An Approach for Valuing Portfolios of Interdependent Real Options under both Exogenous and Endogenous Uncertainties

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Abstract

Although the value of portfolios of real options is often affected by both exogenous and endogenous uncertainties, most existing valuation approaches consider only the former and neglect the latter. In this paper we extend our existing approach for modelling and approximating the value of portfolios of interdependent real options to include endogenous, decision- and state-dependent uncertainties, using stochastic processes. In particular, we study a portfolio of options under conditions of four underlying uncertainties. The options are to defer investment, to stage investment, to temporarily halt expansion, to temporarily mothball the operation, and to abandon the project. Two of the underlying uncertainties, decision-dependent cost to completion and state-dependent salvage value, are endogenous, the other two, annual operating revenues and their growth rate, are exogenous. The directly-modelled dynamics of all four uncertainties and the linear integer constraints modelling the real options' interdependencies are integrated in a multi-stage stochastic integer programme. Using a simulation and regression approach to approximate the value of this optimisation problem, we present an efficient valuation algorithm that is more transparent than those used in existing approaches, as it exploits the problem structure to explicitly account for the path dependencies of the state variables. The applicability of the approach to complex investment projects is illustrated by valuing an urban infrastructure investment in London. In this example we show how the optimal value of the portfolio and its single, well-defined options are affected by the initial level of the annual revenues, and by the degrees of exogenous and endogenous uncertainty.

Keywords: Real options portfolio, Exogenous uncertainty, Decision-dependent

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uncertainty, State-dependent uncertainty, Simulation-and-regression approach

1. Introduction

A fundamental issue in real options analysis and decision-making under uncertainty is how to account correctly and adequately for the multiple sources of uncertainty occurring in most practical real-life situations. In these situations it is generally assumed that the effective sources of uncertainty are purely exogenous and, as such, are independent of both the actions taken by the decision maker and the state of the underlying system affected by these decisions. For example, in the case of investment in a new wind farm, while the wind farm's performance depends on factors such as location, time of day and the wind turbines' height, parameters such as the wind speed to which the turbines will be exposed to, and consequently the amount of power generated, are independent of the investor's decision of whether to build the wind farm or not. Likewise, if the amount of power generated by such wind farm is sufficiently small and/or the relevant wholesale electricity market to which the power is sold is comparatively large, then the underlying wholesale price of electricity, and consequently the investor's revenues are also independent of the investor's decision.

There are, however, many practical situations in which the relevant sources of uncertainty are endogenous, i.e. dependent on the decision maker's actions or the underlying system's state, or both. In the case of the wind farm example, if the above-mentioned conditions are violated, i.e. if the new wind farm is sufficiently large and/or the electricity market relatively small, then the introduction of a new wind farm will affect the wholesale price of electricity and hence the investor's future revenues. Similarly, although the "off-the-shelf" cost of new wind turbines may be known and a feasibility study may provide a construction cost estimate, the actual cost of building a new wind farm will not be known until the investor actually builds it. During the building process, the investor reveals and learns its true capital cost. If the investor wants to sell the wind farm at the end of its lifetime, in the absence of a second hand market, the resale value will depend on its "state", which may include such factors as its lifetime, asset value, wear and tear, and decommissioning cost.

Despite the ubiquity of exogenous and endogenous uncertainties in many real-life situations, there remains a need for a unified approach that accounts for both when real options analysis is used to evaluate practical investment problems. Including both types of uncertainty in a real options approach has rarely been studied in the related literature (Ahsan and Musteen, 2011). Although portfolio of real options approaches have been applied when there is only exogenous uncertainty, there is a need to include both types because that enables decision-makers to manage the two uncertainty types simultaneously (Otim and Grover, 2012). Some authors have therefore suggested that future work should examine the relationship and interactions between different sources of uncertainty and the portfolio's individual options. For example, Tiwana et al. (2006) stated that future research should investigate how the comparative performance of individual real options is affected differentially by multiple sources of uncertainty, and Li et al. (2007) called for studies to investigate how investment decisions are affected individually and interactively by multiple uncertainty sources. More recently, the critical review of Trigeorgis and Reuer (2017) has suggested four extensions, three of which are addressed here: portfolios of interdependent real options, multiple sources of uncertainty, and endogenous resolution of uncertainty through learning.

This paper introduces a valuation approach for portfolios of interdependent real options under both exogenous and endogenous uncertainties. Considering the problem of a sequential and partially reversible investment project, we study a portfolio of options: to defer investment; stage investment; temporarily halt expansion; temporarily mothball the operation; and abandon the project during either construction or operation. In the problem studied here, the portfolio's value is affected by four underlying uncertainties. Of these, the project's actual cost to completion and its salvage value, are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the annual (operating) revenues and their growth rate evolve exogenously. The portfolio of real options approach of Maier et al. (2016) proposed a multi-stage stochastic integer programming approach using influence diagrams and simulation-and-regression. To value such a complex portfolio under both types of uncertainty, we extend this approach to include endogenous sources of uncertainty. The directly-modelled dynamics of all four underlying uncertainties and the linear integer constraints modelling the interdependencies between real options are also integrated in this optimisation problem.

To approximate the value of this optimisation problem, we extend the simulation-and-regression-based valuation algorithm developed by Maier et al. (2016) to include endogenous uncertainty. Unlike the algorithms of Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), which are plain extensions of the algorithm proposed by Longstaff and Schwartz (2001) for American-style options, our algorithm takes into account the numerical implications of the state variables' path-dependencies on the accuracy of the approximation. We account for the negative numerical implications by exploiting the structure of the problem to be solved through dynamically and appropriately adapting the set of basis functions used in the parametric regression. Using an illustrative example of an urban infrastructure investment in London, we investigate the sensitivity of the optimal value of the portfolio and its individual options to the level of the initial annual revenues, as well as to the degrees of exogenous and endogenous uncertainty. In contrast to Miltersen and Schwartz (2007), who noted that the numerical solution techniques used by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008) "cannot easily handle temporary suspensions of the" investment project nor isolate the options' values, this example demonstrates that our approach is flexible and powerful, and can be applied to value both complex portfolios and their individual real options under both types of uncertainty.

The rest of this article is organised as follows: Section 2 reviews the relevant literature with an emphasis on the operational research as well as on the finance and management literature. Section 3 describes the investment problem by specifying both the portfolio of interdependent real options (Subsection 3.1) and set of uncertainties (Subsection 3.2) considered in this work. In Section 4 we present the modeling and valuation approach together with the simulation-and-regression-based valuation algorithm (Subsection 4.3). The approach and the algorithm are then applied to the real-case of a district heating network expansion investment in the London borough of Islington (Section 5). Results are presented and discussed in Subsection 5.4. Finally, some concluding remarks and suggestions for future research are provided in Section 6.

2. Literature review

The classification of uncertainties into exogenous and endogenous has received considerable attention in different branches of literature, and importantly in the operational research as well as in the finance and management literature. With regard to the former, to the best of our knowledge, the work of Jonsbråten et al. (1998) was the first to classify the formulation of stochastic programs into "standard" formulations with decision independent random variables and "manageable" formulations, in which the distribution of the random variables is dependent on decisions. Calling the former "exogenous uncertainty" and the latter "endogenous uncertainty" (Goel and Grossmann, 2004), Goel and Grossmann (2006) specified the way in which decisions can affect the stochastic process – which describes the evolution of an uncertain parameter (see Kirschenmann et al. (2014)) – by presenting two types of endogenous uncertainty. The first is when the decision alters the probability distribution (e.g. parameters of family), whereas the second relates to the decision affecting the timing of uncertainty resolution, a process often described as information revelation.

Considering the above specification of endogenous uncertainties, several relevant works have appeared in the operations research literature over the last few decades. As for the first type of endogenous uncertainty, Pflug (1990) was the first to take into account decision dependent probabilities in a stochastic optimization problem by considering a controlled Markov chain where the transition operator depends on the control, i.e. the decision. Other relevant articles related to this type are in the context of stochastic network problems (Held and Woodruff, 2005; Peeta et al., 2010), global climate policy (Webster et al., 2012) and natural gas markets (Devine et al., 2016). By contrast, the second type of endogenous uncertainty has received considerable more attention in the literature. The first work related to this type was (Goel and Grossmann, 2004), who presented a stochastic programming approach for the planning of an investment into a gas field with uncertain reserves represented through a decision-dependent scenario tree. Similar studies in terms of both uncertainty representation and application domain include (Tarhan et al., 2009; Terrazas-Moreno et al., 2012; Gupta and Grossmann, 2014). Other relevant works include the optimisation of R&D project portfolios (Solak et al., 2010) and clinical trial planning in the pharmaceutical R&D pipeline (Colvin and Maravelias, 2008, 2010, 2011).

Moreover, several works have incorporated both the second type of endogenous uncertainty and exogenous uncertainty in the formulation of stochastic programmes. For generic problem formulations and solution strategies see the rather theoretical works of Dupačová (2006); Goel and Grossmann (2006); Tarhan et al. (2013). Recent advances and summaries over existing computational strategies have been presented by Apap and Grossmann (2016); Grossmann et al. (2016). However, although almost all publications of this branch of literature refer to the classification and specification of Jonsbråten et al. (1998) and Goel and Grossmann (2006), respectively, Mercier and Van Hentenryck (2008) argued that problems in which merely the observation of an uncertainty depends on the decisions, but the actual underlying uncertainty is still exogenous (= second type of endogenous uncertainty) should be classified as "stochastic optimization problems with exogenous uncertainty and endogenous observations". According to their redefined classification, problems with exogenous and the first type of endogenous uncertainty are referred to as *purely exogenous* and *purely endogenous*, respectively.

Unlike the operational research literature, the finance and management literature appears to be rather ambiguous, even somewhat inconsistent when it comes the classification of uncertainties. Indeed, although both the classification of uncertainties into exogenous or endogenous (Hirshleifer and Riley, 1979) and the importance of taking this distinction into account have been widely recognised in this branch of literature, especially in works related to the field of real options (Bowman and Hurry, 1993; Folta, 1998; Li et al., 2007; Li, 2007; Oriani and Sobrero, 2008), there is no clear and widely accepted definition. For example, Pindyck (1993); Dixit and Pindyck (1994) distinguish between technical and input cost uncertainty while noting their different effects on investment decisions as these incentivise investing and waiting, respectively. Building upon this distinction, McGrath (1997) called for a third form of uncertainty that lies in-between, and Folta (1998) stated (italics in their work) that "*exogenous uncertainty can be decreased by actions of the firm*", while "*endogenous uncertainty is largely unaffected by firm actions*". Furthermore, McGrath et al. (2004) refers to the exogenous and endogenous resolution of uncertainty through the passing of time and learning, respectively. By contrast, Van der Hoek and Elliott (2006) took note of uncertainties that are state-dependent rather than dependent on the option holder's actions (i.e. decisions).

A few studies aimed at presenting a more continuous classification of how various sources of uncertainty are affected by the option holder's actions. Based on the overview of Micalizzi and Trigeorgis (1999), Scialdone (2007) presented an "uncertainty-mapping" (similar to Bräutigam et al. (2003)) that indicates the extent to which uncertainty categories (e.g. operational, market demand, price, financial, and industry) are exogenous or endogenous. In addition, the author qualitatively showed the categories' relevance to single well-defined real options (e.g. options to wait, stage, switch and abandon). While these studies have linked different sources of uncertainty to individual option's relative performance, they have presented rather unsatisfactory and ambiguous qualitative approaches; in contrast, this paper takes a fundamentally different approach by presenting a holistic and general portfolio of real options approach that accounts for multiple, possibly interacting, exogenous and endogenous sources of uncertainty, as well as their influence on the performance of the portfolio's interdependent real options.

Various researchers have applied real option approaches to valuation problems with both exogenous and endogenous uncertainty[1]. Generalising the work of Roberts and Weitzman (1981), Pindyck (1993) evaluated a staged-investment with technical (endogenous) and input cost (exogenous) uncertainty using a finite difference method. Other relevant articles considered both types of uncertainty in the context of information technology investment projects (Schwartz and Zozaya-Gorostiza, 2003), patents and R&D projects (Schwartz, 2004), pharmaceutical R&D projects (Hsu and Schwartz, 2008; Pen-

[1]Interestingly, Adner and Levinthal (2004); Cuypers and Martin (2007, 2010) argued that real options theory cannot be applied to problems with endogenous uncertainty since, amongst other things, the real options' discrete nature would be eroded

nings and Sereno, 2011), product platform flexibility planning (Jiao, 2012), and nuclear power plant investments (Zhu, 2012). With regard to state-dependent uncertainty, Sbuelz and Caliari (2012) studied the influence of state-dependent cashflow volatility on the investment decisions related to corporate growth options, whereas Palczewski et al. (2015) examined optimal portfolio strategies under stock price dynamics with state-dependent drift. Despite having applied real option approaches to valuation problems with both types of uncertainty, these quantitative approaches are rather inflexible and restricted. Furthermore, they do not address this paper's problem in such a holistic and unified way we do.

3. The investment problem

In this section, we present the investment problem studied here by specifying both the portfolio of interdependent real options and the underlying set of uncertainties.

3.1. Portfolio of interdependent real options

We study the problem of a decision maker wanting to determine the value of a sequential and partially reversible investment in a project whose stage-wise expansion (development) can be deferred, temporarily halted and/or abandoned altogether, and, once operating, whose cash flow generating asset can be used until the end of the asset's project life in T_3^{max} months, temporarily mothballed and/or abandoned early.

Representing the set of flexibilities as a portfolio of interdependent real options, the portfolio's single, well-defined options are:

(a) Option to defer investment: Instead of starting immediately at time 0, the decision maker may choose to defer the start of the expansion until the expiration of the right to undertake this investment in T_1^{max} months, without incurring any direct costs associated with deferring.

(b) Option to stage investment: As the development takes time to complete, the decision maker can invest at a rate of $0 < C_t \leq I^{max}$ in period t as long as the remaining investment cost at the beginning of period t, K_t, is greater than 0, i.e. while the project is under construction, where I^{max} and K_0 are the maximum rate of investment and the initial (expected) cost of completion, respectively.

  (i) Option to temporarily halt expansion: If conditions turn out to be unfavourable, the decision maker can halt the expansion (i.e. set $C_t = 0$) at a cost of $C^{d,h}$, maintain the halted expansion for a total of T_2^{max} months at a monthly cost of C^h, and, if conditions become favourable again, resume development at a cost of $C^{h,d}$.

(ii) Option to abandon the project during construction (i.e. when $K_t > 0$): Whether developing or halted, the project can be permanently abandoned at any given point in time t for the salvage value X_t, which is assumed to contain any costs that abandonment during construction involves.

(c) Option to temporarily mothball the operation: If operation of the asset becomes uneconomic, the decision maker can mothball the operating asset at a cost of $C^{o,m}$, maintain the mothballed asset at a monthly cost of C^m, and, if conditions become favourable again, reactivate the asset at a cost of $C^{m,o}$.

(d) Option to abandon the project during operation (i.e. when $K_t = 0$): Whether operating or mothballed, the decision maker retains the right to permanently abandon the project at any time t for its salvage value X_t, which is assumed to contain all costs related to abandoning during operation.

The above described individual real options are well-known and have been widely examined in the real options literature, for overviews see Trigeorgis (1993b, 1996). The first one, (a), is arguably the most-widely studied type of real option in the literature[2], e.g. see Trigeorgis (1993a); Tsitsiklis and Van Roy (2001); Longstaff and Schwartz (2001). Sequential investments, as per (b), have been studied in (Roberts and Weitzman, 1981; Majd and Pindyck, 1987; Pindyck, 1993; Trigeorgis, 1993a). Of these, the works of Majd and Pindyck (1987) and Pindyck (1993) explicitly and implicitly, respectively, considered the possibility to temporarily halt and later resume expansion – (b-i) – yet these authors ignored any direct costs associated with these decisions. With regard to (b-ii), these four works also allowed for abandonment during construction, but they neglected the project's salvage value, which is over-simplistic; Trigeorgis (1993b, 1996) referred to (b-ii) as the "option to default during construction". Categorised as an option "to alter operating scale" (Trigeorgis, 1993b), Brennan and Schwartz (1985) valued the option to temporarily shut down operations of a copper mine, which is practically the same as (c). Lastly, several works have analysed the flexibility related to (d). For example, building upon (Robichek and Van Horne, 1967; Dyl and Long, 1969) and considering an existing project with uncertain salvage value, Myers and Majd (1990) valued such option as an American put; Trigeorgis (1993a,b) referred to (d) as the "option to switch use", where the salvage value represents the project's value in its best alternative use.

[2] The continuous- and discrete-time version of this option are generally referred to as American and Bermudan call option, respectively

3.2. *Characterisation of uncertainties*

This study considers four sources of uncertainty – also referred to as stochastic factors or random variables – denoted by K_t, V_t, μ_t and X_t, representing the project's actual cost to completion at time t, the annual revenues (net cash flow) generated by operation in period t, the growth rate of revenues in t and the salvage value at time t, respectively. The first and the fourth uncertainty are decision- and state-dependent, respectively. These uncertainties evolve endogenously, whereas the dynamics of the second and third factor are exogenous. Each of the four factors is described by a discrete diffusion process as follows:

The dynamic of the project's actual cost to completion, K_t , depends on the rate of investment, $0 \leq C_t \leq I^{max}$, chosen by the decision maker, and is given by:

$$K_{t+\Delta} = K_t - C_t\Delta + \sigma_{k_1}\sqrt{C_t K_t \Delta}\epsilon^{k_1}_{t+\Delta} + \sigma_{k_2} K_t \sqrt{\Delta}\epsilon^{k_2}_{t+\Delta}, \qquad (1)$$

where Δ is the time step, and σ_{k_1} and σ_{k_2} are the degrees of technical and input cost uncertainty, respectively. The above equation is a discrete approximation of the controlled diffusion process proposed by Pindyck (1993). As in (Schwartz, 2004), we focus here on the effect of the technical uncertainty (third term), thus set $\sigma_{k_2} = 0$ in the following analysis, thereby eliminate the influence of the input cost uncertainty (fourth term).

The annual revenues received at time t for operation between t and $t + \Delta$, V_t, and their rate of growth, μ_t, evolve exogenously according to:

$$V_{t+\Delta} = e^{-\kappa_v\Delta}V_t + \left(1 - e^{-\kappa_v\Delta}\right)V_0(1 + \mu_t t) + \sigma_v\sqrt{\frac{1 - e^{-2\kappa_v\Delta}}{2\kappa_v}}\epsilon^v_{t+\Delta}, \qquad (2)$$

$$\mu_{t+\Delta} = e^{-\kappa_\mu\Delta}\mu_t + \left(1 - e^{-\kappa_\mu\Delta}\right)\overline{\mu} + \sigma_\mu\sqrt{\frac{1 - e^{-2\kappa_\mu\Delta}}{2\kappa_\mu}}\epsilon^\mu_{t+\Delta}, \qquad (3)$$

where σ_v and σ_μ are the standard deviations of changes in V_t and μ_t, respectively, as well as κ_v and κ_μ are positive mean reversion coefficients that describe the rate at which the corresponding factors converge to their linear trend, $V_0(1 + \mu_t t)$, and long-term average, $\overline{\mu}$, respectively. The nested model (2)-(3) is similar to the one of Schwartz and Moon (2001), who also used an *Ornstein-Uhlenbeck process*[3] to describe the evolution of μ_t. For the evolution of V_t, however, we apply an (arithmetic) Ornstein-Uhlenbeck model with linear – time-varying and stochastic – trend , which is adapted from the geometric

[3] This is the "simplest mean-reverting process" according to Dixit and Pindyck (1994)

mean reversion with exponential – constant and deterministic – trend of Metcalf and Hassett (1995).

The state-dependent salvage value obtained for abandoning the project at time t, X_t, is a function of both the expected asset value at time t, Z_t, which is deterministic and depends on the state of the asset, and a *homoscedastic* noise (i.e. error independent of the state) term[4], which is random and describes the percentage deviation as follows:

$$X_{t+\Delta} = Z_{t+\Delta} + \sigma_x Z_{t+\Delta} \epsilon^{x}_{t+\Delta}, \tag{4}$$

where σ_x is the standard deviation of X_t. Unlike the existing approaches that allow for stochastic salvage (or abandonment) values, e.g see the works of Myers and Majd (1990); Adkins and Paxson (2017) and literature cited therein, which assume these values evolve exogenously, we introduce a state-dependent salvage value, as suggested in (Van der Hoek and Elliott, 2006), thus representing one of the many practical situations in which the salvage value depends on endogenous factors (e.g. see (Trigeorgis, 1993a,b)). It is important to note that by "state" we actually mean its "resource" component (see description in Subsection 4.1), rather than its "information" component, specifically the latter's three stochastic factors of (1)-(3), which are, of course, state-dependent too because Markovian.

While $\epsilon^{k_1}_{t+\Delta}$ is uncorrelated, $\epsilon^{k_2}_{t+\Delta}$, $\epsilon^{v}_{t+\Delta}$, $\epsilon^{\mu}_{t+\Delta}$ and $\epsilon^{x}_{t+\Delta}$ are correlated standard normal random variables (mean 0, variance 1) whose increments are independently and identically distributed, with correlation matrix (= covariance matrix Σ):

$$\begin{pmatrix} 1 & \rho_{k_2,v} & \rho_{k_2,\mu} & \rho_{k_2,x} \\ \rho_{k_2,v} & 1 & \rho_{v,\mu} & \rho_{v,x} \\ \rho_{k_2,\mu} & \rho_{v,\mu} & 1 & \rho_{\mu,x} \\ \rho_{k_2,x} & \rho_{v,x} & \rho_{\mu,x} & 1 \end{pmatrix}$$

4. Methods

This section contains the modelling of the investment problem as a sequential decision-making problem, the formulation of the valuation problem as a multi-stage stochastic integer programme, and the description of the valuation algorithm applied.

[4]For a brief description of the modeling of both *homoscedastic* and *heteroscedastic* (i.e. state-dependent) noise see (Powell, 2011).

4.1. Modelling

The flexibilities available to the decision maker when having the portfolio of interdependent real options of Subsection 3.1 are shown by the influence diagram in Figure 1. It contains nine nodes of which five are decision nodes and four are terminal nodes,

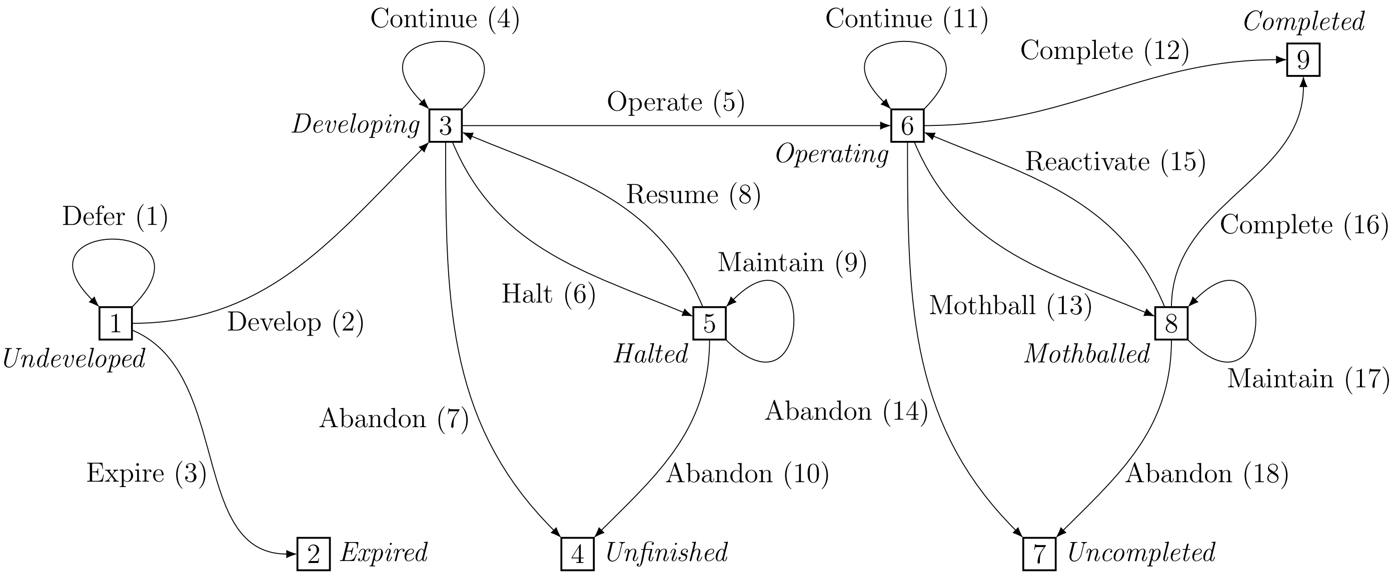


Figure 1: Flexibilities provided by portfolio of interdependent real options.

as well as 18 transitions that link these nodes. The set of nodes and transitions is given by $\mathcal{N} = \{1, 2, \ldots, 9\}$ and $\mathcal{H} = \{1, 2, \ldots, 18\}$, respectively, and the duration of transition $h \in \mathcal{H}$ is Δ_h months.

The state of the investment project at time t is written as:

$$S_t = (\underbrace{t, N_t, T_t, Q_t}_{R_t}, \underbrace{K_t, V_t, \mu_t, X_t}_{I_t}), \tag{5}$$

where $N_t \in \mathcal{N}$ is the (decision or terminal) node at time t; T_t is the time left (in months) at t to defer investment ($h = 1$), halt expansion ($h \in \{6, 9\}$) or use the developed asset ($h \in \{11, 13, 15, 17\}$); Q_t is the amount invested up to time t; and K_t, V_t, μ_t and X_t are as defined in Subsection 3.2. The first four variables of S_t are part of the resource state R_t, which is deterministic, whereas the information state I_t is made up of the problem's four random variables, two of which are exogenous and two are endogenous, decision- and state-dependent.

The binary decision variables associated with the transitions available at decision node N_t at time t, $a_t = (a_{th})_{h \in b^D(N_t)}$, have to satisfy the feasible region $\mathcal{A}_{S_t}$, which is defined by the following set of linear integer constraints:

$$\begin{cases} \sum_{h \in b^D(N_t)} a_{th} = 1, & \forall N_t \in \{1,3,5,6,8\}, & (6) \\ a_{t1} T_1^{max} < T_t + T_1^{max}, & & (7) \\ (1 - a_{t5} - a_{t7}) K_0 < K_t + K_0, & & (8) \\ a_{t5} K_t = 0, & & (9) \\ a_{th} T_2^{max} < T_t + T_2^{max}, & \forall h \in \{6, 9\}, & (10) \\ a_{th} T_t = 0, & \forall h \in \{12, 16\}, & (11) \\ (1 - a_{th}) T_3^{max} < T_t + T_3^{max}, & \forall h \in \{12, 16\}, & (12) \end{cases}$$

where $a_{th} \in \{0, 1\}, \forall h \in \mathcal{H}$, and

$$b^D(N_t) = \begin{cases} \{1,2,3\}, & \text{if } N_t = 1, \\ \{4,5,6,7\}, & \text{if } N_t = 3, \\ \{8,9,10\}, & \text{if } N_t = 5, \\ \{11,12,13,14\}, & \text{if } N_t = 6, \\ \{15,16,17,18\}, & \text{if } N_t = 8, \\ \{\}, & \text{otherwise.} \end{cases} \tag{13}$$

The transition function, which is generically written as $S^M(S_t, a_t, W_{t+\Delta_h})$ and describes the evolution of S_t from t to $t + \Delta_h$ after having made decision a_{th} with respect to $\mathcal{A}_{S_t}$ and learned new information $W_{t+\Delta_h}$, is composed of the resource transition function $S^R(\cdot)\colon R_t \to R_{t+\Delta_h}$ as well as the information transition function $S^I(\cdot)\colon I_t \to I_{t+\Delta_h}$. With regard to the former, the transition of t is trivial as it simply evolves to $t+\Delta_h$; the transition of N_t is implicitly given by the adjacency matrix of the directed (and cyclic) graph $(\mathcal{N}, \mathcal{H})$ underlying the influence diagram (matrix not shown here in order to save space); the transition of T_t is given by:

$$T_{t+\Delta_h} = \begin{cases} \max\{T_t - \Delta_h, 0\}, & \text{if } a_{th} = 1, h \in \mathcal{H}_1, \\ T_2^{max}, & \text{if } a_{t2} = 1, \\ T_3^{max} - \Delta_5, & \text{if } a_{t5} = 1, \\ T_t, & \text{otherwise,} \end{cases} \tag{14}$$

where $T_0 = T_1^{max}$ and $\mathcal{H}_1 = \{1, 6, 9, 11, 13, 15, 17\}$; and the transition of Q_t is given by:

$$Q_{t+\Delta_h} = \begin{cases} Q_t + I^{max}\Delta_h, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ Q_t, & \text{otherwise,} \end{cases} \tag{15}$$

where $Q_0 = 0$. In contrast to the deterministic transitions of the variables of R_t, the information state variables evolve generally stochastically according to:

$$K_{t+\Delta_h} = \begin{cases} \max\left\{K_t - I^{max}\Delta_h + \sigma_{k_1}\sqrt{I^{max}K_t\Delta_h}\epsilon^{k_1}_{t+\Delta_h}, 0\right\}, & \text{if } a_{th} = 1, h \in \{2, 4, 8\}, \\ K_t, & \text{otherwise,} \end{cases} \tag{16}$$

$$V_{t+\Delta_h} = e^{-\kappa_v\Delta_h}V_t + \left(1 - e^{-\kappa_v\Delta_h}\right)V_0(1 + \mu_t t) + \sigma_v\sqrt{\frac{1 - e^{-2\kappa_v\Delta_h}}{2\kappa_v}}\epsilon^v_{t+\Delta_h}, \tag{17}$$

$$\mu_{t+\Delta_h} = e^{-\kappa_\mu\Delta_h}\mu_t + \left(1 - e^{-\kappa_\mu\Delta_h}\right)\overline{\mu} + \sigma_\mu\sqrt{\frac{1 - e^{-2\kappa_\mu\Delta_h}}{2\kappa_\mu}}\epsilon^\mu_{t+\Delta_h}, \tag{18}$$

$$X_{t+\Delta_h} = Z_{t+\Delta_h}(S_{t+\Delta_h}) + \sigma_x Z_{t+\Delta_h}(S_{t+\Delta_h})\epsilon^x_{t+\Delta_h}, \tag{19}$$

where $Z_t(S_t)$, the expected asset value at time t, is given by:

$$Z_t(S_t) = \begin{cases} -\alpha I^{max}, & \text{if } N_t = 3 \wedge K_t > 0, \\ \gamma Q_t, & \text{if } N_t = 3 \wedge K_t = 0, \\ -\beta I^{max}, & \text{if } N_t = 5, \\ \gamma Q_t e^{-\xi(T_3^{max} - T_t)}, & \text{if } N_t = 6, \\ \delta Q_t e^{-\xi(T_3^{max} - T_t)}, & \text{if } N_t = 8, \\ 0, & \text{otherwise,} \end{cases} \tag{20}$$

where $\alpha \geq 0$ and $\beta \geq 0$ define the expected abandonment cost when *Developing* or *Halted*, respectively; $\gamma \geq 0$ and $\delta \geq 0$ are pay-out ratios determining the expected asset value when *Operating* or *Mothballed*, respectively; and ξ is the monthly depreciation rate describing the asset value's decline over time. It is important to note that since the process (16) and the processes (17)-(19) are uncorrelated ($\sigma_{k_2} = 0$), the optimal rate of investment, C_t, in (1) and consequently in (16) is either I^{max} or 0, as analytically shown by Pindyck (1993); Schwartz and Zozaya-Gorostiza (2003) and referred to as "bang-bang policy" by Schwartz (2004).

Lastly, the pay-off function is represented by:

$$\begin{aligned}\Pi_t(S_t, a_t) = &- I^{max}(\Delta_2 a_{t2} + \Delta_4 a_{t4}) + V_t(a_{t5} + a_{t11}) + X_t(a_{t7} + a_{t10} + a_{t14} + a_{t18}) \\ &- C^{d,h} a_{t6} - (C^{h,d} + I^{max}\Delta_8)a_{t8} - C^h \Delta_9 a_{t9} + X_t(a_{t12} + a_{t16}) \\ &- C^{o,m} a_{t13} + (V_t - C^{m,o})a_{t15} - C^m \Delta_{17} a_{t17}.\end{aligned} \tag{21}$$

Note that, for the sake of simplicity, it is assumed that completing the project after T_3^{max} months of use – by making either transition 12 (when *Operating*) or transition 16 (when *Mothballed*) – results in a pay-off of X_t, which thus represents the project's residual value.

4.2. Valuation problem

Having fully modeled the sequential decision-making problem, as described by Maier et al. (2016), the value of the portfolio of interdependent real options at time 0 given state S_0, $G_0(S_0)$, is obtained by solving the following multi-stage stochastic integer programme:

$$G_0(S_0) \;=\; \max_{(a_t)_{t\in\mathcal{T}}} \mathbb{E}\Big[\textstyle\sum_{t\in\mathcal{T}} e^{-rt}\Pi_t(S_t, a_t)\big|S_0\Big], \tag{22}$$

where $S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$, $a_t = (a_{th})_{h\in b^D(N_t)}$, $\mathcal{T}$ is the set of decision times (or decision epochs), $S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h})$, and r is the risk-free rate.

Applying Bellman's "principle of optimality", the optimisation problem in (22) can be solved recursively with the optimal value of being in state S_t given by:

$$G_t(S_t) \;= \max_{a_t} \Pi_t(S_t, a_t) + \mathbb{E}\Big[e^{-r\Delta_h} G_{t+\Delta_h}(S_{t+\Delta_h})\big|S_t, a_t\Big] \tag{23}$$

$$\text{s.t.}\;\; a_{th} \in \{0, 1\}, \qquad \forall h \in b^D(N_t), \tag{24}$$

$$a_t \in \mathcal{A}(S_t), \tag{25}$$

$$S_{t+\Delta_h} = S^M(S_t, a_t, W_{t+\Delta_h}), \quad \forall h \in b^D(N_t), \tag{26}$$

where $W_{t+\Delta_h} = \big(\epsilon^{k_1}_{t+\Delta}, \epsilon^{v}_{t+\Delta}, \epsilon^{\mu}_{t+\Delta}, \epsilon^{x}_{t+\Delta}\big)$ describes the information that arrives between time t and $t+\Delta_h$. The aim is then to determine $G_0(S_0)$, given the boundary (or terminal) condition $G_t(S_t) = 0, \forall t \in \mathcal{T}, N_t \in \{2, 4, 7, 9\}$.

4.3. The simulation-and-regression-based valuation algorithm

In order to approximate the value of the portfolio of interdependent real options given by the optimisation problem (23)-(26), we extend the simulation-and-regression-based valuation algorithm developed by Maier et al. (2016) to include endogenous sources of uncertainty. Furthermore, our proposed algorithm is both a generalisation and formalisation of the solution procedures offered by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), which are plain extensions of the algorithm proposed by Longstaff and Schwartz (2001) for single American-style options. While our algorithm also consists of an induction procedure with a forward and a backward pass as in (Maier et al., 2016), the procedure's individual steps were adapted to include endogenous uncertainty and to explicitly account for the numerical implications of the state variables' path dependencies on the accuracy of the approximation. See Appendix A for a description of the solution procedure's steps in which we assumed, for the sake of simplicity, that $\Delta_h = \Delta_1, \forall h \in \{2, 4, 6, 8, 9\}$ and $\Delta_h = \Delta_5, \forall h \in \{11, 13, 15, 17\}$.

The forward induction procedure generates the discrete state space $\mathcal{S}_t$ through "exploration" of the resource state space $\mathcal{R}_t$ and simulation (Monte Carlo sampling) of the information state space $\mathcal{I}_t$ for all $t \in \mathcal{T}$. However, in addition to the path dependency of R_t because of the sequential decision process underlying the portfolio of real options (Maier et al., 2016), now both R_t and I_t are path-dependent because of the decision-dependent cost to completion, K_t. In fact, whether a resource state and its corresponding information state can be reached at time t (and are therefore part of $\mathcal{R}_t$ and $\mathcal{I}_t$, respectively) does not solely depend on the sequence of decisions made up to this point, but also on how K_t evolves stochastically; for instance, it might be that a particular R_t can be reached in only $\Omega_{R_t} \subseteq \Omega$, where $|\Omega_{R_t}| < |\Omega|$. Moreover, since the stochastic cost to completion can be directly translated into a stochastic time to completion, the decision times in $\mathcal{T}$ are also path-dependent.

As a strategy in our procedure to overcome the curse of dimensionality related to both $\mathcal{I}_t$ and the outcome space (for a discussion see Powell (2011); Maier et al. (2016); Nadarajah et al. (2017)), whenever needed we approximate the conditional expectation in (23), which represents the continuation value, by the following parametric model:

$$\hat{\Phi}_t^{L_{S_t}}(S_t, a_t) = \sum_{l=0}^{L_{S_t}} \hat{\alpha}_{tl}(S^R(R_t, a_t))\phi_{S_t l}(I_t), \tag{27}$$

where L_{S_t} is the model's dimension, $\{\phi_{S_t l}\}_{l=0}^{L_{S_t}}$ are called basis functions (or features), and the coefficients $\big(\hat{\alpha}_{tl}(S^R(R_t, a_t))\big)_{l=0}^{L_{S_t}}$ are obtained by the least-squares regression in (A.4). Unlike the parametric model of Maier et al. (2016), here L_{S_t} and $\phi_{S_t l}$ depend on S_t, which enables us to reduce the model's dimension if $N_t = 1$ ($N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\}$) by omitting functions of K_t and X_t (K_t) in the regression, thus reducing computational cost. Importantly, the continuous function (27) is determined separately for each relevant and feasible decision a_t, given state S_t, whilst taking into account the set of paths Ω_{R_t} in which R_t can actually be reached. By contrast, in the setting of Maier et al. (2016), every R_t can be reached along each path $\omega \in \Omega$ as it only considered exogenous uncertainty.

The valuation procedure shown in Algorithm 1 applies a standard backward induction to approximate the value of the multi-stage stochastic integer programme (23)-(26). Starting at $t = \max \mathcal{T}$ and moving backwards to $t = \min \mathcal{T} \setminus 0$, for each state $S_t \in \mathcal{S}_t$ perform the following three steps: (i) approximate the continuation values by (27) and (A.4) separately for all feasible a_t that do not lead to a terminal node, otherwise set them to 0 (*lines 3-9*); (ii) compute the pathwise optimisers $\hat{a}_t(\omega)$ for all $\omega \in \Omega_{R_t}$ (*line 11*); (iii) using these pathwise optimisers, determine the approximation $\overline{G}_t(S_t(\omega))$ for each path $\omega \in \Omega_{R_t}$ (*line 12*). At $t = 0$, we have $(K_0, V_0, \mu_0, X_0) = (K_0(\omega), V_0(\omega), \mu_0(\omega), X_0(\omega))$, so we can simply calculate the value of the conditional expectation in (23) by taking averages of the path-wise continuation values over all $|\Omega|$ paths, and make optimal decisions based on these average values, giving $\overline{G}_0(S_0)$ (*line 17*).

4.4. Computational efficiency and numerical accuracy

While the efficiency and the accuracy of simulation and (parametric) regression approaches generally depend on a range of factors (e.g., see Maier et al. (2016) for a recent discussion), here the actual number of paths ($|\Omega_{R_t}|$) available in the regression for state $S_t = (R_t, I_t)$ is particularly critical. Indeed, although disregarded by Miltersen and Schwartz (2004); Schwartz (2004); Hsu and Schwartz (2008); Zhu (2012), the additional path-dependency of both R_t and I_t caused by the decision-dependent uncertainty K_t may result in $|\Omega_{R_t}| \ll |\Omega|$, which, in turn, generally reduces the accuracy of the parametric regression model[5]. Considering polynomials as basis functions in the parametric model, Glasserman and Yu (2004) examined the relationship between the number of simulated paths and the number of basis functions (L_{S_t}), and showed that the required $|\Omega_{R_t}|$ for ensuring convergence increases exponentially in L_{S_t}. However, Cortazar et al. (2008) have shown that taking advantage of the problem structure and carefully choosing an appropriate set of basis functions (e.g. call and put options on the expected spot

[5]A fundamentally different approach is to use the simulated evolution of K_t to determine the probability distribution that describes the probability that construction will be completed after a certain amount of cumulative investment. Probability distributions for technical uncertainty have been considered in several works including (Cortazar et al., 2001; Gamba, 2003; Pennings and Sereno, 2011). Here, such probabilities can be easily integrated in the influence diagram via chance nodes.

price (Andersen and Broadie, 2004; Nadarajah et al., 2017)), rather than simply using high-order polynomials of information state variables as in (Glasserman and Yu, 2004), allows one to substantially reduce the required L_{S_t} for a given level of accuracy, and is computationally more efficient. Hence, in general by exploiting the structure of the problem to be solved and choosing the set of basis functions appropriately, both the efficiency of the algorithm and the accuracy of the approximation are improved.

5. An illustrative example

This section describes the computational implementation of our valuation algorithm, provides specific details about the numerical example and presents the stochastic input data.

5.1. Model implementation

The valuation algorithm presented in Subsection 4.3 was implemented in MATLAB.

5.2. Expansion of district heating network

We consider the real case of an investment into the expansion of the district heating network in the London borough of Islington. We focus here on the development of the network's "north extension", as identified in a recent report (Grainger and Etherington, 2014) which investigated the development of a borough-wide network on behalf of the local council. According to this report, the capital expenditure of this expansion and the initial, annual operating revenues are estimated at £9.94 millions (K_0) and £564,600 (V_0), respectively. The report also noted that the asset can be used for up to 25 years (i.e. T_3^{max}=300). The risk-free rate, used to discount monetary values, is 3.5% per year (i.e. $r = 3.5\%/12$), as recommended by HM Treasury (2011). In addition, we assume the following: a maximum rate of investment of £1.0 million per month (I^{max}); the possibility of deferring development for up to one year (i.e. T_1^{max}=11); the possibility of halting expansion for up to one year (i.e. T_2^{max}=11); and the following durations of transitions (in months): $\Delta_h = 1, \forall h \in \{1, 2, 4, 6, 8, 9\}$; $\Delta_h = 12, \forall h \in \{5, 11, 13, 15, 17\}$; and 0 for the remainder of the transitions. Table 1 summarises the chosen input values for this example.

5.3. Generated state space and utilised basis functions

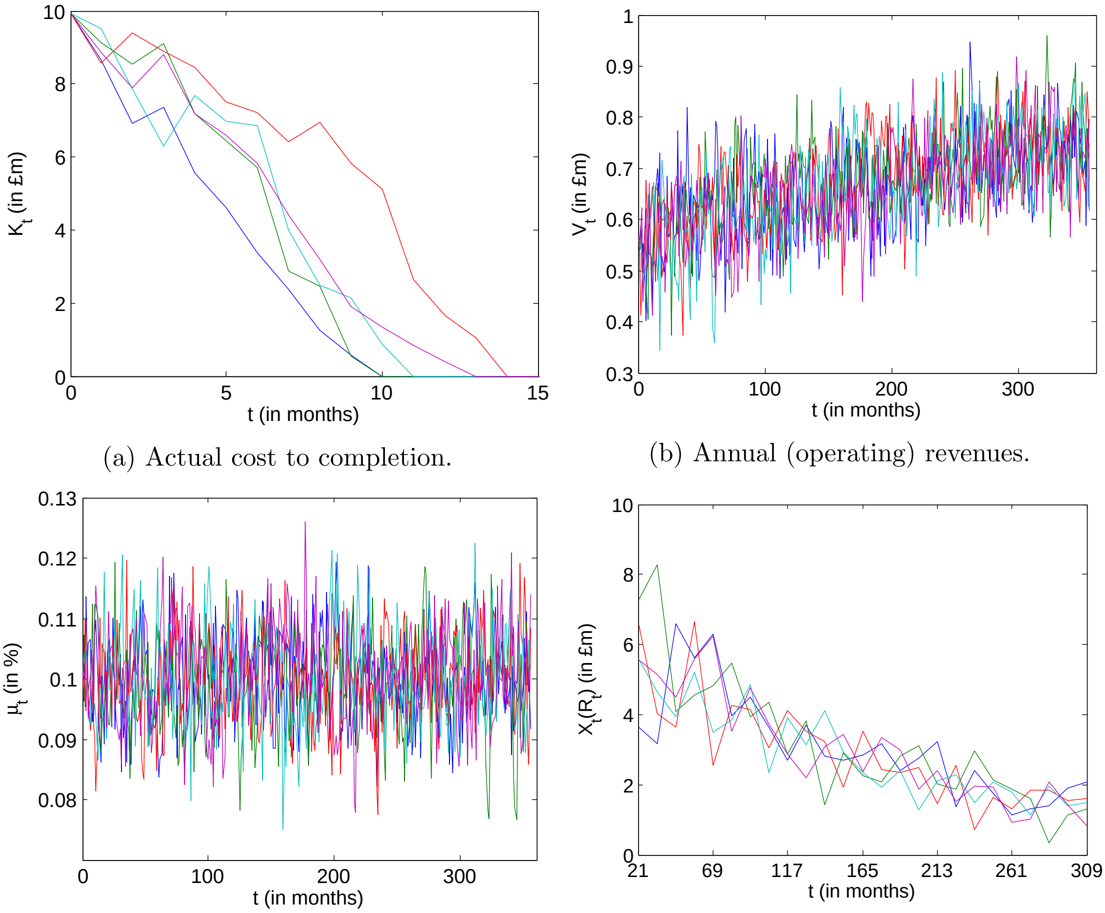
The discrete state space was generated by applying the forward induction procedure described in Subsection 4.3 (and Appendix A) and using the data of Subsection 5.2. More specifically, 100,000 paths ($|\Omega|$) were generated to describe the stochastic evolution

Table 1: Input data for district heating network expansion adapted from Grainger and Etherington (2014); HM Treasury (2011) and own estimates.

Description	Parameter	Value	Unit
Network expansion			
Cost of halting	$C^{d,h}$	$10 \cdot 10^3$	£
Cost of resuming	$C^{h,d}$	$10 \cdot 10^3$	£
Maintenance cost (halted)	C^h	$10 \cdot 10^3$	£/month
Cost of mothballing	$C^{o,m}$	$20 \cdot 10^3$	£
Cost of reactivating	$C^{m,o}$	$20 \cdot 10^3$	£
Maintenance cost (mothballed)	C^m	$20 \cdot 10^3$	£/month
Risk-free rate	r	$0.035/12$	month^{-1}
Expiration of development right	T_1^{max}	11	month
Maximum period to halt expansion	T_2^{max}	11	month
Project life of developed asset	T_3^{max}	300	month
Investment cost			
Initial (expected) cost to completion	K_0	$9.94 \cdot 10^6$	£
Maximum rate of investment	I^{max}	$1.0 \cdot 10^6$	£/month
Degree of technical uncertainty	σ_{k_1}	35%	–
Revenue[a]			
Initial annual operating revenue	V_0	$564,600$	£
Speed of mean reversion in revenue	κ_v	0.90	–
Standard deviation of revenue	σ_v	10%	month^{-1}
Initial revenue growth rate	μ_0	0.10%	month^{-1}
Speed of mean reversion in growth rate	κ_μ	0.90	–
Long-run mean growth rate level	$\overline{\mu}$	0.10%	month^{-1}
Standard deviation of growth rate	σ_μ	0.01%	month^{-1}
Salvage[a]			
Depreciation rate	ξ	0.50%	month^{-1}
Cost ratios	α, β	0.30	–
Pay-out ratios	δ, γ	0.70	–
Standard deviation of salvage value	σ_x	25%	–

[a] The correlations between processes in the four-factor model are: $\rho_{k_2,v} = \rho_{k_2,\mu} = \rho_{k_2,x} = 0$, $\rho_{v,\mu} = -0.8$, $\rho_{v,x} = 0$, and $\rho_{\mu,x} = 0$.

of the four factors K_t, V_t, μ and X_t for all $t \in \mathcal{T}$. Figure 2 shows the evolution of these for 5 equally likely paths. As can be seen in Figure 2a, while the expected duration of the expansion is 10 months, the actual time to build can vary substantially. Figure 3 and 4 show the total number of resource states $|\mathcal{R}_t|$ for all $t \in \mathcal{T}$ and the number of paths $|\Omega_{R_t}|$ in which every $R_t \in \mathcal{R}_t$ can be reached at $t \in \mathcal{T}$, respectively, supporting



(a) Actual cost to completion.

(b) Annual (operating) revenues.

(c) Growth rate of annual revenues.

(d) Salvage Value at $R_t = (t, 6, T_t, 9)$.

Figure 2: Selection of 5 equally likely paths for the evolution of the four stochastic factors.

the claim made in Subsection 4.3 that resource states may not be reachable in every simulation path.

With regard to the parametric model in (27), we apply as basis functions polynomials of the information state variables as well as both call and put options on the expected value of these variables partially based on (Longstaff and Schwartz, 2001; Andersen and Broadie, 2004; Cortazar et al., 2008; Nadarajah et al., 2017). In case $(N_t = 3 \wedge K_t > 0) \vee N_t = 5$, we use a set of $L_{S_t} = 51$ basis functions composed of a constant term, the four information state variables, polynomials of degree two (i.e. the squares of each variable and their cross products), polynomials of degree three, as well as the value of call and put options on the expected value of each variable and the square of this value. Otherwise, if $N_t = 1$ $(N_t = 3 \wedge K_t = 0$ or $N_t \in \{6, 8\})$, as mentioned in Subsection 4.3,

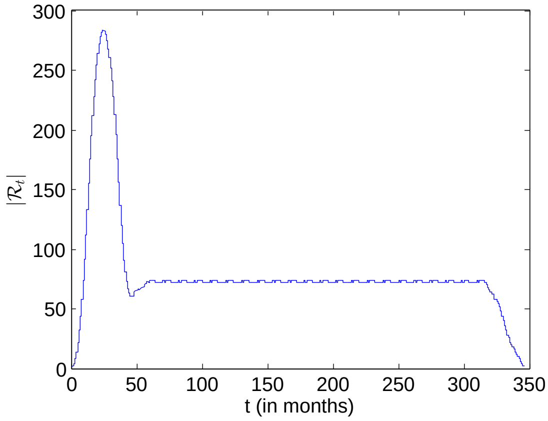


Figure 3: Total number of resource states reachable at every time $t \in \mathcal{T}$.

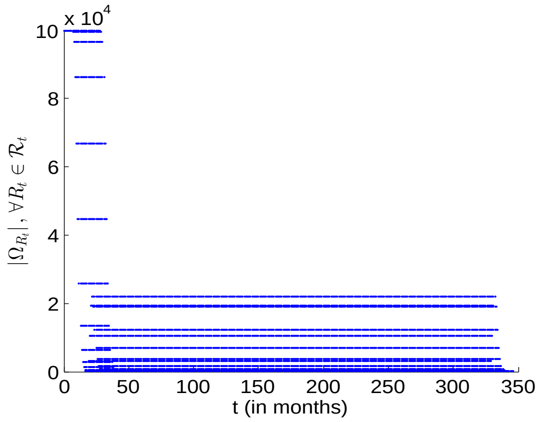


Figure 4: Number of paths in which each resource state $R_t \in \mathcal{R}_t$ is reachable at every time $t \in \mathcal{T}$.

we can reduce L_{S_t} to 18 (32) by eliminating all the functions of K_t and X_t (K_t) because $K_t = K_0$ and X_t is non-existent ($K_t = 0$), so these variables do not add any information value to the least-squares regression. In order to avoid numerical problems the basis functions were properly scaled before performing the least-squares regression, which is based on a singular value decomposition (SVD) algorithm.

5.4. Results and discussion

In order to illustrate the extent to which the profitability of the district heating investment project depends on the initial value of the annual revenues, V_0, Table 2 shows the sensitivity of the value of different portfolio configurations to varying levels of V_0. As can be seen, for values of V_0 of £0.50 millions and below, the value of the investment project without options, configuration (-), is 0. This is because the expected NPV of the project is -£2.2060 millions, -£1.2751 millions, and -£0.3441 millions for values of V_0 of £0.40 millions, 0.45 millions, and 0.50 millions, respectively, so the optimal "now-or-never strategy", which does not take any flexibility into account, is to leave the project undeveloped. The same strategy is optimal for the project with portfolio of options (a,b,c,d) for the lowest value of V_0 under consideration. However, for levels of V_0 of £0.45 millions and 0.50 millions, the value of the project with (a,b,c,d) is positive, reflecting the substantial value of having the flexibility provided by the portfolio of interdependent real options. Interestingly, in the first case, although the portfolio with all options achieves a positive value there is no individual option that provides sufficient added value on its own (i.e. in isolation), whereas in the case V_0 = £0.50 millions, having the option to defer alone – configuration (a) – also results in an economically viable project.

Table 2: Value of investment project (in £millions) for different levels of initial annual revenues.

Annual Revenue (£m)	Value Without Options	Value of Option to Defer	During expansion			During operation			Value with Portfolio of Options
			Value of Option to			Value of Option to			
			Halt	Abandon	Stage	Mothball	Abandon	Switch	
V_0	(–)	(a)	(b-i)	(b-ii)	(b)	(c)	(d)	(c,d)	(a,b,c,d)
0.40	0*	0	0	0	0	0	0	0	0*
		(-)**	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.45	0*	0	0	0	0	0	0	0	£42
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.50	0*	0.0006	0	0	0	0	0	0	0.1448
		(-)	(-)	(-)	(-)	(-)	(-)	(-)	(-)
0.55	0.5868	0.0760	0.0045	0.1321	0.1618	0	0.1643	0.1643	0.9110
		(12.95)	(0.77)	(22.51)	(27.57)	(0)	(28.00)	(28.00)	(55.24)
0.5646	0.8586	0.0702	0.0043	0.1112	0.1419	0	0.1412	0.1412	1.1438
		(8.18)	(0.50)	(12.95)	(16.53)	(0)	(16.44)	(16.44)	(33.22)
0.60	1.5178	0.0556	0.0040	0.0754	0.1029	0	0.0978	0.0978	1.7296
		(3.66)	(0.26)	(4.97)	(6.78)	(0)	(6.45)	(6.45)	(13.96)
0.65	2.4487	0.0353	0.0035	0.0442	0.0658	0	0.0590	0.0590	2.5838
		(1.44)	(0.14)	(1.80)	(2.69)	(0)	(2.41)	(2.41)	(5.52)
0.70	3.3797	0.0161	0.0029	0.0260	0.0433	0	0.0361	0.0361	3.4621
		(0.48)	(0.08)	(0.77)	(1.28)	(0)	(1.07)	(1.07)	(2.44)
0.75	4.3106	0.0008	0.0027	0.0153	0.0306	0	0.0222	0.0222	4.3588
		(0.02)	(0.06)	(0.36)	(0.71)	(0)	(0.51)	(0.51)	(1.12)

* No investment.

** Added value of option(s) in %.

Note: the sets of transitions available in the different settings are as follows: $\mathcal{H}^- = \{2,3,4,5,11,12\}$ in (–); $\mathcal{H}^- \cup \{1\}$ in (a); $\mathcal{H}^- \cup \{6,8,9\}$ in (b-i); $\mathcal{H}^- \cup \{7,10\}$ in (b-ii); $\mathcal{H}^- \cup \{6,\ldots,10\}$ in (b); $\mathcal{H}^- \cup \{13,15,16,17\}$ in (c); $\mathcal{H}^- \cup \{14,18\}$ in (d); $\mathcal{H}^- \cup \{13,\ldots,18\}$ in (c,d); and $\mathcal{H}$ in (a,b,c,d).

As can be seen from Table 2, beginning at a V_0 of £0.55 millions, the values of both the project without any flexibilities and almost all portfolio configurations are positive. In most cases the value of the project with (a,b,c,d) is considerable larger than without options (-), revealing the significant added value that is obtained by considering such a complex portfolio. While the values of the project without any options and the portfolio with all options both increase in V_0, the values of almost all of the individual options in isolation show a different trend. Indeed, the values of the options to defer (a), to halt (b-i), and to abandon the project during construction (b-ii) and operation (d) are decreasing in V_0, meaning there is less value in deferring, halting, and abandoning as the value of initial annual revenues increases. This is because the annual revenues, although still uncertain (i.e. stochastic), revert now to a linear trend that is shifted upwards, so their level is generally higher, which makes deviating from the static now-or-never strategy, and consequently the flexibility provided by individual real options less valuable. For all values of V_0 under consideration, the option to temporarily mothball the operation – configuration (c) – is of no value as the simulated values of V_t are always positive. Interestingly, the portfolio of options (a,b,c,d) is sub-additive with respect to the value of its individual options (a), (b) and (c,d), whereas the option to stage (b), which can be interpreted as the portfolio of options (b-i) and (b-ii), is super-additive with regard to the combined value of (b-i) and (b-ii).

The effects of the degrees of exogenous and endogenous uncertainty on both the value of the portfolio of options and the comparative performance of the portfolio's individual options are particularly important for understanding the influence of different underlying uncertainties. In order to illustrate these effects for the exogenous annual revenues, V_t, and the endogenous, decision-dependent cost to completion, K_t, Figure 5 shows for $C^{o,m} = C^{m,o} = C^m = 0$ the way in which the standard deviation of changes in revenues, σ_v, and the degree of technical uncertainty, σ_{k_1}, effect the value of the investment project. While the effects of changes of σ_v on the value of the project without options is negligible, the value of the portfolio is generally increasing in σ_v, particularly steep for higher levels of σ_v and it seems the increase is more pronounced for lower values of σ_{k_1}. This increase in project value results from the flexibilities provided by the portfolio of real options, which allow a decision maker to exploit the upside potential of increased annual revenues, as compared to the negligibly affected value of the investment project without options, which applies a static now-or-never strategy.

On the other hand, increasing σ_{k_1} from 0 to 0.05 (i.e. introducing some construction cost uncertainty) results in a sharp decline in values of the investment project, but the decline is smaller for the project with the portfolio of real options. The reason for

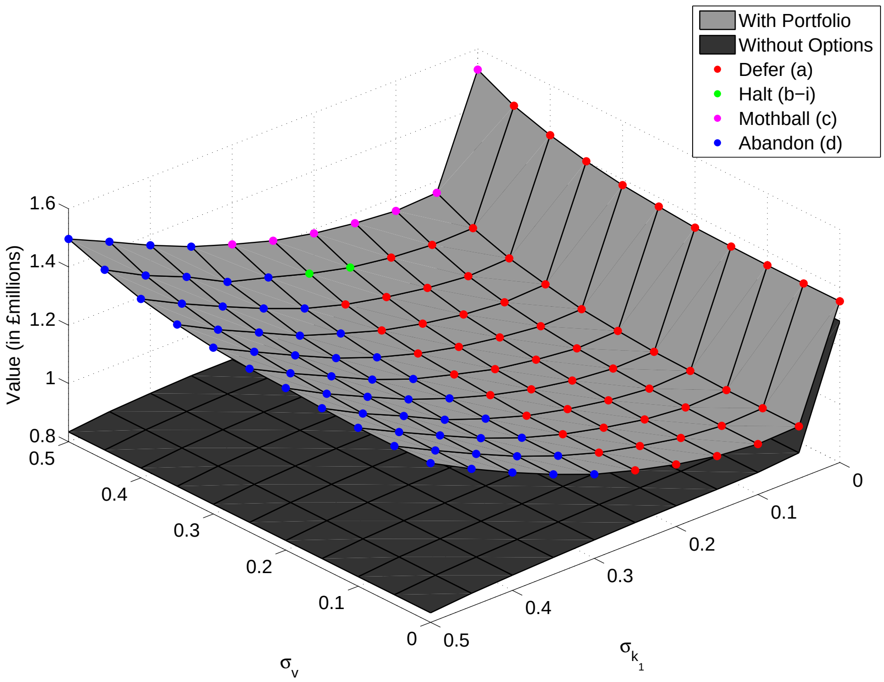


Figure 5: Value of investment project with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of degrees of revenue (σ_v) and technical (σ_{k_1}) uncertainty.

this sharp decline is mainly due to the increase in actual cost of completion caused by the introduction of technical uncertainty, but also because of the discretised investment expenditures. Unlike the investment project without options, whose value is always decreasing in σ_{k_1}, beginning at a σ_{k_1} of 0.1, the value of the portfolio is increasing in σ_{k_1}. This is because the flexibility provided by the portfolio, particularly by its option to abandon during operation (d), allows one to partially reverse the investment by recovering increased investment expenditures in situations with high values of σ_{k_1}, thereby taking advantage of relatively high state-dependent salvage values. This seems to explain why option (d) is the portfolio's most valuable individual option when the degree of technical uncertainty is high, whereas in most other situations, the option to defer (a) is the portfolio's most valuable option. Interestingly, for high values of σ_v, there are even situations in which options (b-i) and (c) are most-valuable, reflecting the ability of such a complex portfolio of real options to manage exogenous and endogenous uncertainties simultaneously in a wide range of uncertain environments.

To show the effect of the endogenous, state-dependent salvage value, X_t, on investment decisions, Figure 6 shows the extent to which the value of the investment project is affected by the pay-out ratios γ and δ as well as by the standard deviation σ_x. The value

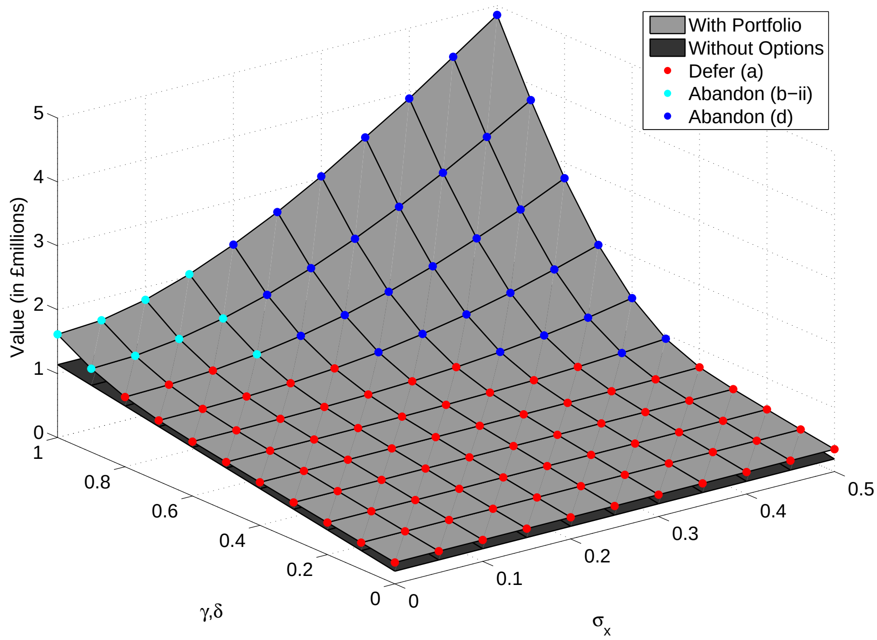


Figure 6: Value of investment project with portfolio of real options and without options as well as portfolio's most valuable individual option (filled circles), as a function of pay-out ratios (γ, δ) and standard deviation of salvage value, σ_x.

of the project without options – where X_t is received as residual value when completing the project after 25 years of operation – is positive for all parameters under consideration. Furthermore, its value increases virtually linearly in (γ, δ) because of the linear dependence of the expected asset value, Z_t, on (γ, δ), but is practically unaffected by changes in σ_x simply because the expected value of X_t does not change. Although the value of the project with the portfolio of options is always greater than the value of the project without options, the difference remains relatively constant for low values of (γ, δ) and for both low σ_x and moderate (γ, δ), with the option to defer (a) being the portfolio's most valuable individual option in these situations. As can be seen, however, for high expected asset values and fairly high yet risky salvage values, the portfolio considered here is capable of extracting considerable value from flexibilities, especially from abandoning the project during either construction (b-ii) or operation (d). The above results therefore highlight the importance of applying such a portfolio of real options approach when there is both exogenous and endogenous uncertainty.

6. Conclusions

This paper presents an approach for approximating the value of portfolios of interdependent real options under both exogenous and endogenous uncertainties. The approach is illustrated by valuing a complex urban infrastructure investment in London. Unlike existing valuation approaches, which have considered only exogenous uncertainty or rather inflexible and restricted portfolios, this work has studied a complex yet practical portfolio of real options under conditions of four underlying uncertainties. The options were: to defer investment; stage investment; temporarily halt expansion; temporarily mothball the operation; and abandon the project. Two of the underlying uncertainties, decision-dependent cost to completion and state-dependent salvage value, were endogenous, whereas the other two, annual revenues and their growth rate, were exogenous. We have extended our existing approach for valuing portfolios of interdependent real options to include endogenous uncertainties. In the new approach, the directly-modelled dynamics of all four uncertainties and the linear integer constraints modelling the real options' interdependencies are integrated in a multi-stage stochastic integer programme.

This study has presented an efficient valuation algorithm to approximate the value of this portfolio using simulation and parametric regression. In contrast to existing valuation algorithms, ours explicitly accounts for the negative numerical implications of the state variables' path dependencies on the accuracy of the approximation. We do so by exploiting the structure of the investment problem to be solved by dynamically and appropriately adapting the basis functions used in the parametric model. The illustrative example shows that our approach is flexible and powerful, and can be used to value both complex portfolios and their individual real options under both types of uncertainty. Future work will explore ways to model the dynamics of other sources of endogenous uncertainty as well as investigate how these can be integrated into the valuation framework presented here.

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Appendix A. Solution procedure

The forward induction procedure consists of the following steps:

1. Starting at time 0 and using (16), sample $|\Omega|$ paths of K_t conditional on $a_{t2} = 1$ or $a_{t4} = 1$ until $K_t(\omega) = 0, \forall \omega \in \Omega$, where $\Delta^{con}(\omega) = \{\min t : K_t(\omega) = 0\}$ and $\mathcal{T}^{con} = \{\Delta^{con}(\omega) \colon \omega \in \Omega\}$ denote the construction time in path ω and the set of construction times, respectively.
2. Determine the set of decision times, $\mathcal{T}_{N_t}$, for all decisions nodes $N_t \in \{1, 3, 5, 6, 8\}$, forming subsets of $\mathcal{T}$:

$$\mathcal{T}_{N_t} = \begin{cases} \big\{ i\Delta_1 \colon i \in \mathbb{Z}_{\geq 0}, 0 \leq i\Delta_1 \leq T_1^{max} \big\}, & \text{if } N_t = 1, \\ \big\{ \tau_1 + \Delta_1(1 + i + 2j + m) \colon \tau_1 \in \mathcal{T}_1, i, j, m \in \mathbb{Z}_{\geq 0}, \Delta_1(1 + i + j) \leq \max \mathcal{T}^{con}, & \\ \qquad \Delta_1(j + m) \leq T_2^{max} \max(0, \min(1, j)) \big\}, & \text{if } N_t = 3, \\ \big\{ \tau_1 + \Delta_1(2 + i + 2j + m) \colon \tau_1 \in \mathcal{T}_1, i, j, m \in \mathbb{Z}_{\geq 0}, \Delta_1(1 + i + j) < \max \mathcal{T}^{con}, & \\ \qquad \Delta_1(1 + j + m) \leq T_2^{max} \big\}, & \text{if } N_t = 5, \\ \big\{ \tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(1 + j) \colon \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i, j \in \mathbb{Z}_{\geq 0}, & \\ \qquad \Delta_1 i \leq T_2^{max}, \Delta_5(1 + j) \leq T_3^{max}, & \text{if } N_t = 6, \\ \big\{ \tau_1 + \tau^{con} + \Delta_1 i + \Delta_5(2 + j) \colon \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, i, j \in \mathbb{Z}_{\geq 0}, & \\ \qquad \Delta_1 i \leq T_2^{max}, \Delta_5(2 + j) \leq T_3^{max}, & \text{if } N_t = 8, \end{cases} \tag{A.1}$$

3. Use (17) and (18) to sample $|\Omega|$ paths of V_t and μ_t, respectively, giving $\big(V_t(\omega), \mu_t(\omega)\big)_{\omega \in \Omega}, \forall t \in \mathcal{T}$
4. Generate the possible resource state space $\mathcal{R}_t$ for each decision node and decision time according to:

$$\mathcal{R}_t = \begin{cases} (t, 1, T_1^{max} - t/\Delta_1, 0), & \text{if } N_t = 1, t \in \mathcal{T}_1, \\ \big\{(t, 3, T, Q)\colon \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, & \\ \qquad \tau_1 < t, \Delta_1 \leq Q/I^{max} \leq \Delta^{con}(\omega), \exists \omega \in \Omega, & \\ \qquad 0 \leq T_2^{max} - T \leq \max(t - \tau_1 - 2\Delta_1, 0)\big\}, & \text{if } N_t = 3, t \in \mathcal{T}_3, \\ \big\{(t, 5, T, Q)\colon \tau_1 \in \mathcal{T}_1, T, Q \in \mathbb{Z}_{\geq 0}, t = \tau_1 + Q/I^{max} + T_2^{max} - T, & \\ \qquad \tau_1 < t, \Delta_1 \leq Q/I^{max} < \Delta^{con}(\omega), \exists \omega \in \Omega, & \\ \qquad \Delta_1 \leq T_2^{max} - T \leq \max(t - \tau_1 - \Delta_1, \Delta_1)\big\}, & \text{if } N_t = 5, t \in \mathcal{T}_5, \\ \big\{(t, 6, T, Q)\colon \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, & \\ \qquad T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - \Delta_5, & \\ \qquad T \bmod \Delta_5 = 0, i \leq T_2^{max}\big\}, & \text{if } N_t = 6, t \in \mathcal{T}_6, \\ \big\{(t, 8, T, Q)\colon \tau_1 \in \mathcal{T}_1, \tau^{con} \in \mathcal{T}^{con}, T, Q, i \in \mathbb{Z}_{\geq 0}, Q = \tau^{con} I^{max}, & \\ \qquad T = T_3^{max} - t + \tau_1 + \tau^{con} + \Delta_1 i, T \leq T_3^{max} - 2\Delta_5, & \\ \qquad T \bmod \Delta_5 = 0, i \leq T_2^{max}\big\}, & \text{if } N_t = 8, t \in \mathcal{T}_8, \end{cases} \tag{A.2}$$

5. For all $R_t \in \mathcal{R}_t, t \in \mathcal{T}$, compute the set of paths Ω_{R_t} in which resource state R_t is reachable by:

$$\Omega_{R_t} = \begin{cases} \Omega, & \text{if } N_t = 1, \\ \big\{\omega \in \Omega : t - \tau_1 - T_2^{max} + T \leq \Delta^{con}(\omega), Q/I^{max} \leq \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\big\}, & \text{if } N_t = 3, \\ \big\{\omega \in \Omega : t - \tau_1 - T_2^{max} + T < \Delta^{con}(\omega), Q/I^{max} < \Delta^{con}(\omega), \tau_1 \in \mathcal{T}_1\big\}, & \text{if } N_t = 5, \\ \big\{\omega \in \Omega : \Delta^{con}(\omega) = Q/I^{max}\big\}, & \text{if } N_t \in \{6, 8\}. \end{cases} \tag{A.3}$$

6. Use (19) and (20) to sample $|\Omega_{R_t}|$ realisations of X_t giving $\big(X_t(\omega)\big)_{\omega \in \Omega_{R_t}}, \forall R_t \in \mathcal{R}_t, t \in \mathcal{T}$.

The backward induction procedure is shown by Algorithm 1, with the optimal values of the coefficients $\big(\alpha_{tl}(S^R(R_t, a_t))\big)_{l=0}^{L_{S_t}}$, given resource state R_t and action a_t, in *line 7* determined by (A.4).

$$\big(\hat{\alpha}_{tl}(R_{t+\Delta_h})\big)_{l=0}^{L_{S_t}} = \operatorname*{arg\,min}_{(\alpha_{tl}(\cdot))_{l=0}^{L_{S_t}}} \left\{ \sum_{\omega \in \Omega_{R_t}} \left[e^{-r\Delta_h} \overline{G}_{t+\Delta_h}(S_{t+\Delta_h}(\omega)) - \sum_{l=0}^{L_{S_t}} \alpha_{tl}(R_{t+\Delta_h}) \phi_{S_t l}(I_t(\omega)) \right]^2 \right\}, \tag{A.4}$$

where $R_{t+\Delta_h} = S^R(R_t, a_t)$ and $S_{t+\Delta_h}(\omega) = (R_{t+\Delta_h}, I_{t+\Delta_h}(\omega))$.

Algorithm 1: Approximation of optimal value of problem (23)-(26)

Data: All the above

Result: $\overline{G}_0(S_0)$

for $t = \max\{\mathcal{T} \setminus 0\}$ **do**

  forall $S_t \in \mathcal{S}_t$ **do**

    forall $a_t \in \mathcal{A}_{S_t}$ **do**

      if $a_{th} = 1, h \in \{7, 10, 12, 14, 16, 18\}$ **then**

        $F_t(S_t(\omega), a_t) \leftarrow 0, \forall \omega \in \Omega_{R_t}$

      else

        Use both (27) and (A.4) to determine:

        $F_t(S_t(\omega), a_t) \leftarrow \hat{\Phi}_t^{L_{S_t}}(S_t(\omega), a_t), \forall \omega \in \Omega_{R_t}$

      end

    end

    forall $\omega \in \Omega_{R_t}$ **do**

      Compute pathwise optimisers:

      $\hat{a}_t(\omega) \leftarrow \underset{a_t(\omega) \in \mathcal{A}_{S_t(\omega)}}{\arg\max} \Big\{ \Pi_t(S_t(\omega), a_t(\omega)) + F_t(S_t(\omega), a_t(\omega)) \Big\}$

      Approximate optimal portfolio value along each path ω:

      $\overline{G}_t(S_t(\omega)) \leftarrow \Pi_t(S_t(\omega), \hat{a}_t(\omega)) + e^{-r\Delta_h} \overline{G}_{t+\Delta_h} \Big(S^M \big(S_t(\omega), \hat{a}_t(\omega), W_{t+\Delta_h}(\omega) \big) \Big)$

    end

  end

  $\mathcal{T} \leftarrow \mathcal{T} \setminus t$

end

At $t = 0$, $S_0 = (0, 1, T_1^{max}, 0, K_0, V_0, \mu_0, X_0)$, determine:

$$\overline{G}_0(S_0) \leftarrow \max_{a_0 \in \mathcal{A}_{S_0}} \Big\{ \Pi_0(S_0, a_0) + \frac{1}{|\Omega|} \sum_{\omega \in \Omega} e^{-r\Delta_h} \overline{G}_{\Delta_h} \Big(S^M \big(S_0, a_0, W_{\Delta_h}(\omega) \big) \Big) \Big\}$$

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