F}_0$ at the revised $\{X_0, K_0\}$ is determined as before. The absolute percentage error is evaluated from the two option values and ΔX_0, (45). Some illustrative values are presented below.

X_0	K_0	$\hat{X}_0$	$\hat{K}_0$	$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_0$	$\hat{F}_0$	$\%\lvert err_{X_0}\rvert$
10.50	4.1147	10.5000	4.1147	1.6574	-0.0260	0.0000	158.3853	
10.49	4.1147	10.5001	4.1159	1.6574	-0.0260	1.8563	158.1353	0.0312
15.00	54.6831	15.0000	54.6831	1.7021	-0.2482	0.0000	220.3169	
14.99	54.6831	15.0004	54.6877	1.7021	-0.2482	1.8829	220.0670	0.0234
20.00	107.8213	20.0000	107.8213	1.7113	-0.3690	0.0000	292.1787	
19.99	107.8213	20.0002	107.8229	1.7113	-0.3690	1.8553	291.9288	0.0178
25.00	159.8284	25.0000	159.8284	1.7115	-0.4377	0.0000	365.1716	
24.99	159.8284	24.9999	159.8270	1.7115	-0.4377	1.8242	364.9216	0.0142
30.00	211.3260	30.0000	211.3260	1.7097	-0.4817	0.0000	438.6740	
29.99	211.3260	29.9996	211.3220	1.7097	-0.4817	1.7986	438.4240	0.0118
35.00	262.5509	35.0000	262.5509	1.7075	-0.5123	0.0000	512.4491	
34.99	262.5509	34.9994	262.5449	1.7075	-0.5123	1.7785	512.1991	0.0101
40.00	313.6131	40.0000	313.6131	1.7054	-0.5348	0.0000	586.3869	
39.99	313.6131	39.9993	313.6055	1.7054	-0.5348	1.7625	586.1370	0.0088
45.00	364.5703	45.0000	364.5703	1.7034	-0.5520	0.0000	660.4297	
44.99	364.5703	44.9991	364.5614	1.7034	-0.5520	1.7497	660.1797	0.0078
50.00	415.4559	50.0000	415.4559	1.7017	-0.5656	0.0000	734.5441	
49.99	415.4559	49.9990	415.4461	1.7017	-0.5656	1.7392	734.2941	0.0070
55.00	466.2906	55.0000	466.2906	1.7002	-0.5766	0.0000	808.7094	
54.99	466.2906	54.9989	466.2799	1.7002	-0.5766	1.7305	808.4595	0.0064
60.00	517.0876	60.0000	517.0876	1.6989	-0.5857	0.0000	882.9124	
59.99	517.0876	59.9989	517.0762	1.6989	-0.5857	1.7232	882.6624	0.0058

Figure 6b

Absolute percentage error relating to the investment cost smooth-pasting condition based on a finite-difference approximation with $\Delta K_0 = 0.04$



The calculations supporting this figure are processed in the following way. Using an arbitrarily feasible K_0 value, we compute the corresponding X_0 value belonging to the discriminatory boundary. Then the option value $\hat{F}_0$ at $\{X_0, K_0\}$ is determined from (24), (25) and $Q\left(\hat{\beta}_0, \hat{\gamma}_0\right) = 0$, (5); since $\{X_0, K_0\} \in \partial\Omega_P$, this can be alternatively obtained from $\left(\hat{X}_0 / \delta_X - f / r - \hat{K}_0\right)$. The K_0 value is reset to $K_0 + \Delta K_0$ where $\Delta K_0 = 0.04$, while the X_0 remains unchanged. The option value $\hat{F}_0$ at the revised $\{X_0, K_0\}$ is determined as before. The absolute percentage error is evaluated from the two option values and ΔX_0, (45). Some illustrative values are presented below.

K_0	X_0	$\hat{K}_0$	$\hat{X}_0$	$\hat{\beta}_0$	$\hat{\gamma}_0$	$\hat{\lambda}_0$	$\hat{F}_0$	$\%\left|err_{K_0}\right|$
20.00	11.8678	20.0000	11.8678	1.6791	-0.1132	0.0000	176.6945	
20.04	11.8678	20.0413	11.8714	1.6792	-0.1134	0.6591	176.6545	0.0219
40.00	13.6553	40.0000	13.6553	1.6952	-0.1986	0.0000	201.3818	
40.04	13.6553	40.0417	13.6591	1.6952	-0.1988	0.6847	201.3418	0.0209
60.00	15.4921	60.0000	15.4921	1.7039	-0.2640	0.0000	227.3018	
60.04	15.4921	60.0416	15.4959	1.7039	-0.2641	0.6985	227.2618	0.0195
80.00	17.3620	80.0000	17.3620	1.7085	-0.3149	0.0000	254.0505	
80.04	17.3620	80.0413	17.3659	1.7085	-0.3150	0.7052	254.0106	0.0181
100.00	19.2550	100.0000	19.2550	1.7108	-0.3554	0.0000	281.3745	
100.04	19.2550	100.0408	19.2589	1.7108	-0.3555	0.7079	281.3345	0.0167
120.00	21.1644	120.0000	21.1644	1.7117	-0.3882	0.0000	309.1099	
120.04	21.1644	120.0403	21.1683	1.7117	-0.3883	0.7083	309.0699	0.0154
140.00	23.0859	140.0000	23.0859	1.7119	-0.4152	0.0000	337.1482	
140.04	23.0859	140.0399	23.0898	1.7119	-0.4153	0.7075	337.1082	0.0143
160.00	25.0166	160.0000	25.0166	1.7115	-0.4379	0.0000	365.4148	
160.04	25.0166	160.0394	25.0204	1.7115	-0.4379	0.7059	365.3748	0.0133
180.00	26.9543	180.0000	26.9543	1.7109	-0.4570	0.0000	393.8574	
180.04	26.9543	180.0390	26.9581	1.7109	-0.4571	0.7041	393.8174	0.0124
200.00	28.8975	200.0000	28.8975	1.7102	-0.4734	0.0000	422.4383	
200.04	28.8975	200.0386	28.9013	1.7102	-0.4735	0.7021	422.3983	0.0116
220.00	30.8452	220.0000	30.8452	1.7093	-0.4877	0.0000	451.1295	
220.04	30.8452	220.0383	30.8489	1.7093	-0.4877	0.7000	451.0895	0.0109
240.00	32.7964	240.0000	32.7964	1.7085	-0.5001	0.0000	479.9101	
240.04	32.7964	240.0380	32.8001	1.7085	-0.5001	0.6980	479.8701	0.0103
260.00	34.7506	260.0000	34.7506	1.7076	-0.5110	0.0000	508.7643	
260.04	34.7506	260.0377	34.7543	1.7076	-0.5111	0.6960	508.7243	0.0097
280.00	36.7072	280.0000	36.7072	1.7067	-0.5208	0.0000	537.6794	
280.04	36.7072	280.0374	36.7108	1.7067	-0.5208	0.6942	537.6394	0.0092
300.00	38.6658	300.0000	38.6658	1.7059	-0.5294	0.0000	566.6457	
300.04	38.6658	300.0372	38.6695	1.7059	-0.5294	0.6924	566.6057	0.0088
320.00	40.6262	320.0000	40.6262	1.7051	-0.5372	0.0000	595.6554	
320.04	40.6262	320.0370	40.6298	1.7051	-0.5372	0.6907	595.6154	0.0083
340.00	42.5881	340.0000	42.5881	1.7043	-0.5443	0.0000	624.7022	
340.04	42.5881	340.0368	42.5917	1.7043	-0.5443	0.6892	624.6622	0.0080
360.00	44.5512	360.0000	44.5512	1.7036	-0.5506	0.0000	653.7808	
360.04	44.5512	360.0366	44.5548	1.7036	-0.5507	0.6877	653.7408	0.0076
380.00	46.5155	380.0000	46.5155	1.7029	-0.5565	0.0000	682.8872	
380.04	46.5155	380.0364	46.5191	1.7029	-0.5565	0.6863	682.8472	0.0073
400.00	48.4807	400.0000	48.4807	1.7022	-0.5618	0.0000	712.0176	
400.04	48.4807	400.0362	48.4843	1.7022	-0.5618	0.6850	711.9776	0.0070
500.00	58.3177	500.0000	58.3177	1.6993	-0.5828	0.0000	857.9425	
500.04	58.3177	500.0356	58.3212	1.6993	-0.5828	0.6795	857.9025	0.0058

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