Random Start American Perpetual Options

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Abstract

We consider the valuation of American perpetual options with the property that they are only possible to exercise after a random time has occured. One situation where this feature is present is when we want to value the real option of when to build on vacant land and we are waiting for a permit. The random time in this case is the time at which the permit is given. This and the value of a version of an abandonment option are given as two applications of this modelling framework.

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1 Introduction

During the lifetime of many investments there are events which the investor can not control, but which are crucial in the development of the project. It could be waiting for a permit or unexpectedly be exposed to some form of news. In this paper we consider a situation where there is an exogenously given random time marking the first time at which an action can be taken. This means that even though the investor wants to take the action, e.g. initiate a project, he is not necessarily allowed to do so. There is a resemblance between this type of random time and the default time of a bond, and we will partly use similar models as the ones used in credit risk modelling as described in Bielecki & Rutkowski [2] and Jeanblanc et al [8].

To calculate the value of the random start option, we start by calculating the value of the standard American perpetual option using the same gain function as in the random start option problem. We will assume that the underlying process, e.g. the value of developed land, evolves according to a geometric Brownian motion. This makes it possible to get analytical expressions for many American perpertual options, see Karatzas [9], Karatzas & Shreve [10], Øksendal [14], Peskir & Shiryaev [15] and Shiryaev [16] for theory and applications, and this is indeed the case for the two gain functions considered in this paper.

As applications we consider the optimal time to initiate a project (e.g. to start building on vacant land) given the constraint of a pending application to start, and a version of an abandonment option. These are two examples of real options, i.e. investment opportunities where there is an element of optionality. For a broad introduction to real options, see Dixit & Pindyck [4].

The rest of the paper is organised as follows. In Section 2 we discuss the general modelling assumptions. Section 3 contains the model applied to two random start investment problems, Section 4 outline some generalisations of the models described in Section 2, and Section 5 concludes and summarises.

2 The model

2.1 Generalities

Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ be a complete filtered probability space where the filtration satisfies the usual assumptions of being right-continuous and $\mathcal{F}_0$ containing all P-null sets of $\mathcal{F}$. A random time τ is a non-negative random variable:

$$\tau : \Omega \to [0, \infty].$$

A random time τ is a stopping time with respect to the filtration $(\mathcal{F}_t)$ if it fulfills

$$\{\tau \leq t\} \in \mathcal{F}_t \text{ for every } t \geq 0.$$

We assume that there exists a bank account with constant rate $r >$ whose value evoles according to

$$dB(t) = rB(t)dt \text{ with } B(0) = 1.$$

We further assume the existence of a pricing measure Q locally equivalent to P such that the value of a stream of cash flows is the discounted expected value under Q and where the risk-free rate is used as discount rate.

Given is also a real-valued and continuous time-homogeneous Markov process (X_t) which is adapted to $(\mathcal{F}_t)$, and a non-negative function $G : \mathbb{R} \to \mathbb{R}_+$. Our first task will be to calculate the value of the standard American perpetual option with (X_t) as the underlying and G as gain function. The value at time $t \geq 0$ of this American contract is

$$U_t = \operatorname*{ess\,sup}_{\tau \in \mathcal{S}_t} E^Q \left[e^{-r(\tau - t)} G(X_\tau) \middle| \mathcal{F}_t \right],$$

where $\mathcal{S}_t$ is the set of stopping times greater than or equal to t. Fakeev [6] has shown that when (X_t) is a time-homogenous Markov process, then $U_t = V(X_t)$ where V is the function

$$V(x) = \sup_\tau E_x^Q \left[e^{-r\tau} G(X_\tau) \right],$$

the supremum is taken over all stopping times and where

$$E_x^Q [\,\cdot\,] = E^Q [\,\cdot\,| X_0 = x].$$

Hence, it is enough to calculate the function V. We allow for $\tau = \infty$, and define

$$e^{-r\tau} G(X_\tau) = \limsup_{t \to \infty} e^{-rt} G(X_t) \text{ on } \{\tau = \infty\}.$$

A stopping time $\tau^\star$ such that

$$V(x) = E_x^Q \left[e^{-r\tau^\star} G(X_{\tau^\star}) \right]$$

is called an optimal stopping time. For more on optimal stopping see e.g. Peskir & Shiryaev [15], and for more on optimal stopping and American options in models driven by a Brownian motion see Karatzas [9] and Karatzas & Shreve [10]. We finally let τ_S denote the random time at which we at the earliest can exercise an American perpetual option.

To solve our type of problems we proceed according to the following program:

1. Calculate the value function V for the standard perpetual American option with gain function G:
$$V(x) = \sup_\tau E_x^Q \left[e^{-r\tau} G(X_\tau) \right].$$

2. If $t \geq \tau_S$, then the value at t of the random start option with gain function G is given by $V(X_t)$.

3. If $t < \tau_S$, the value is given by
$$E^Q \left[e^{-r(\tau_S - t)} V(X_{\tau_S}) \middle| \mathcal{F}_t \right]. \tag{1}$$
Note that we in general need to keep track of the time t here, since it might influence when the time τ_S occurs.

In our applications the function V is known and the goal of the major part of this paper is to show how we can compute the value of the option at times t when $t < \tau_S$; i.e. evaluate expressions of the type given in Equation (1) above.

We are interested in properties of the random time τ_S under both the objective measure P and the pricing measure Q. To this we will start to specify the properties of τ_S under the objective measure P, and then determine what happens to these properties when we change measure from P to Q. The investor in our model has no possibility of influencing the time τ_S, and we assume that τ_S and the underlying process (X_t) are independent under P. We assume that the randomness generated by τ_S can not be traded, so we have an incomplete model. This means that there is not one unique, but infinitely many, potential pricing measures Q, and we need to choose one of these. One way of doing this is to assume that the distribution of τ_S under Q is the same as under P and that τ_S is independent of (X_t) under Q as well. Choosing Q to have these properties means that we use what is called the minimal martingale measure, and this is the approach we will use. It has previously been used by e.g. Møller [12] in applications to insurance and by Armerin & Song [1] in a real options model. See Föllmer & Schweizer [7] and references therein for more on the minimal martingale measure. Explicitly, we make the following assumptions on the random time τ_S:

- $P(\tau_S > t) = Q(\tau_S > t) > 0$ for every $t \geq 0$,
- $P(\tau_S < \infty) = Q(\tau_S < \infty) = 1$, and
- τ_S is independent of X under both P and Q.

Note that a constant starting time does not fulfill the first of these requirements, and must thus be considered seperately.

2.2 The hazard function

To be able to calculate the value of the American random start option we need the result in Lemma 2.1 below. We let

$$F_t = Q(\tau_S \leq t),$$

and introduce

$$\Gamma_t = -\ln(1 - F_t) \quad \Leftrightarrow \quad F_t = 1 - e^{-\Gamma_t},$$

where the assumption $Q(\tau_S > t) > 0$ for every $t \geq 0$ from above guarantees that Γ is well defined for every $t \geq 0$. Now fix $T > 0$. Using the previous notation, we have the following result:

Lemma 2.1 *Assume that Z is an $(\mathcal{F}_t)$-predictable process such that the random variable $Z_{\tau_S}\mathbf{1}(\tau_S \le T)$ is integrable. Then we have, for every $t \le T$,*

$$\begin{aligned} E^Q\left[Z_{\tau_S}\mathbf{1}(t<\tau_S\le T)|\mathcal{F}_t\right] &= e^{\Gamma_t}E^Q\left[\int_{(t,T]} Z_u dF_u \middle| \mathcal{F}_t\right]\mathbf{1}(t<\tau_S) \\ &= e^{\Gamma_t}\int_{(t,T]} E^Q\left[Z_u|\mathcal{F}_t\right]dF_u\,\mathbf{1}(t<\tau_S). \end{aligned}$$

For a proof see Bielecki & Rutkowski [2] or Jeanblanc et al [8]. If Z is non-negative, then we can let $T \to \infty$, use monotone convergence, and get

$$\begin{aligned} E^Q\left[Z_{\tau_S}\mathbf{1}(t<\tau_S)|\mathcal{F}_t\right] &= e^{\Gamma_t}E^Q\left[\int_{(t,\infty)} Z_u dF_u \middle| \mathcal{F}_t\right]\mathbf{1}(t<\tau_S) \\ &= e^{\Gamma_t}\int_{(t,\infty)} E^Q\left[Z_u|\mathcal{F}_t\right]dF_u\,\mathbf{1}(t<\tau_S). \end{aligned}$$

Here we have used the fact that by assumption $Q(\tau_S < \infty)$. When Γ can be written

$$\Gamma_t = \int_0^t \gamma_s ds$$

for a function γ, then we say that τ_S has intensity γ and we have

$$dF_t = \gamma_t e^{-\int_0^t \gamma_s ds}dt$$

in this case. We can then write

$$E^Q\left[Z_{\tau_S}|\mathcal{F}_t\right]\mathbf{1}(t<\tau_S) = \int_t^\infty E^Q\left[Z_u|\mathcal{F}_t\right]\gamma_u e^{-\int_t^u \gamma_s ds}du\,\mathbf{1}(t<\tau_S).$$

Using this result when

$$Z_t = e^{-rt}f(X_t)$$

for a function $f : \mathbb{R} \to \mathbb{R}_+$ yields the following result.

Proposition 2.2 *With notation and assumptions above, we have*

$$E^Q\left[e^{-r(\tau_S-t)}f(X_{\tau_S})\middle|\mathcal{F}_t\right] = \int_t^\infty E^Q\left[f(X_u)|\mathcal{F}_t\right]\gamma_u e^{-\int_t^u (r+\gamma_s)ds}du$$

when $t < \tau_S$.

Note that the left-hand side with $f = V$ in the expression above is Equation (1). The case when $\tau_S = T > 0$ is deterministic is not covered by the previous Proposition. In this case we use that

$$E^Q\left[e^{-r(\tau_S-t)}f(X_{\tau_S})\middle|\mathcal{F}_t\right] = e^{-r(T-t)}E^Q\left[f(X_T)|\mathcal{F}_t\right]$$

for $t < T$.

3 Applications

3.1 The model

We will now give examples of the technique described so far. In all examples below we use the following model. Here X_t denotes the present value at time $t \geq 0$ of a developed project or investment.

- Under P the value follows the geometric Brownian motion

  $$dX_t = \mu X_t dt + \sigma X_t dW_t$$

  with $X_0 > 0$, $\mu \in \mathbb{R}$ and $\sigma > 0$. The process W is a standard Brownian motion under P.

- Under Q the value follows the geometric Brownian motion

  $$dX_t = (r - \delta)X_t dt + \sigma X_t dW_t^Q,$$

  where W^Q is a Q-Brownian motion. Here $\delta > 0$ is the cash flow yield generated by the investment.

- The intensity function is a constant $\gamma > 0$; i.e. τ_S is exponentially distributed with mean $1/\gamma$ under both P and Q.

In the examples below the two constants

$$\beta_1 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}} > 1$$

and

$$\beta_2 = \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}} < 0$$

will be used. They are the solutions to the quadratic equation

$$\frac{1}{2}\sigma^2\beta(\beta-1) + (r-\delta)\beta - r = 0, \tag{2}$$

which in turn comes from the fact that we use the geometric Brownian motion above when modelling the underlying value. In the continuation region (i.e. the values of X_t for which we choose not to stop) the value function V satisfies a second order ODE, and in general the solutions to this ODE are the linear combinations of the two functions x^{β_1} and x^{β_2}. The following Proposition will be used to calculate the value of random start options.

Proposition 3.1 *Let X be the geometric Brownian motion*

$$dX_t = (r-\delta)X_t dt + \sigma X_t dW_t^Q,$$

where W^Q is a standard Brownian motion under Q, and let τ_S be exponentially distributed with mean $1/\gamma > 0$ and independent of X under Q. For any $a, b \in \mathbb{R}$ and $t < \tau_S$ we have

$$\begin{aligned} E^Q\left[e^{-r(\tau_S - t)} X_{\tau_S}^a \mathbf{1}(X_{\tau_S} \le b) \middle| \mathcal{F}_t\right] &= \gamma X_t^a J\Big(r + \gamma - a(r - \delta + (a-1)\sigma^2/2), \\ &\quad \frac{1}{\sigma}\ln\left(\frac{b}{X_t}\right), \frac{\sigma}{2} - \frac{r-\delta}{\sigma} - a\sigma\Big), \end{aligned}$$

where

$$J(k, L, M) = \begin{cases} \frac{1}{2k} e^{-L\left(M - \sqrt{M^2+2k}\right)}\left(\frac{M}{\sqrt{M^2+2k}} + 1\right) & \text{if } L < 0 \\ \frac{1}{k} + \frac{1}{2k} e^{-L\left(M + \sqrt{M^2+2k}\right)}\left(\frac{M}{\sqrt{M^2+2k}} - 1\right) & \text{if } L \ge 0. \end{cases}$$

When $a = \beta_i$ for $i = 1, 2$ (with β_1 and β_2 as above), then

$$\begin{aligned} E^Q\left[e^{-r(\tau_S - t)} X_{\tau_S}^{\beta_i} \mathbf{1}(X_{\tau_S} \le b) \middle| \mathcal{F}_t\right] &= \gamma X_t^{\beta_i} J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{b}{X_t}\right),\right. \\ &\quad \left. -\operatorname{sgn}(\beta_i)\sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}}\right). \end{aligned}$$

For a proof of the proposition, see Appendix A.1.

3.2 The objective measure P

We are mainly interested in the pricing measure Q, but in some cases we need to use the objective meaure P. One example is when we want to compare the time until we exercise the investment option in the non-random case (i.e. when $\tau_S = 0$) with the random start case we need the distribution of X_{τ_S} under P. The solution to the GBM

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

is

$$X_t = X_0 e^{\left(\mu - \sigma^2/2\right)t + \sigma W_t}, \; t \ge 0.$$

It follows that

$$\begin{aligned} \ln X_{\tau_S} &= \ln X_0 + \left(\mu - \frac{\sigma^2}{2}\right)\tau_S + \sigma W_{\tau_S} \\ &= \ln X_0 + \left(\mu - \frac{\sigma^2}{2}\right)\tau_S + \sigma\sqrt{\tau_S} \cdot \frac{W_{\tau_S}}{\sqrt{\tau_S}}, \end{aligned}$$

where we set $W_{\tau_S}/\sqrt{\tau_S} = 0$ when $\tau_S = 0$. Since τ_S is independent of W under P we have $W_{\tau_S}/\sqrt{\tau_S} \sim N(0,1)$ and we can write

$$\ln X_{\tau_S} \stackrel{D}{=} \ln X_0 + \left(\mu - \frac{\sigma^2}{2}\right)\tau_S + \sigma\sqrt{\tau_S} \cdot Z,$$

where $Z \sim N(0,1)$ is independent of τ_S. Hence, we revover the well known fact that the random variable $\ln X_{\tau_S}$ has a normal mean-variance mixture distribution. When τ_S is exponentially distributed, then $\ln X_{\tau_S}$ is skew-Laplace distributed. We recall that a random variable is skew-Laplace distributed if its density function is given by

$$f(x) = \frac{\sqrt{2}}{\Sigma} \cdot \frac{\kappa}{1+\kappa^2} \begin{cases} e^{-\frac{\sqrt{2}\kappa}{\Sigma}|x-\theta|} & \text{if } x \geq \theta \\ e^{-\frac{\sqrt{2}}{\Sigma\kappa}|x-\theta|} & \text{if } x < \theta \end{cases}$$

for some $\theta \in \mathbb{R}$, $\kappa > 0$ and $\Sigma \geq 0$. We write $\mathrm{AL}(\theta, \Sigma, \kappa)$ to denote this distribution. If τ_S is exponentially distributed with mean $1/\gamma$, then $\ln X_{\tau_S}$ is skew-Laplace distributed with parameters

$$\begin{aligned} \theta &= \ln X_0 \\ \Sigma &= \frac{\sigma}{\sqrt{\gamma}} \\ \kappa &= \frac{\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu}{\sqrt{2\gamma}\sigma}, \end{aligned}$$

where

$$\nu = \mu - \frac{\sigma^2}{2}$$

is the expected growth rate of the GBM. Expressing the parameters in the skew-Laplace's density function using σ, γ, ν and $\ln X_0$ we get

$$f_{\tau_S}(x) = \frac{\sigma^2\gamma}{\sqrt{\nu^2 + 2\gamma\sigma^2}} \begin{cases} e^{-\frac{\sqrt{\nu^2+2\gamma\sigma^2}-\nu}{\sigma^2}|x-\ln X_0|} & \text{if } x \geq \ln X_0 \\ e^{-\frac{2\gamma}{\sqrt{\nu^2+2\gamma\sigma^2}-\nu}|x-\ln X_0|} & \text{if } x < \ln X_0. \end{cases}$$

For more on normal mean-variance mixture models and the skew-Laplace distribution see Kotz et al [11].

Finally we note that when $\tau_S = T$ is a deterministic time, then

$$\ln X_T \sim N\left(\ln X_0 + \left(\mu - \frac{\sigma^2}{2}\right) T, \sigma\sqrt{T}\right).$$

3.3 Valuation of a project – an optimal timing option

This is the main example we have in mind when studying random start American options. When buying land in order to build on it, usually a building permit is needed. Hence, even though the investor wants to build on the land he is not allowed to do so until he has received the permit. In this application τ_S is the time at which the building permit is given. The gain function in this case is given by

$$G(x) = x - I,$$

where I is the investment cost of the project. Since it is never optimal to exercise the option when the value of the project is smaller than 0, this problem is equivalent to the one where

$$G(x) = \max(x - I, 0),$$

i.e. when we have a perpetual American call option. Hence, the problem we initially want to solve is

$$V(x) = \sup_{\tau} E_x^Q \left[\max(X_\tau - I, 0)\right].$$

With dynamics of X as above we have

$$V(x) = \begin{cases} (L_c - I)\left(\frac{x}{L_c}\right)^{\beta_1} & \text{when } x \in [0, L_c) \\ x - I & \text{when } x \in [L_c, \infty), \end{cases}$$

where the critical level L_c is given by

$$L_c = \frac{\beta_1}{\beta_1 - 1} I.$$

For a proof of this see e.g. Chapter VIII, §2a in Shiryaev [16] or pp. 209-211 in Øksendal [14]. To calculate the value at a time $t < \tau_S$ of the random start option, we write the value V as

$$\begin{aligned} V(x) &= (L_c - I)\left(\frac{x}{L_c}\right)^{\beta_1} \mathbf{1}(x < L_c) + (x - I)\mathbf{1}(x \geq L_c) \\ &= (L_c - I)\left(\frac{x}{L_c}\right)^{\beta_1} \mathbf{1}(x < L_c) + x - I + x\mathbf{1}(x < L_c) - I\mathbf{1}(x < L_c). \end{aligned}$$

We now use Proposition 3.1 to find the value of this random start option at a time $t < \tau_S$ with $X_t = x$. By using Proposition 3.1 on the first, fourth and fifth of the five terms in the expression for $V(x)$ we get

$$\begin{aligned} \text{Value} &= (L_c - I)\left(\frac{x}{L_c}\right)^{\beta_1} \gamma J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{L_c}{x}\right), -\sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}}\right) \\ &\quad + x\frac{\gamma}{\gamma + \delta} - I\frac{\gamma}{\gamma + r} \\ &\quad + \gamma x J\left(\gamma + \delta, \frac{1}{\sigma}\ln\left(\frac{L_c}{x}\right), -\frac{r - \delta + \sigma^2/2}{\sigma}\right) \\ &\quad - \gamma I J\left(\gamma + r, \frac{1}{\sigma}\ln\left(\frac{L_c}{x}\right), -\frac{r - \delta - \sigma^2/2}{\sigma}\right). \end{aligned}$$

This formula is not so intuitive, but looking at a concrete example as in Figure 1 we get a better picture. For small values of the geometric Brownian motion

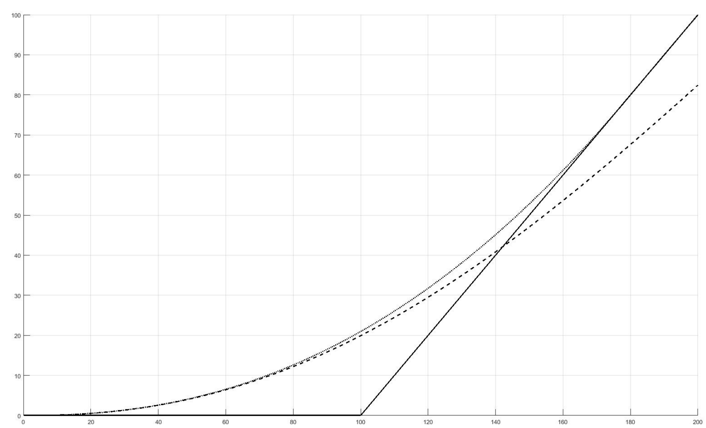

Figure 1: *Gain function (solid curve), value of the standard American call option (dotted curve) and value of the random start American call option (dashed curve) all with* $I = 100$ *as function of the initial value* x. *The parameter values are* $r = 0.01$, $\delta = 0.02$, $\sigma = 0.15$ *and* $\gamma = 0.10$.

the value of the random start option is not much smaller than the value of the standard American perpetuate call option. As the value of the underlying increases towards the critical value, the two option values starts to diverge, and at one point the value of random start option crosses the gain function; this of couse will never happen to the standard American option. The value at time t of the random start option is equal to value on the dashed curve if $t < \tau_S$, and is equal to the value of dotted curve if $t \geq \tau_S$.

We now proceed to calculate, at time 0, the mean time until the project is initiated. We start with some notation. Let $\tau_c^\star$ denote the optimal stopping time in the standard perpetuate American call option case, i.e.

$$\tau_c^\star = \inf\{t \geq 0 \,|\, X_t \geq L_c\}.$$

We let

$$\tau^\star = \inf\{t \geq \tau_S \,|\, X_t \geq L_c\}$$

be the optimal stopping time for the random start option, and we finally let

$$\tau_S^\star = \tau^\star - \tau_S$$

denote the time we wait until we optimally start the project after the random time τ_S has occured. With this notation we have

$$\tau_S^\star | X_{\tau_S} = x \quad \stackrel{d}{=} \quad \tau_c^\star | X_0 = x. \tag{3}$$

We are interested in the actual mean time until the option is exercised, so we use the objective measure P here, and we want to calculate

$$E_x\left[\tau^\star\right] = E_x\left[\tau_S\right] + E_x\left[\tau_S^\star\right].$$

We have $E_x\left[\tau_S\right] = 1/\gamma$ and use relation (3) to calculate $E_x\left[\tau_S^\star\right]$. We recall that under P the value process X has dynamics

$$dX_t = \mu X_t dt + \sigma X_t dW_t.$$

We now assume that

$$\nu = \mu - \frac{\sigma^2}{2} > 0.$$

The reason for doing this is that if this inequality does not hold, then if $x < L_c$ the expected time until we hit the critical level is infinite. One can show that if $\nu > 0$, then

$$E_x\left[e^{-\alpha\tau_c^\star}\right] = \begin{cases} 1 & \text{when } x \geq L_c \\ (x/L_c)^{\sqrt{(\mu/\sigma^2-1/2)^2+2\alpha/\sigma^2}-\mu/\sigma^2+1/2} & \text{when } x < L_c \end{cases}$$

(see e.g. Borodin & Salminen [3] p. 622). From this it follows that

$$E_x\left[\tau_c^\star\right] = \begin{cases} \frac{1}{\nu}\ln\left(L_c/x\right) & \text{when } x \leq L_c \\ 0 & \text{when } x > L_c, \end{cases}$$

and using relation (3) we get

$$\begin{aligned} E_x\left[\tau_S^\star\right] &= E_x\left[E_x\left[\tau_S^\star | X_{\tau_S}\right]\right] \\ &= E_x\left[\frac{1}{\nu}\ln(L_c/X_{\tau_S})\mathbf{1}(X_{\tau_S} \leq L_c)\right] \\ &= -\frac{1}{\nu}E_x\left[\ln(X_{\tau_S}/L_c)\mathbf{1}(\ln(X_{\tau_S}/L_c) \leq 0)\right]. \end{aligned}$$

We now give two examples of the distribution of τ_S, and also comment briefly on the general case.

An exponentially distributed starting time

In this case we need to calculate

$$-E\left[Y\mathbf{1}(Y \leq 0)\right] \text{ where } Y \sim \text{AL}\left(\ln(x/L_c), \frac{\sigma}{\sqrt{\gamma}}, \frac{\sqrt{\nu^2+2\gamma\sigma^2}-\nu}{\sqrt{2\gamma}\sigma}\right).$$

We have to distinguish between the two cases

(a) $\ln(x/L_c) \geq 0 \;\Leftrightarrow\; x \geq L_c$, and

(b) $\ln(x/L_c) < 0 \;\Leftrightarrow\; x < L_c$.

In case (a) we use the fact that if $h \geq 0$ and $a > 0$ then

$$\int_{-\infty}^{0} y e^{a(y-h)} dy = -\frac{e^{-ah}}{a^2},$$

and in case (b) that if $h < 0$ and $a, b > 0$ then

$$\int_{-\infty}^{h} y e^{a(y-h)} dy + \int_{h}^{0} y e^{-b(y-h)} dy = \frac{h}{a} - \frac{1}{a^2} + \frac{h}{b} + \frac{e^{bh} - 1}{b^2}.$$

In both cases we use these results with

$$h = \ln(x/L_c) \text{ and } a = \frac{2\gamma}{\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu},$$

and in case (b) we additionally set

$$b = \frac{\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu}{\sigma^2}.$$

Using these results together with Equation (3) we get the following expected times until the options is optimally exercised.

(a) When $x \geq L_c$:

$$E_x\left[\tau_S^\star\right] = \frac{\nu^2 + \gamma\sigma^2 - \nu\sqrt{\nu^2 + 2\gamma\sigma^2}}{2\nu\gamma^2} \left(\frac{x}{L_c}\right)^{-2\gamma/(\sqrt{\nu^2+2\gamma\sigma^2}-\nu)}.$$

(b) When $x < L_c$:

$$\begin{aligned} E_x\left[\tau_S^\star\right] \quad = \quad & \frac{\left(\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu\right)\ln\left(\frac{x}{L_c}\right)}{2\gamma} - \frac{\left(\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu\right)^2}{4\gamma^2} \\ & + \frac{\sigma^2}{\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu} \cdot \ln\left(\frac{x}{L_c}\right) \\ & + \frac{\sigma^4}{\left(\sqrt{\nu^2 + 2\gamma\sigma^2} - \nu\right)^2} \cdot \left[\left(\frac{x}{L_c}\right)^{\frac{\sqrt{\nu^2+2\gamma\sigma^2}-\nu}{\sigma^2}} - 1\right]. \end{aligned}$$

To get the mean time $E_x\left[\tau^\star\right]$ until the option is optimally exercised we simply add $E_x\left[\tau_S^\star\right] = 1/\gamma$ to the expression for $E_x\left[\tau_S^\star\right]$ above.

A deterministic starting time

When τ_S is a given fixed time,

$$\tau_S = T > 0,$$

then

$$E_x\left[\tau_S^\star\right] = -\frac{1}{\nu} E_x\left[\ln(X_T/L_c)\mathbf{1}(\ln(X_T/L_c) \leq 0)\right].$$

We now use the following well known result.

Lemma 3.2 *It is well known that if $Y \sim N(\mu, \sigma)$, then*

$$E\left[Y\mathbf{1}(Y \leq 0)\right] = \mu\Phi(-\mu/\sigma) - \sigma\varphi(-\mu/\sigma).$$

Since

$$\ln(X_T/L_c) = \ln(x/L_c) + \nu T + \sigma W_T \sim N\left(\ln(x/L_c) + \nu T, \sigma\sqrt{T}\right)$$

when $X_0 = x$, we get

$$E_x\left[\tau_S^\star\right] = \frac{1}{\nu}\left[\sigma\sqrt{T}\varphi\left(-\frac{\ln(x/L_c) + \nu T}{\sigma\sqrt{T}}\right) - \left(\ln(x/L_c) + \nu T\right)\Phi\left(-\frac{\ln(x/L_c) + \nu T}{\sigma\sqrt{T}}\right)\right].$$

Hence, the mean of the time until the option is exercised is

$$E_x\left[\tau^\star\right] = T + \frac{1}{\nu}\left[\sigma\sqrt{T}\varphi\left(-\frac{\ln(x/L_c) + \nu T}{\sigma\sqrt{T}}\right) - \left(\ln(x/L_c) + \nu T\right)\Phi\left(-\frac{\ln(x/L_c) + \nu T}{\sigma\sqrt{T}}\right)\right].$$

The general case
In general, when τ_S is any random variable, we can use Lemma 3.2 above. Using the fact that τ_S is independent of (X_t) we get

$$\ln(X_{\tau_S}/L_c)|\tau_S \sim N\left(\ln(x/L_c) + \nu\tau_S, \sigma\sqrt{\tau_S}\right)$$

when $X_0 = x$. It follows that

$$E_x\left[\ln(X_{\tau_S}/L_c)\mathbf{1}\left(\ln(X_{\tau_S}/L_c) \leq 0\right)\right] =$$

$$E_x\left[\ln(x/L_c) + \nu\tau_S\Phi\left(-\frac{\ln X_0 + \nu\tau_S}{\sigma\sqrt{\tau_S}}\right) - \sigma\sqrt{\tau_S}\varphi\left(-\frac{\ln x + \nu\tau_S}{\sigma\sqrt{\tau_S}}\right)\right].$$

If analytical methods are not working, the right-hand side can be calculated using simulation techniques.

3.4 An abandonment option

We start by describing the standard American version of this example. At time $t = 0$ we pay a sunk cost for the right to invest in a project at any future time. There is also a possibility to abandon the right to carry out the project, and in this case we get the recovery amount K. Hence we want to find

$$V(x) = \sup_\tau E_x^Q\left[e^{-r\tau}\max(X_\tau, K)\right],$$

where X_t is the value of the project if it is initiated at time t. Note that since the cost for the investment is paid for at the start, it does not enter into the optimal timing problem.

Under the dynamics given at the beginning of this Section, the optimal value is given by

$$V(x) = \begin{cases} K & \text{when } x \in [0, L_1] \\ K\left[\frac{-\beta_2}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_1} + \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_2}\right] & \text{when } x \in (L_1, L_2) \\ x & \text{when } x \in [L_2, \infty). \end{cases}$$

where

$$L_1 = K \cdot \frac{\beta_2}{\beta_2 - 1}\left(-\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2}\right)^{(1-\beta_1)/(\beta_1-\beta_2)}$$

and

$$L_2 = K \cdot \frac{\beta_2}{\beta_2 - 1}\left(-\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1 - \beta_2}\right)^{-\beta_1/(\beta_1-\beta_2)}.$$

A proof of this is given in Appendix A.2. See also Yu [17]. Let us now turn to the problem of valuing the random start version of this option. We start by writing the optimal value of the standard American perpetuate option as

$$\begin{aligned} V(x) &= K\mathbf{1}(x \le L_1) \\ &\quad + K\left[\frac{-\beta_2}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_1} + \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_2}\right]\mathbf{1}(L_1 < x < L_2) \\ &\quad + x\mathbf{1}(x \ge L_2) \\ &= K\mathbf{1}(x \le L_1) \\ &\quad + K\left[\frac{-\beta_2}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_1} + \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_2}\right](\mathbf{1}(x < L_2) - \mathbf{1}(x \le L_1)) \\ &\quad + x - x\mathbf{1}(x < L_2) \end{aligned}$$

Again we can use Proposition 3.1 to get the value of the random start version of the perpetuate option.

$$\begin{aligned}
\text{Value} \quad = \quad & \gamma K J\left(\gamma + r, \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), -\frac{r-\delta-\sigma^2/2}{\sigma}\right) \\
& + \frac{\gamma K}{\beta_1-\beta_2}\Bigg(-\beta_2\left(\frac{x}{L_1}\right)^{\beta_1}\left[J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right), -\sqrt{\left[\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right]^2+\frac{2r}{\sigma^2}}\right)\right. \\
& \left. - J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), -\sqrt{\left[\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right]^2+\frac{2r}{\sigma^2}}\right)\right] \\
& + \beta_1\left(\frac{x}{L_1}\right)^{\beta_2}\left[J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right), \sqrt{\left[\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right]^2+\frac{2r}{\sigma^2}}\right)\right. \\
& \left. - J\left(\gamma, \frac{1}{\sigma}\ln\left(\frac{L_1}{x}\right), \sqrt{\left[\frac{1}{2}-\frac{r-\delta}{\sigma^2}\right]^2+\frac{2r}{\sigma^2}}\right)\right]\Bigg) \\
& + x\frac{\gamma}{\gamma+\delta} - \gamma x J\left(\gamma+\delta, \frac{1}{\sigma}\ln\left(\frac{L_2}{x}\right), -\frac{r-\delta+\sigma^2/2}{\sigma}\right).
\end{aligned}$$

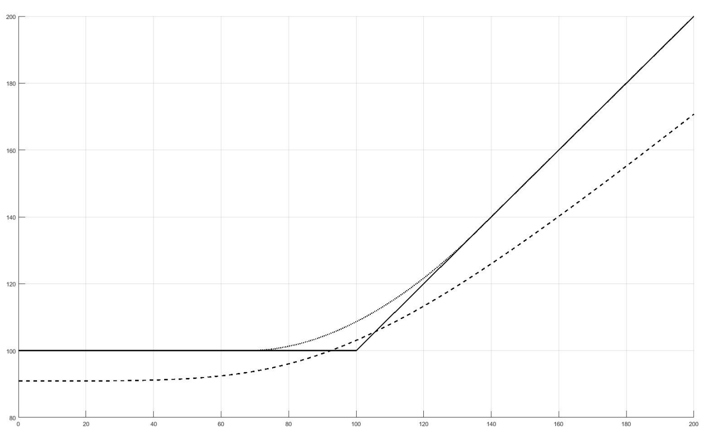

Figure 2: *Gain function (solid curve), value of the standard abandonment option (dotted curve) and value of the random start abandonment option (dashed curve) all with* $K = 100$ *as function of the initial value* x*. The parameter values are* $r = 0.01$*,* $\delta = 0.02$*,* $\sigma = 0.15$ *and* $\gamma = 0.1$*.*

See Figure 2 for an example of the value of the standard and the random start abandonment option respectively.

To calculate the mean time until the random start abandonment option is exercised we will use the fact that the expected time until a Brownian motion with drift μ per unit of time, volatility σ and started at x exits from the interval (a, b) is given by

$$m(x; a, b) = \frac{b-x}{\mu} - \frac{b-a}{\mu} \cdot \frac{e^{-\frac{2\mu x}{\sigma^2}} - e^{-\frac{2\mu b}{\sigma^2}}}{e^{-\frac{2\mu a}{\sigma^2}} - e^{-\frac{2\mu b}{\sigma^2}}} \quad \text{for } a \leq x \leq b$$

(see e.g. Dominé [5]). We now sketch how the expected time until the abandonment option is optimally exercise can be calculated. If we again let

$$\tau_c^\star = \inf\{t \geq 0 \,|\, X_t \notin (L_1, L_2)\}$$

be the optimal stopping time of the standard version of the perpetuate American option under consideration, and let

$$\tau^\star = \inf\{t \geq \tau_S^\star \,|\, X_t \notin (L_1, L_2)\}$$

be the optimal stopping time of the random start version of the option. We have

$$X_t \notin (L_1, L_2) \quad \Leftrightarrow \quad \ln X_t \notin (\ln L_1, \ln L_2),$$

and since (X_t) is a GBM we also have

$$\ln X_t = \ln x + \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t = \ln x + \nu t + \sigma W_t.$$

With

$$a = \ln L_1 \text{ and } b = \ln L_2$$

it follows that

$$\begin{aligned}
E_x\left[\tau_c^\star\right] &= m(\ln x; \ln L_1, \ln L_2) \\
&= \frac{\ln L_2 - \ln x}{\nu} - \frac{\ln L_2 - \ln L_1}{\nu} \cdot \frac{e^{-\frac{2\nu \ln x}{\sigma^2}} - e^{-\frac{2\nu \ln L_2}{\sigma^2}}}{e^{-\frac{2\nu \ln L_1}{\sigma^2}} - e^{-\frac{2\nu \ln L_2}{\sigma^2}}} \\
&= \frac{1}{\nu} \ln \frac{L_2}{x} - \frac{1}{\nu} \ln \frac{L_2}{L_1} \cdot \frac{L_2^{-2\nu/\sigma^2} - x^{-2\nu/\sigma^2}}{L_2^{-2\nu/\sigma^2} - L_1^{-2\nu/\sigma^2}} \\
&= \frac{1}{\nu} \ln \frac{L_2}{x} + \frac{1}{\nu} \cdot \frac{1 - (x/L_2)^{-2\nu/\sigma^2}}{1 - (L_1/L_2)^{-2\nu/\sigma^2}} \ln \frac{L_1}{L_2}.
\end{aligned}$$

Remark 3.3 *For $a > 0$ we have*

$$\lim_{z \downarrow 0} \frac{\ln z}{1 - z^{-a}} = 0,$$

so letting $L_1 \downarrow 0$ in the expression for $E_x\left[\tau_c^\star\right]$ above under the assumption that $\nu > 0$ we get

$$E_x\left[\tau_c^\star\right] = \frac{1}{\nu} \ln \frac{L_2}{x}.$$

With $L_2 = L_c$ this is consistent with the expression for the expectation in the optimal investment problem above.

Again we use

$$\tau_S^\star | X_{\tau_S} = x \quad \stackrel{d}{=} \quad \tau_c^\star | X_0 = x,$$

and get

$$\begin{aligned} E_x\left[\tau_S^\star\right] &= E_x\left[E_x\left[\tau_S^\star | X_{\tau_S}\right]\right] \\ &= E_x\left[m(\ln X_{\tau_S}; \ln L_1, \ln L_2)\mathbf{1}(L_1 \le X_{\tau_S} \le L_2)\right] \\ &= \frac{1}{\nu}\left(\ln L_2 + \frac{1}{1-(L_1/L_2)^{-2\nu/\sigma^2}} \ln\frac{L_1}{L_2}\right) P(L_1 \le X_{\tau_S} \le L_2) \\ &\quad -\frac{1}{\nu} E\left[\left(\ln X_{\tau_S} + \frac{(X_{\tau_S}/L_2)^{-2\nu/\sigma^2}}{1-(L_1/L_2)^{-2\nu/\sigma^2}} \ln\frac{L_1}{L_2}\right)\mathbf{1}(L_1 \le X_{\tau_S} \le L_2)\right]. \end{aligned}$$

Using the fact that $\ln X_{\tau_S}$ has a known distribution makes it possible to explicitly evaluate the expression on the right-hand side.

Remark 3.4 *A more realistic model is perhaps to consider the payoff function*

$$G(x) = \max(K, x - I).$$

In this model the investment can be terminated for a payoff of K, or initiated at a cost of I – in this case paid at the time the project is undertaken. The parameters β_1 and β_2 are the same (since they are determined by the dynamics of the underlying diffusion) as above, but the matching condition at the level at which we choose to initiate the project is different from the one above. It does not seem to exist an analytical solution in this case, so we have to use some numerical method to get the value of the standard American perpetual option.

4 Extensions

In this section we briefly comment on some possible ways of extending the model used here.

4.1 Lévy processes with negative jumps

Instead of assuming a geometric Brownian motion, as we did above, we can assume a more general model driven by a Lévy process Z which is assumed to have finite exponential moments and only negative jumps. In this case the solution to the standard American call option is known and has the same form as in the GBM case (see Mordecki [13] for details). More explicitly we assume that for $t \ge 0$ we have

$$X_t = X_0 e^{Z_t}$$

under Q (we focus on the valuation problem here), and as in the GBM case we will use Proposition 2.2 to calculate the value of the random start option when

$t < \tau_S$. In the Lévy process case we get for $u \geq t$

$$\begin{aligned} E^Q\left[X_u^a \mathbf{1}(X_u \leq b) | \mathcal{F}_t\right] &= X_t^a E^Q\left[e^{a(Z_u - Z_t)} \mathbf{1}(X_u \leq b)\right] \\ &= X_t^a E^Q\left[e^{aZ_{u-t}} \mathbf{1}\left(Z_{u-t} \leq \ln\left(\frac{b}{X_t}\right)\right)\right] \\ &= X_t^a \int_{-\infty}^{\ln(b/X_t)} e^{az} dF_{u-t}(z), \end{aligned}$$

where

$$F_t(z) = Q(Z_t \leq z).$$

To continue we need to be able to calculate this expression, and then proceed to prove a new version of Proposition 3.1. Even under the assumption of a constant intensity it seems hard to get explicit expressions for the value of any interesting random start options, and we will have to use numerical methods.

4.2 A more general model

If we move away from the Markovian case, then we need to use the general formula for the value of an American perpetual option with gain function $G : \mathbb{R} \to \mathbb{R}_+$ given by

$$U_t = \operatorname*{ess\,sup}_{\nu \in \mathcal{S}_t} E^Q\left[e^{-r(\nu - t)} G(X_\nu) \Big| \mathcal{F}_t\right].$$

Again $\mathcal{S}_t$ is the set of stopping times greater than or equal to t. In this case the value of the random start American perpetual option is given by

$$\text{Value} = \begin{cases} U_t \text{ on } \{\tau_S \leq t\} \\ E^Q\left[e^{-r(\tau_S - t)} U_{\tau_S} \big| \mathcal{F}_t\right] \text{ on } \{\tau_S > t\} \end{cases}$$

To get explicit expressions could be hard, but the important point to make is that we do not need to make the assumption of a Markovian model; the same principle holds for random start options in the general case.

4.3 A more general random time τ_S

Instead of assuming that τ_S is independent of the driving process(es) under both P and Q we can use constructions that are used in credit risk models. Let $H_t = \mathbf{1}(\tau_S \leq t)$ and define $\mathcal{H}_t = \sigma(H_u,\ 0 \leq u \leq t)$. One approach is to assume that the full information available at time $t \geq 0$ is given by the σ-algebra $\mathcal{G}_t$, which in turn is assumed to be decomposed according to $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$. Here $\mathcal{F}_t$ represents all information up to and including time t in excess of knowing if the random time τ_S has occured or not (this information is given by $\mathcal{H}_t$). In these type of models it is assumed that τ_S is not an $(\mathcal{F}_t)$-stopping time (it is obviously a $(\mathcal{G}_t)$-stopping time). In credit risk modelling this is known as the reduced form approach (see e.g. Jeanblanc et al [8] for more on reduced form modelling). If we are only interested in the value of the random start option, then it is quite straightforward to use this approach. If we want to use properties of τ_S under P, e.g. to calculate the mean time until an option is exercised, then we need to extend the reduced form models to also take care of the properties of τ_S under P.

5 Conclusions

We have considered a model in which an American option cannot be exercised until a random time has occured. The main application we have in mind is when an irreversible investment should be done (an example of a timing option), but where we have to wait for a permit before the investment can be done. The value of this optionalty is explicitly calculated, and we also determine the expected time until this random option is optimally exercised. As another application of the modelling framewaork presented, we also consider a version of an abandonment option. Again, it was possible to calculate the value of the random start version of this real option.

A Proofs

A.1 Proof of Proposition 3.1

We begin with two lemmas.

Lemma A.1 *Define for $k > 0$ and $L, M \in \mathbb{R}$*

$$J(k, L, M) = \int_0^\infty \Phi\left(M\sqrt{x} + \frac{L}{\sqrt{x}}\right) e^{-kx} dx,$$

where Φ is the distribution function of a standard normal distributed random variable. Then

$$J(k, L, M) = \begin{cases} \frac{1}{2k} e^{-L\left(M - \sqrt{M^2+2k}\right)} \left(\frac{M}{\sqrt{M^2+2k}} + 1\right) & \text{if } L < 0 \\ \frac{1}{k} + \frac{1}{2k} e^{-L\left(M + \sqrt{M^2+2k}\right)} \left(\frac{M}{\sqrt{M^2+2k}} - 1\right) & \text{if } L \geq 0 \end{cases}$$

For a proof, see Armerin & Song [1].

Lemma A.2 *If X is the geometric Brownian motion*

$$dX_t = \mu X_t dt + \sigma X_t dW_t$$

and $a, b \in \mathbb{R}$ are two constants, then for $0 \leq t < u$ it holds that

$$E\left[X_u^a \mathbf{1}(X_u \leq b) | \mathcal{F}_t\right] = x^a e^{a\left(\mu + (a-1)\sigma^2/2\right)(u-t)} \Phi\left(D(u-t)\right),$$

where

$$
\begin{aligned}
D(z) &= \frac{\ln(b/x) - \left(\mu - \sigma^2/2\right) z}{\sigma\sqrt{z}} - a\sigma\sqrt{z} \\
&= \frac{1}{\sigma}\ln\left(\frac{b}{x}\right) \cdot \frac{1}{\sqrt{z}} + \left(\frac{\sigma}{2} - \frac{\mu}{\sigma} - a\sigma\right)\sqrt{z}.
\end{aligned}
$$

Proof. Using

$$X_u = X_t e^{\left(\mu-\sigma^2/2\right)(u-t)+\sigma(W_u-W_t)}$$

we get

$$\mathbf{1}(X_u \le b) = \mathbf{1}\left(\frac{W_u - W_t}{\sqrt{u-t}} \le \frac{\ln(b/X_t) - \left(\mu - \sigma^2/2\right)(u-t)}{\sigma\sqrt{u-t}}\right).$$

Since

$$\frac{W_u - W_t}{\sqrt{u-t}} \sim N(0,1),$$

and letting $d(z) = D(z) + a\sigma\sqrt{z}$, we get

$$
\begin{aligned}
E\left[X_u^a \mathbf{1}(X_u \le b)|\mathcal{F}_t\right] &= X_t^a e^{a\left(\mu-\sigma^2/2\right)(u-t)} E\left[e^{a\sigma(W_u-W_t)}\mathbf{1}(X_u \le b)\middle|\mathcal{F}_t\right] \\
&= X_t^a e^{a\left(\mu-\sigma^2/2\right)(u-t)} \int_{-\infty}^{d(u-t)} e^{a\sigma\sqrt{u-t}z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\
&= X_t^a e^{\alpha\left(\mu-\sigma^2/2\right)(u-t)} \int_{-\infty}^{d(u-t)} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left[(z-a\sigma\sqrt{u-t})^2-a^2\sigma^2(u-t)\right]}dz \\
&= X_t^a e^{a\left(\mu+(a-1)\sigma^2/2\right)(u-t)}\Phi\left(d(u-t) - a\sigma\sqrt{u-t}\right) \\
&= X_t^a e^{a\left(\mu+(a-1)\sigma^2/2\right)(u-t)}\Phi\left(D(u-t)\right).
\end{aligned}
$$

$\square$

Here is know the proof of Proposition 3.1.

Proof. For general $a, b \in \mathbb{R}$ we have

$$
\begin{aligned}
E^Q\left[e^{-r(\tau_S - t)} X^a_{\tau_S} \mathbf{1}(X_{\tau_S} \le b) \middle| \mathcal{F}_t\right] &= \{\text{Use Proposition 2.2 with } f(x) = x^a \mathbf{1}(x \le b)\} \\
&= \int_t^\infty E^Q\left[X^a_u \mathbf{1}(X_u \le b) | \mathcal{F}_t\right] \gamma e^{-(r+\gamma)(u-t)} du \\
&= \{\text{Use Lemma A.2}\} \\
&= \int_t^\infty X^a_t e^{a(r-\delta+(a-1)\sigma^2/2)(u-t)} \Phi(D(u-t)) \gamma e^{-(r+\gamma)(u-t)} du \\
&= \gamma X^a_t \int_0^\infty e^{-[r+\gamma-a(r-\delta+(a-1)\sigma^2/2)]v} \Phi(D(v)) dv \\
&= \{\text{Use Lemma A.1}\} \\
&= \gamma X^a_t J\Big(r+\gamma-a(r-\delta+(a-1)\sigma^2/2), \\
&\quad \frac{1}{\sigma}\ln\frac{b}{X_t}, \frac{\sigma}{2} - \frac{r-\delta}{\sigma} - a\sigma\Big).
\end{aligned}
$$

With $a = \beta_i$ we get

$$r + \gamma - \beta_i(r - \delta + (\beta_i - 1)\sigma^2/2) = \gamma$$

and

$$\frac{\sigma}{2} - \frac{r-\delta}{\sigma} - \beta_i\sigma = -\mathrm{sgn}(\beta_i)\sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}},$$

and the proof is complete. □

A.2 The value of the abandonment option

We want to solve the problem

$$V(x) = \sup_\tau E^Q_x\left[e^{-r\tau}\max(K, X_\tau)\right],$$

where X is a geometric Brownian motion with dynamics given by

$$dX_t = (r-\delta)X_t dt + \sigma X_t dW^Q_t.$$

Here W^Q is a Q-Wiener process and we assume that $r > 0$, $\delta > 0$ and $\sigma > 0$. We also want to find, if it exists, an optimal stopping time $\tau^\star$ of this problem.

Theorem A.3 *The optimal value and the optimal stopping time to the optimal stopping problem above is given by*

$$V(x) = \begin{cases} K & \text{when } x \in [0, L_1] \\ K\left[\frac{-\beta_2}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_1} + \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_2}\right] & \text{when } x \in (L_1, L_2) \\ x & \text{when } x \in [L_2, \infty) \end{cases}$$

and

$$\tau^\star = \inf\{t \geq 0 \,|\, X_t = L_1 \text{ or } X_t = L_2\}$$

respectively, where

$$\begin{aligned}
\beta_1 &= \frac{1}{2} - \frac{r-\delta}{\sigma^2} + \sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}} > 1, \\
\beta_2 &= \frac{1}{2} - \frac{r-\delta}{\sigma^2} - \sqrt{\left[\frac{1}{2} - \frac{r-\delta}{\sigma^2}\right]^2 + \frac{2r}{\sigma^2}} < 0, \\
L_1 &= K\left(-\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1-\beta_2}\right)^{(1-\beta_1)/(\beta_1-\beta_2)} \frac{\beta_2}{\beta_2 - 1},
\end{aligned}$$

and

$$L_2 = K\left(-\frac{\beta_2}{\beta_1} \cdot \frac{\beta_1 - 1}{1-\beta_2}\right)^{-\beta_1/(\beta_1-\beta_2)} \frac{\beta_2}{\beta_2 - 1}.$$

To prove this, we will use the following observation from Mordecki [13]. We formulate it under the pricing measure Q, but this is only because it fits with our application, and for a general gain function G.

Observation A.4 *Let*

$$V(x) = \sup_\tau E_x^Q\left[e^{-r\tau}G(X_\tau)\right],$$

where the supremum is taken over the set of stopping times. If a function $\hat{V}$ and a stopping time $\hat{\tau}$ fulfills

$$\begin{array}{rlcl}
\text{(i)} & \hat{V}(x) & = & E_x^Q\left[e^{-r\hat{\tau}}G(X_{\hat{\tau}})\right] \\
\text{(ii)} & \hat{V}(x) & \geq & E_x^Q\left[e^{-r\tau}G(X_\tau)\right] \text{ for every stopping time } \tau,
\end{array}$$

then

$$V = \hat{V} \text{ and } \hat{\tau} \text{ is an optimal stopping time.}$$

Lemma A.5 *Assume that $G(x) \geq 0$ for every $x \in \mathbb{R}$. Sufficient conditions for* (ii) *in Observation A.4 to hold are*

- *$\hat{V}(x) \geq G(x)$ for every $x \in \mathbb{R}$, and*
- *$e^{-rt}\hat{V}(X_t)$ is a Q-supermartingale.*

Proof. Let τ be a stopping time. Since $e^{-rt}\hat{V}(X_t)$ is a supermartingale by assumption, for any $n \in \mathbb{Z}_+$ we have

$$E_x^Q\left[e^{-r(\tau\wedge n)}\hat{V}(X_{\tau\wedge n})\right] \leq \hat{V}(x).$$

It follows that

$$\liminf_{n\to\infty} E_x^Q\left[e^{-r(\tau\wedge n)}\hat{V}(X_{\tau\wedge n})\right] \le \hat{V}(x),$$

and using Fatou's lemma (since G is non-negative and $\hat{V} \ge G$, the process $e^{-rt}\hat{V}(X_t)$ is also non-negative) we get

$$E_x^Q\left[e^{-r\tau}\hat{V}(X_\tau)\right] \le \hat{V}(x).$$

Using $\hat{V}(x) \ge G(x)$ for every $x \in \mathbb{R}$ we finally get

$$E_x^Q\left[e^{-r\tau}G(X_\tau)\right] \le \hat{V}(x),$$

which is condition (ii). □

We now turn to the proof of Theorem A.3.

Proof. We use Observation A.4 with

$$\hat{V}(x) = \begin{cases} K & \text{when } x \in [0, L_1] \\ K\left[\frac{-\beta_2}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_1} + \frac{\beta_1}{\beta_1-\beta_2}\left(\frac{x}{L_1}\right)^{\beta_2}\right] & \text{when } x \in (L_1, L_2) \\ x & \text{when } x \in [L_2, \infty) \end{cases}$$

and

$$\hat{\tau} = \inf\{t \ge 0 \,|\, X_t = L_1 \text{ or } X_t = L_2\}.$$

We start by noting that $Q(\hat{\tau} < \infty) = 1$ and that

$$M_t^i = e^{-rt}X_t^{\beta_i},\ i = 1, 2,$$

are non-negative Q-martingales. It follows that

$$\begin{aligned} E_x^Q\left[e^{-r\hat{\tau}}\max(K, X_{\hat{\tau}})\right] &= E_x^Q\left[e^{-r\hat{\tau}}\hat{V}(X_{\hat{\tau}})\right] \\ &= E_x^Q\left[e^{-r\hat{\tau}}(k_1 X_{\hat{\tau}}^{\beta_1} + k_2 X_{\hat{\tau}}^{\beta_2})\right] \\ &= E_x^Q\left[k_1 M_{\hat{\tau}}^1 + k_2 M_{\hat{\tau}}^2\right]. \end{aligned}$$

Since for $i = 1, 2$ and every integer n we have

$$0 \le M_{\hat{\tau}\wedge n}^i \le L_2^{\beta_i}$$

we can use bounded convergence to get

$$E_x^Q\left[e^{-r\hat{\tau}}\max(K, X_{\hat{\tau}})\right] = \lim_{n\to\infty} E_x^Q\left[k_1 M_{\hat{\tau}\wedge n}^1 + k_2 M_{\hat{\tau}\wedge n}^2\right] = k_1 M_0^1 + k_2 M_0^2 = \hat{V}(x).$$

This shows that $\hat{V}$ and $\hat{\tau}$ satisfies condition (i). To prove that condition (ii) is satisfied we use Lemma A.5. We have

$$\hat{V}(x) \ge \max(K, x) = G(x)$$

and need to show that $e^{-rt}\hat{V}(X_t)$ is a Q-supermartingale. To do this we begin by defining the function

$$F(x) = \begin{cases} Kx^{-\beta_1} & \text{if } x \in (0, L_1] \\ A_1 + A_2 x^{\beta_2-\beta_1} & \text{if } x \in (L_1, L_2) \\ x^{1-\beta_1} & \text{if } x \in [L_2, \infty). \end{cases}$$

The function F is decreasing and concave, and we have

$$\hat{V}(x) = x^{\beta_1} F(x) \text{ when } x \in (0, \infty).$$

Now take $0 \le s \le t$ and introduce the measure Q^1 on $\mathcal{F}_t$ by using M_t^1 as Radon-Nikodym derivative with respect to Q:

$$\left.\frac{dQ^1}{dQ}\right|_{\mathcal{F}_t} = M_t^1.$$

We now get

$$\begin{aligned} E^Q\left[e^{-rt}\hat{V}(X_t)\middle|\,\mathcal{F}_s\right] &= E^Q\left[e^{-rt}X_t^{\beta_1}F(X_t)\middle|\,\mathcal{F}_s\right] \\ &= M_s^1 E^Q\left[\frac{M_t^1}{M_s^2}F(X_t)\middle|\,\mathcal{F}_s\right] \\ &= M_s^1 E^1\left[F(X_t)|\mathcal{F}_s\right] \\ &\le M_s^1 F\left(E^1\left[X_t|\mathcal{F}_s\right]\right) \\ &= M_s^1 F\left(\frac{E^Q\left[M_t^1 X_t\middle|\,\mathcal{F}_s\right]}{M_s^1}\right) \\ &\le M_s^1 F(X_s) \\ &= e^{-rs}X_s^{\beta_1}F(X_s) \\ &= e^{-rs}\hat{V}(X_s) \end{aligned}$$

The first inequality above follows from Jensen's inequality (since F is concave), and the second from the facts that $(M_t^1 X_t) = \left(e^{-rt}X_t^{\beta_1+1}\right)$ is a Q-submartingale and that F is decreasing. □

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