The Choice of Mean Reversion Stochastic Process for Real Option Valuation

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Abstract:

A main issue in financial derivatives and real options valuation is the choice of an adequate stochastic model to describe the price dynamics of the underlying asset. Particularly, in investment projects where there is significant managerial flexibility, the choice of stochastic process can have significant impacts on the project value and, therefore, the investment rule. This paper discusses several criteria for stochastic processes selection between Brownian Motion and Mean Reversion and the related theoretical, as well as practical, issues concerning the application of these to real options valuation. Empirical examples are used to illustrate the methods described along with guidelines for their implementation.

Key-words:

Mean Reversion Model, Geometric Brownian Motion, Stochastic Processes, Real Options

JEL codes:

C15, C53, G13, M21.

1 Introduction

Investment decisions involving stocks, financial derivatives and corporate projects are influenced by several sources of uncertainties. One way of dealing with these a priori unexplained market movements is the statistical modeling of the asset prices that relate to the specific problem at hand. The choice of an adequate statistical time series model (or, should one prefer, *stochastic process*; or *stochastic model*) plays a central role, for instance, in real option valuation, impacting not only the project value itself, but also the investment rule (Dixit & Pindyck, 1994; Schwartz, 1997).

In financial derivatives valuation, the geometric Brownian motion (GBM) is frequently assumed to be an appropriate choice to describe the behavior of stock prices and indexes (Black & Scholes, 1973; Cox et al., 1979). GBM has also been largely used in corporate project valuation using real options analysis (Trigeorgis, 1993). On the other hand, with commodities related projects and derivatives valuation, mean reversion models (MRM) have become a recurrent choice (Schwartz 1997, Gibson & Schwartz, 1990; Pindyck, 1999; Schwartz & Smith, 2000). The rationale behind this derives from micro-economics: while commodity price wanders randomly in the short term, they tend to converge to an equilibrium level in the long term, mirroring their marginal cost of production. Nevertheless, it is rarely easy to determine which model – GBM, or MRM – is a more adequate choice for modeling uncertain variables. Some of the questions that must be taken into account in the choice of an adequate stochastic process for modeling a particular derivative on an underlying asset are: (i) the economic features of the corresponding underlying asset; (2) the derivative lifetime; (3) the numerical and statistical issues regarding parameter estimation; and (4) the applicability of the chosen process in solutions (analytical or numerical) of the models used to valuation, among other factors.

This paper discusses several criteria for selecting an appropriate single factor stochastic process in real option valuation. Some of the generally adopted stochastic processes are briefly presented (GBM and MRM) and some statistical methods for model selection are revisited (unit root tests and variance ratio tests). Furthermore we propose the use of parameter ratios to better determine the adequacy of each process. Empirical examples involving nine commodities time series (three energy, three metals and three agricultural) are also provided for assessing the performance of the methods with real data sets. As an important complement, some guidelines regarding economic theory considerations, assets lifetime, numerical issues in parameter estimation, and the difficulties in real option valuation are suggested.

This paper is structured as follows: after this introduction, Section 2 offers, without claiming exhaustiveness, a review of the literature on some single factor stochastic processes frequently considered in real option analyses and their basic mathematical properties. Section 3 describes in some detail the statistical framework under which stochastic model selection is implemented; in this section, some empirical examples are given to illustrate the techniques. Section 4 suggests guidelines to future users of the methods discussed and in Section 5 we conclude by listing the main findings and a suggestion of themes for future research.

2 Single Factor Stochastic Processes in Real Option Valuation

We can define stochastic process as variables that move discretely or continuously in time unpredictably or, at least, partially randomly. Formally, let Ω be a set that represents the randomness, where $w \in \Omega$ denotes a state of the world and ***f*** a function which represents a stochastic process. The function ***f*** depends on $x \in R$ e $w \in \Omega$: $R \times \Omega \rightarrow R$ or $f(x,w)$, and it has the following property: given $w \in \Omega$, $f(\circ,w)$ becomes a function of only ***x***. Thus, for different values of $w \in \Omega$ we get different functions of ***x***. When ***x*** represents time, we can interpret $f(x,w_1)$ and $f(x,w_2)$ as two different trajectories that depend on different states of the world, as we can see in Figure 1:

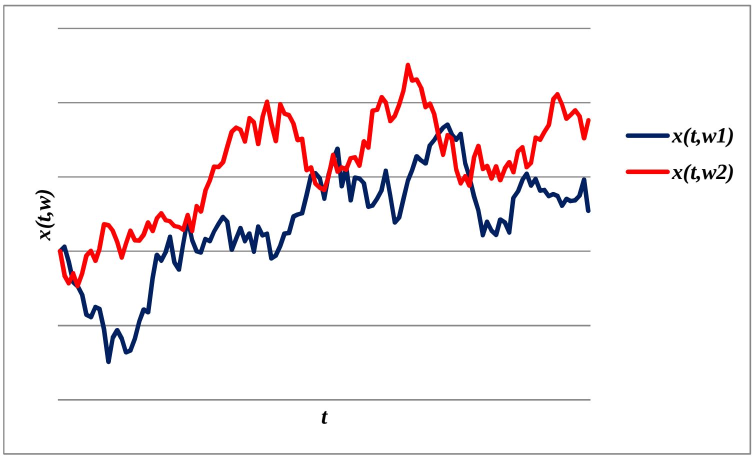


Figure 1 – Stochastic process trajectories.

In investment decisions, such as with real option valuation, the appropriate choice of the stochastic process used for modeling the dynamics of the asset prices involved is a matter of great relevance, since the forecasting and/or simulation of future cash flows crucially depends on conditional probability distributions of future trajectories. In the case of real options valuation, in which the uncertainties are directly considered in the future cash flow of the assets, the relevance is even greater.

A class of stochastic processes that plays an important role in financial modeling are Markov Processes. In a Markov Process, only the latest observed value is considered to be relevant to forecast future values. Such condition is commonly referred as the Markov property and is consistent with the Weak Form of Market Efficiency. One of the most popular types of Markov Processes is the Geometric Brownian Motion, which has been traditionally used in the modeling of financial options (Black & Scholes, 1973) and real option (Brennan & Schwartz, 1985; McDonald & Siegel, 1985, 1986; Paddock, Siegel, & Smith, 1988).

GBM is a continuous time process and is defined by equation (1).

$$dx = \alpha x dt + \sigma x dz, \tag{1}$$

where:

x is the asset price;
α is the *drift* parameter;
σ is the volatility parameter;
dt is the time increment;
dz is the standard Wiener increment.

Some interesting features of GBM, which are part of the reason for its popularity, are the small number of parameters that need to be estimated and the ease of application of analytical and numerical solutions for asset valuation. However, it has as a major drawback the fact that, under this model with a nonzero drift parameter, asset prices trajectories diverge with probability 1 as time increases. And, even considering a null drift parameter, its variance grows proportionally with time. Therefore, projected or simulated price trajectories could create unrealistic scenarios, which is an undesirable property in the case of long term assets. Figure 2 displays several simulated price trajectories from a GBM with a positive drift (on the left side) and the corresponding frequency distribution of a 20.000 events simulation; one can readily notice both the mean and the variance of the process increasing with time.

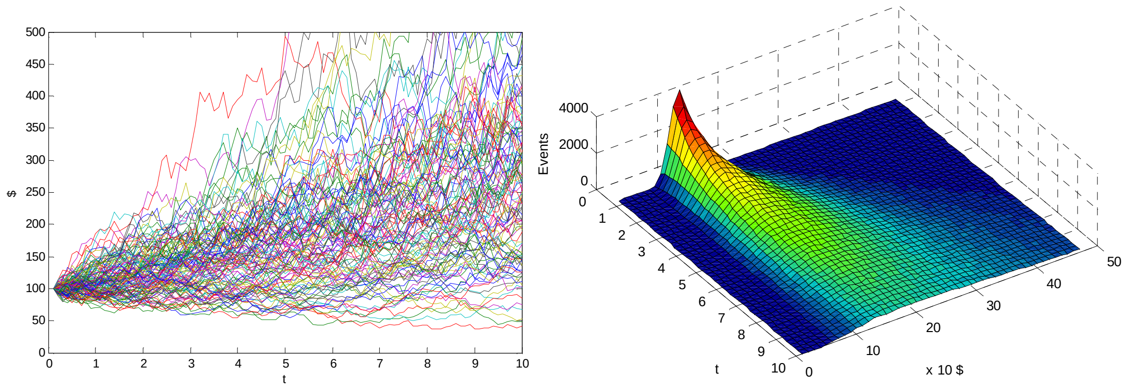


Figure 2 – GBM Price simulation 120 trajectories (left), with: Δ*t* = 0.1; α = 0.1 and σ = 0.2; and corresponding density distributions for 20,000 simulations (right).

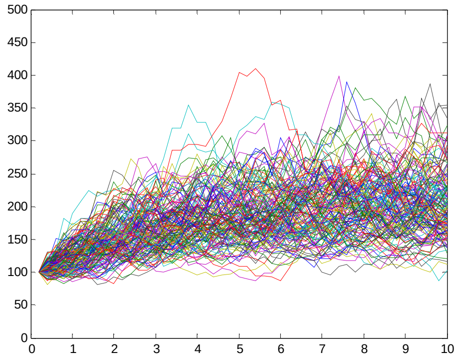
But with some variables, for which there are evidences and economic considerations supporting the claim that uncertainties in prices maintain an equilibrium level, such as in the case of commodities prices and interest rates, the appropriateness of the GBM has been vastly questioned (Bhattacharya, 1978; Brennan & Schwartz, 1985; Lund, 1993; Metcalf & Hasset, 1995; Smith & McCardle, 1998; Pindyck, 1999, 2001; Geman, 2005; Al-Harthy, 2007). Focusing on commodities, such as oil, copper, sugar or ethanol, it is common practice to assume that prices are at least partially driven by mean-reversion components, which makes these wander randomly in short term, but tend to converge to the equilibrium level in the long term, reflecting their marginal costs of production. A well-known mean reversion model (MRM) is the Ornstein Uhlenbeck (OU) model, whose expression, for its continuous-time version, is displayed in equation (2):

$$dx = \eta(\bar{x} - x)dt + \sigma dz \tag{2}$$

where:

x is the asset price;
$\bar{x}$ is the equilibrium level to which the process reverts in the long run;
η is the speed of reversion parameter;
σ is the volatility parameter;
dz is the standard Wiener increment.

Although the MRM is also a Markov process, its increments are not independent, as opposed to the case of the GBM. Also, as it is an arithmetic model, the use of the Ornstein-Uhlenbeck model for simulating future price paths might result in negative values. To avoid such inconvenience, it is useful to consider that the logarithm of prices follow an OU process, such as in Schwartz (1997), and prices therefore follow a geometric MRM. Figure 3 presents simulated trajectories for such an MRM.



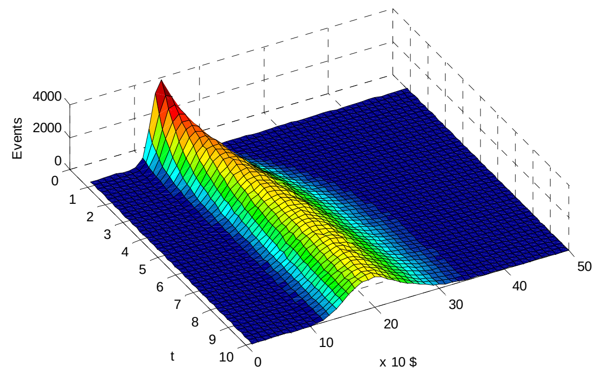


Figure 3 –MRM Price simulation 120 trajectories (left), with: Δ*t* = 0.1; *η* = 0.5;*σ* = 0.2; x_0 =100; $\bar{x}$= 200; and corresponding density distributions for 20,000 simulations (right).

Other authors (Gibson & Schwartz, 1990; Schwartz, 1997; Pindyck, 1999; Schwartz & Smith, 2000) focused on the stochastic behavior of commodity prices.

These authors claim that, besides a MRM factor, commodities prices may also have a stochastic trend factor. From the practical standpoint, such trend factor would alter the equilibrium level to which the process reverts in the long run. These changes would have additional motivations to momentary mismatches of supply and demand (captured by MRM) and they could be caused by the progressive exhaustion of natural resources and incremental costs related to new requirements of environmental laws, among other issues yielding a steady increase in the equilibrium level. On the other hand, improvements in the exploration and production technologies might yield a downward trend. Schwartz & Smith (2000) give support to these economic standpoints and propose a two-factor model, with a GBM and a MRM correlated components. The sum of these two stochastic factor results in the logarithm of the asset price, as it can be seen in eq.(6):

$$\ln S_t = \chi_t + \xi_t \tag{6}$$

where:

S_t is the spot price of the commodity;
χ_t is the factor which represents the changes of the prices in short term;
ξ_t is the factor which represents the tendency of the prices in the long run.
The differential equations of the two stochastic processes are:
$d\chi_t = -\kappa\, \chi_t dt + \sigma_\chi dz_\chi$
$d\xi_t = \mu_\xi dt + \sigma_\xi dz_\xi$

$dz_\chi dz_\xi = \rho dt$

where:

κ is the speedy of reversion parameter of the MRM component;
σ_χ is the volatility parameter of the short run changes in prices;

dz_χ is the Wiener increment of the short run changes in prices;
μ_ξ is the drift parameter of the long run price tendency;
σ_ξ is the volatility parameter of the long run price tendency;
dz_ξ is the Wiener increment of the long run price tendency;
ρ is the correlation parameter of the two factor increments.

In order to estimate the necessary parameters, the authors use future prices of commodities and applied the state space modeling approach combined with the Kalman filter (cf. Harvey, 1989; and Durbin & Koopman, 2001). But these examples of stochastic processes are multiple factor processes, and thus involve a significant level of complexity in parametrization as well as modeling, and will not be covered in this paper. Table 1 summarizes the different types of stochastic processes that have been briefly revisited so far in this paper.

Table 1 - Stochastic processes used in real options valuation.

Type of Stochastic Process	References
One Factor Models	
GBM	Paddock, Siegel & Smith (1988)
MRM	Dixit & Pindyck (1994), Schwartz (1997, model 1)
MRM with drift	Ozorio, Bastian-Pinto, Baidya & Brandao (2012)
Multiple Factor Models	
Combination of MRM and GBM	Gibson & Schwartz (1990), Schwartz (1997, models 2 & 3), Schwartz & Smith (2000)

3 Approaches for selecting stochastic processes

Some statistical tools, as well as time series parameter estimation, can be useful in the investigation of how much adequate a stochastic models is for asset price dynamics description. In this section we discuss three of these tools in some detail: (1) a unit root test; (2) the variance ratio test; and (3) parameter estimation and calibration.

3.1 Dickey-Fuller, or Unit Root test

The Dickey-Fuller (DF) unit root test (cf. Wooldridge, 2000; Enders, 2004) is frequently used to statistically assess if a commodity price is non-stationary. This test involves estimating a first-order autoregressive model for the lagged log of prices (Eq. (7)) and testing whether the autoregressive coefficient is 1. Confirmation of this hypothesis implies that the price is driven by a non-stationary process:

$$\ln(x_t) = a + b\ln(x_{t-1}) + \varepsilon_t \qquad (7)$$

where x_t is the asset price in the time t. Failure to reject the null hypothesis of $b = 1$ by means of a *t*-test should be taken as evidence of a GBM component. The critical values of DF test can be seen in the Table 2.

Table 2 - Asymptotic critical value of the t-test for the DF test.

Significance Level	1%	2.5%	5%	10%
Critical Values	-3.43	-3.12	-2.86	-2.57

Source: Wooldridge, 2000, p. 580

In case of autocorrelation between the log of prices and residues of the regression, the use of an Augmented Dickey-Fuller Test (ADF) is recommended. It is a direct extension of the DF test, where lags of $\Delta\ln(x_t) = \ln(x_t) - \ln(x_{t-1})$ are added as explanatory variables to the model in eq.(7), in order to diminish (or even eliminate) serial correlation of the random shock ε_t. Critical values remain the same as those given in Table 2.

Yet in the case of log-returns ($\Delta\ln(x_t)$) not presenting stationarity, the use of another enhancement on the ADF test is recommended, with the inclusion of a determinist trend, as shown in eq.(8):

$$\ln(x_t) = a + b\ln(x_{t-1}) + ct + \varepsilon_t \qquad (8)$$

This time, the corresponding critical values are changed, as shown in Table 3.

Table 3 - Asymptotic critical value of the t-test for the ADF test with a deterministic trend.

Significance Level	1%	2.5%	5%	10%
Critical Values	-3.96	-3.66	-3.41	-3.12

Source: Wooldridge, 2000, p. 583

It is usually difficult to reject the hypothesis that the price process dynamics follows a GBM through the use of ADF test, as demonstrated by Pindyck (1999). Nevertheless an adequate choice of process does not need be non-stationary. For instance, should the coefficient b in eq.(7) or eq.(8) be strictly less than 1 (but not that different from unity – which would mean strong persistence), there might be a presence of MRM dynamics, even in cases where the GBM has not been formally rejected. In order to illustrate these difficulties Dixit & Pindyck (1994) applied these unit root tests with 30 and 40-year price series and were not able to reject the hypothesis that oil prices would follow a GBM. It was necessary to make tests with 120-year series for the formal rejection of a unit root.

One way to verify the consistency of ADF test results is with the use of a complementary test: the Kwiatkowski–Phillips–Schmidt–Shin (KPSS) test, which can be used for checking a null hypothesis that a variable is stationary around a deterministic trend. KPSS type tests are intended to complement unit root tests, such as the Dickey–Fuller tests. Therefore when testing both the unit root (DF) and the stationarity (KPSS) hypothesis, one can distinguish series that appear to be stationary, series that appear to have a unit root, and series for which the data (or the tests) are not sufficiently informative to be sure whether they are in fact stationary.

Table 4 - Asymptotic critical value of the t-test for the KPSS test with a deterministic trend.

Significance Level	1%	2.5%	5%	10%
Critical Values	0.216	0.184	0.146	0.199

If it is not possible to consistently reject the presence of a unit root at a significant level, which would imply adequacy of a GBM modeling, running the KPSS test checks the stationarity of the series: rejection of it confirms the presence of some degree of mean reversion.

3.2 Variance Ratio Test

A second approach that can be used for investigating how adequate a stochastic model is for the price generating process, is to verify whether the variance of the log of prices increases proportionally to the lag considered. This approach measures how much the level of the stochastic shocks are persistent. With autoregressive processes – such as MRM – shocks tend to dissipate when there is a permanent reversion strength. In contrast, with GBM – which is not an autoregressive process – the shocks in prices are persistent, which is a necessary condition for the adequacy of the GBM. For such a task, Pindyck (1999) proposed a variance ratio test which consists of verifying if the variance of the log of prices increases proportionally in time. Rigorously speaking, this is not a statistical test in the formal sense. Instead, this is a simple, albeit quite powerful, graphical observation mean that works as follows. The “test statistic” measures the level to where the variance converges as the lags in the log returns increase, as shown in Eq. (9):

$$R_k = \frac{1}{k}\frac{Var\left(\ln P_{t+k} - \ln P_t\right)}{Var\left(\ln P_{t+1} - \ln P_t\right)}, \tag{9}$$

where $Var(.)$ denotes the sample variance. Should GBM be an adequate dynamic for P, then we would expect that variance increases proportionally and linearly with the lag of k periods, implying that R_k would converge to 1 as k grows indefinitely. Alternatively, in case of a MRM the variance is bounded to a certain level, implying that R_k will decrease, tending to zero, as k grows.

3.3 Parameter Indicators: Half-Life ($T_{1/2}$) and Normalized Variance (*NVar*)

Parameter estimation for different stochastic models can be quite revealing and useful for determination of the stochastic behavior of a time series. As seen with the variance ratio test, GBM assumes that stochastic shocks are permanent and do not dissipate with time. Also variance of the variable return should grow proportionally to time. In the case MRM the opposite happens: not only is the variance limited to a boundary as time increases but stochastic shocks also tends to dissipate under the effect of the mean reversing force or speed. Thus, estimation of the mean reverting speed parameter η and its intensity can clearly indicate how much mean reverting the time series has been behaving as.

As the Mean Reversion Speed Parameter η is estimated in a time basis but is not very indicative of its intensity (does a parameter of $\eta = 0.2$ indicate a strong mean reversion? And 0.5?) we can convert it to a less elusive magnitude. We can think of the half-life of a Mean Reverting time series in the same way we think of the half-life of a radioactive compound. The radioactive half-life for a given radioisotope is the time required for half the radioactive nuclei in a sample to undergo radioactive decay. Similarly, the half-life ($T_{1/2}$) of a time series indicates the expected time required for the S value of a series to reach half the separation it has now from the equilibrium level $\bar{S}$, given the mean reversion deterministic dynamics. Or, equivalently, the time for the stochastic shock to dissipate half of its effect on the time series values. The half-life ($T_{1/2}$) is inversely proportional to the mean reversion speed parameter and can be calculated by equation (10).

$$T_{1/2} = \frac{Log(2)}{\eta} \tag{10}$$

As it is a time measure, it provides a more intuitive indication of mean reversion force than the mean reversion parameter itself.

Another parameter which is also highly indicative of the presence of mean reversion is the value to which the variance of the series converges as the lime lag increases. In the presence of Brownian

Motion, variance tends to increase continuously in time, while under Mean Reversion, variance is restricted to limit value since the reversion force tends to dissipate the volatility of the series. This limit value is also inversely proportional to twice the mean reversion speed parameter and directly proportional to the square of the volatility parameter, and therefore is referred to as the Normalized Variance (*NVar*) parameter of the series. The *NVar* can be estimated by equation (11).

$$NVar = \frac{\sigma^2}{2\eta} \tag{11}$$

A graphical representation of the measurement of those parameters on the expected value of a time series is shown in Figure 4.

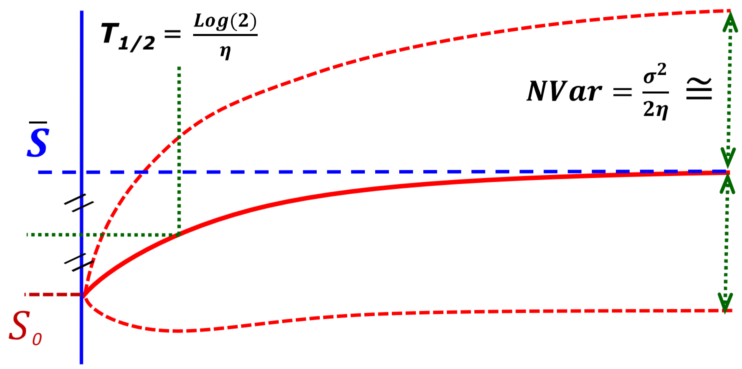


Figure 4 – Half-Life ($T_{1/2}$) Indication and Normalized Variance (*NVar*) of a Mean Reversion Time Series

Although both measures are inversely proportional to η, each indicates one characteristic of mean reversion: the half-life indicates about half the cycle involved in the dynamics present in a series. If is it too long (i.e. several years) mean reversion is hardly present in these dynamics. Likewise, if the normalized variance is too high it is unlikely that such a series are reverting to an equilibrium level, since its stochastic process (proportional to σ) dominates the dampening effect of the mean reversion (quantified by η).

4 Testing with Spot Prices Time Series

If order to illustrate application of the measures explained in chapter 3, we test nine different types of data sets of commodity prices: thirty three years of monthly closing spot prices (from December 1982 to December 2015) for Natural Gas (Henry Hub, data available only from 1991 on), Oil (Brent), Coal (Australian Thermal), Copper, Nickel, Aluminum, Soybeans, Cotton and Coffee (Other Mild Arabicas). These series are available through CBE group site and are displayed in Figures 5.

It is interesting to note that these series are highly volatile, and with great amplitude of their range as, in some cases, the high values are up to ten times greater than the low values during the time span studied. They also cover energy (black lines), metal (blue lines) and agricultural (green) commodities, covering a wide range of different products. Although they are historical data, the effect of inflation, will be covered in a specific chapter later on this paper. Also note that all graph in Figure 5 have 0 as their minimal value in the *y* axis, therefore they can be directly compared one to another.

A first visual inspection suggests that Aluminum and Coffee clearly display mean reversion behavior, while Cotton only marginally so. All others show clear trend or level change during the time span.

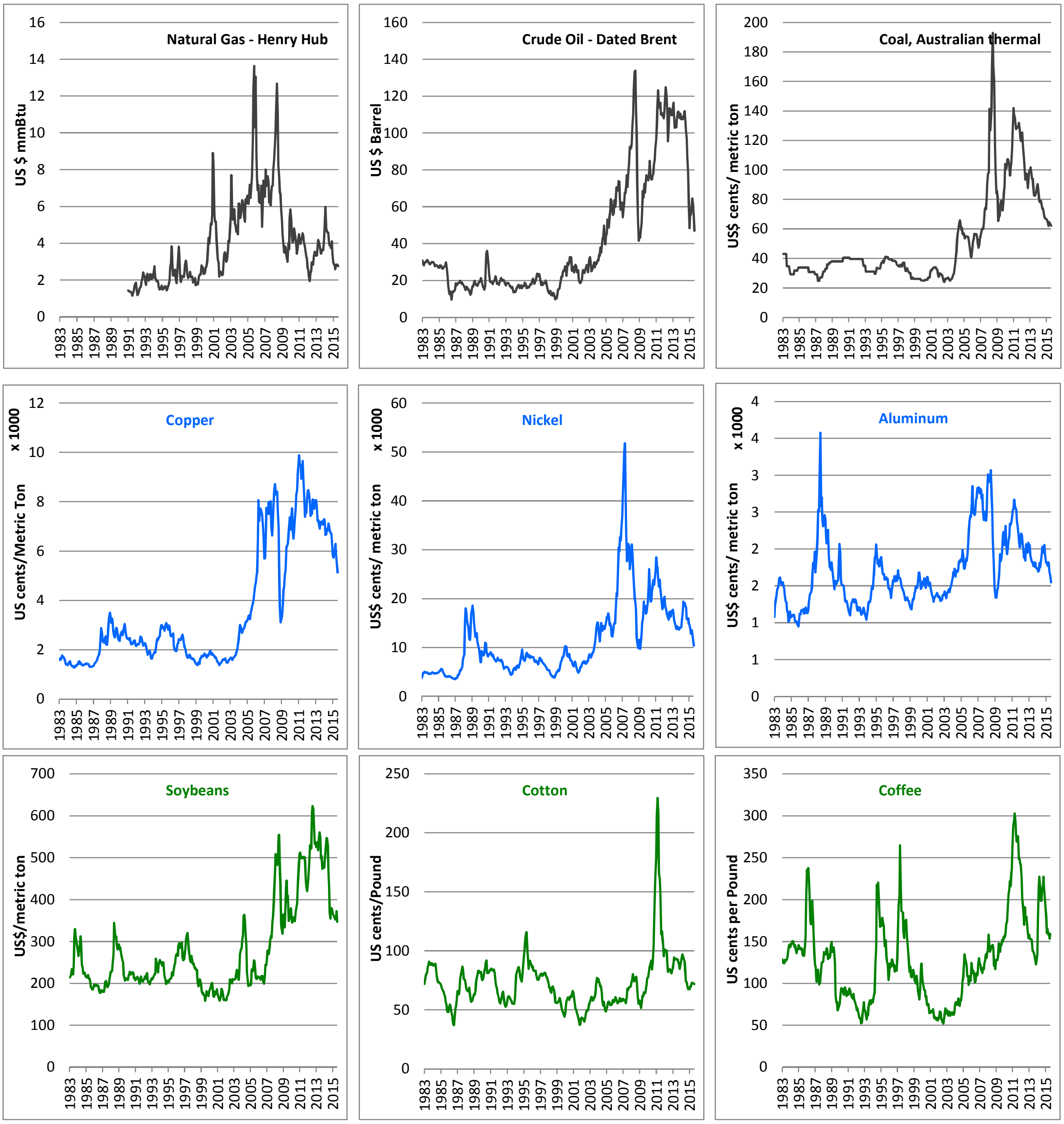


Figure 5 –Commodity monthly prices - Source: CBE Group

All nine series (log returns) are tested for unit foot presence, using DF, ADF and also KPSS, for trend and intercept in E-Views software. Results of these are displayed in Table 5.

Table 5 – Unit Root Tests – Series 1982 - 2015.

Test	Nat Gas	Oil Brent	Coal Austr	Copper	Nickel	Aluminum	Soybeans	Cotton	Coffee
DF	-2.956*	-2.072	-1.923	-2.400	-2.567	-2.500	-2.565	-3.608^{+}	-2.459
ADF	-2.924	-2.859	-2.539	-2.444	-2.666	-2.931	-2.954	-3.635*	-2.616
KPSS	0.251^{+}	0.502^{+}	0.413^{+}	0.303^{+}	0.158*	0.099	0.372^{+}	0.195*	0.282^{+}

Statistical Significance: $^{+}$: 1% ; *: 5%

These same ADF results are displayed graphically in Figure 6 for a better estimation of significance.

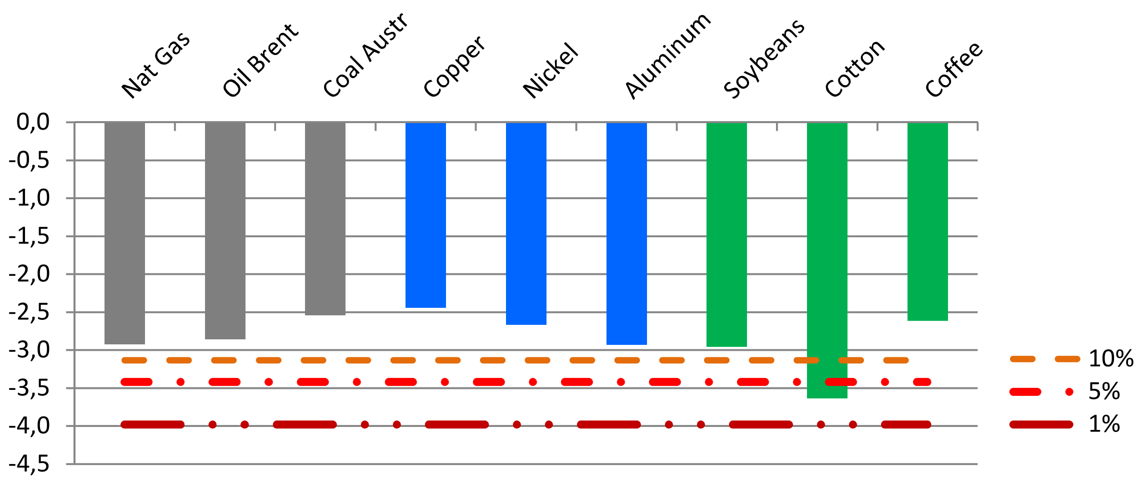


Figure 6 –ADF test results

Contrary to expectation for ADF, only Cotton series reject the presence of a unit root and only at a 5% level of significance, suggesting that all other series of price should not be modeled as MRM. Aluminum, Natural Gas, Oil and Soybeans come close to a 10% level and Coffee contrary to expectations stays clearly out of rejection range.

Use of the variance ratio test is also applied to the same time series. Figure 7 displays the results of the variance ratio test with these, and as can be observed after initially rising above the unity, for most series the variance ratio drops below 1 after a lag of about 30 to 60 months. Differently Coal and Copper stay above unity for all the time span of 120 months (10 years) clearly sustaining that for these prices, variance keeps growing with time indicating the presence of a GBM. In the case of Natural Gas, Oil and Soybeans, the ration drops below unity rather rapidly but only with Natural Gas it keeps going down below 0.5. With Oil and Soybeans prices variance ratio stabilizes above 0.5 for the duration of the analyzed time span. With Nickel, Aluminum, Cotton and Coffee, the ratio drops below 1 after a higher value of k (lag) but only with Aluminum does it continue dropping clearly below 0.5.

A variance ratio below 1 is a strong indication that the prices are certainly not exclusively driven by GBM dynamics, and that an MRM type of process might prove to be more suitable for stochastic modeling of theses series, or at least be also present.

These results indicate that Coal and Copper are clearly GBM type of series, Natural Gas and Aluminum can be modeled with MRM type of dynamics, and all others display presence of both type of dynamics in their behavior (GBM and MRM).

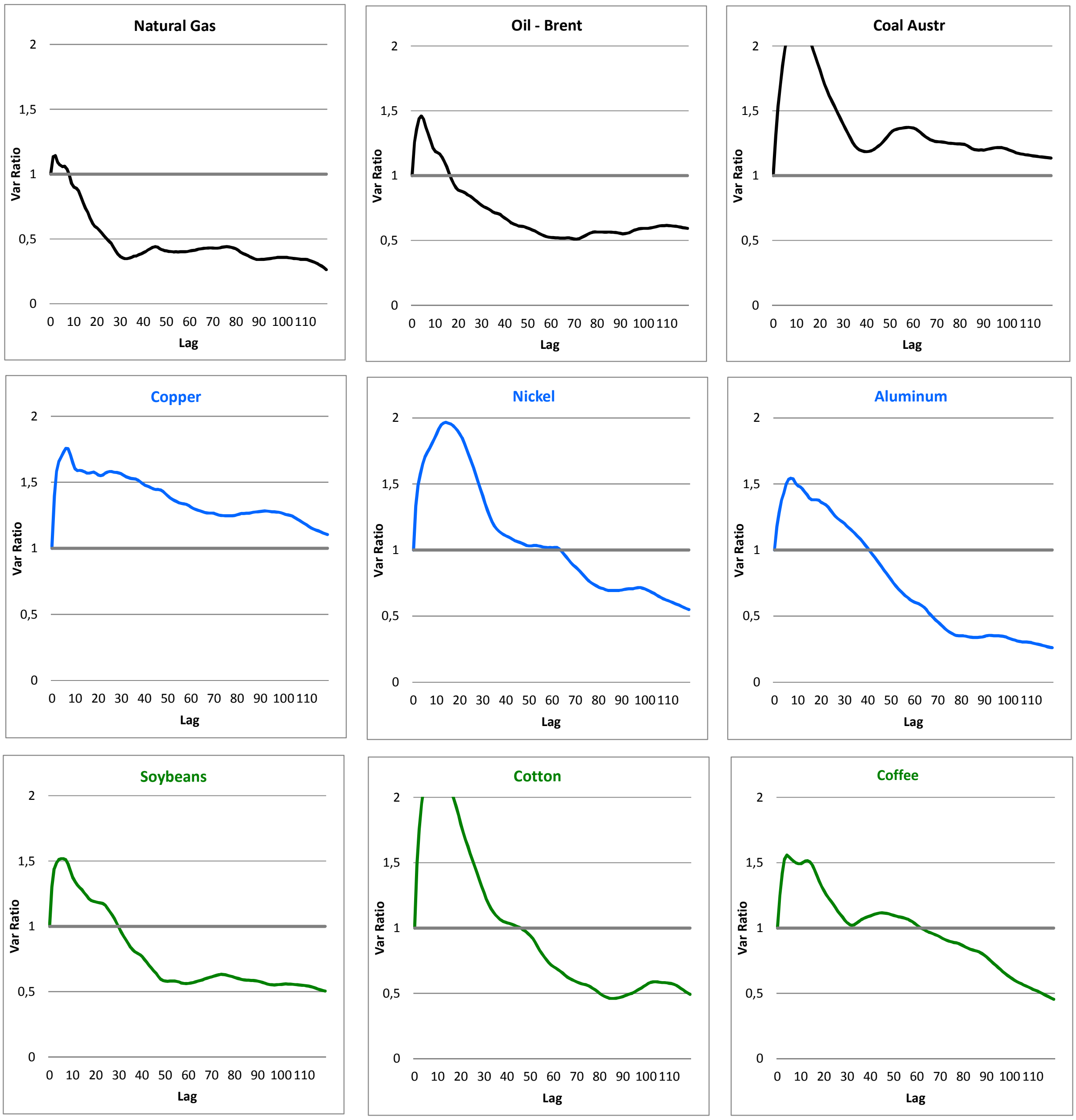


Figure 8 –Variance Ratio tests of Spot Price Series

As a tie breaker, we then estimate the parameters indicators for all nine time series for modeling these as GBM and MRM. For this last stochastic model we consider the Schwartz (1997) model 1 type of single factor geometric MRM. Basic parameters (drift and volatility, α, σ for GBM and reversion speed, volatility and equilibrium level) where calculated based on the log return of the series, and equations of Appendix 1. With these we calculated the Half-life time and Normalized Variance for all nine time series. These are displayed in Table 6, all parameters are in yearly values.

Table 6 – Parameters – Series 1982 - 2015

	Natural Gas	Oil Brent	Coal Austr.	Copper	Nickel	Alumin.	Soybean	Cotton	Coffee
MGB									
Drift α	3.1%	1.2%	1.0%	3.8%	3.3%	1.4%	1.6%	0.1%	0.5%
Vol σ	40.1%	30.9%	18.8%	22.0%	29.7%	19.8%	19.9%	19.7%	27.4%
MRM									
Rev Speed η	0.41	0.09	0.07	0.07	0.16	0.34	0.17	0.27	0.23
Vol σ	46.2%	31.1%	18.9%	22.1%	29.8%	20.0%	20.0%	19.9%	27.6%
Parameter Indicator									
Half-life: $T_{1/2}$	1.7	7.5	10.3	10.2	4.2	2.1	4.1	2.6	3.0
$NVar$: $\sigma^2/2\eta$	0.26	0.52	0.27	0.36	0.27	0.06	0.12	0.07	0.17

In order to better compare these results, we plotted them in Figure 9 with Half-life as x and Normalized Volatility as y axis. The closer the series are situated to the axis origins, the more MRM characteristic they bear. Oppositely the farther they are from the origins the more they resemble a GBM. With this approach we can see that all three agricultural commodities, as well as aluminum and Natural gas, bear MRM resemblance, whereas Oil, Coal and Copper are GBM. Nickel stands in an intermediate region.

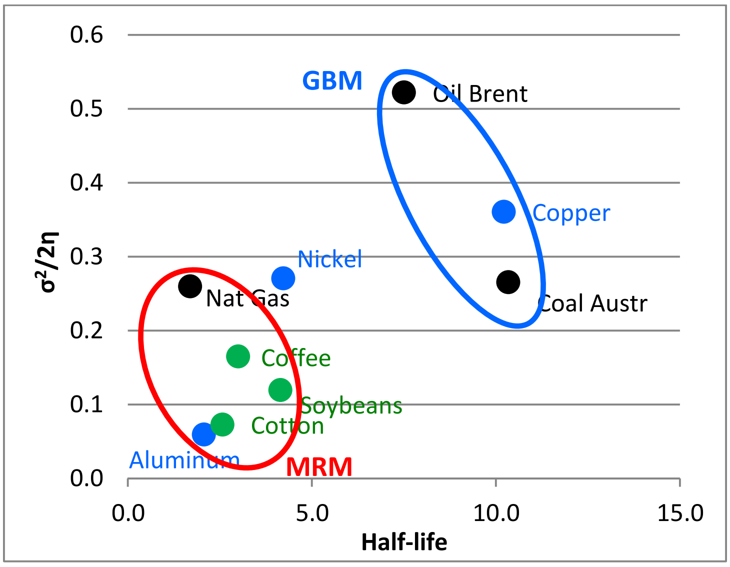


Figure 8 – *Half life* x *NVar* of Spot Price Series

As a summary of the approaches used on all nine price series, we can see in Table 7 that not all results are coincident. Parameters Indicators, such as the ones measured ($T_{1/2}$ and $NVar$), show more applicability: almost all series are clearly classified as either GBM or MRM, and only Nickel stands marginally out of the MRM group. Also using two indicators allows us to show how different the characteristics of each series are: Natural Gas and Coal have a comparable value of Normalized Variance but very different Half-Lifes, clearly differentiating both series.

Table 7 – Summary of results of methods in time series

	Nat Gas	Oil Brent	Coal Austr	Copper	Nickel	Aluminum	Soybeans	Cotton	Coffee
ADF	Undef.	Undef.	Undef.	Undef.	Undef.	Undef.	Undef.	MRM	Undef.
Var Ratio	MRM	Undef.	GBM	GBM	Undef.	MRM	Undef.	Undef.	Undef.
Parameters	MRM	GBM	GBM	GBM	Undef.	MRM	MRM	MRM	MRM

5 Theoretical considerations on the selection of stochastic process for real option valuation

Aside from the statistical approaches discussed in section 3, theoretical considerations supported by economic theory should be considered when one is choosing a model for asset prices in real options valuation. For instance, the assumption of an equilibrium mechanism for the prices would justify the use of MRM dynamics to represent the behavior of some asset price. As another example, evidences of gradual increases in the production marginal cost and the occurrence of rare events (such as crisis and wars) would support some mixed models involving MRM, GBM and JDP (Jump Diffusion Models) components.

Another relevant issue to be considered when choosing a stochastic model is the lifetime of the real option to be valued. Generally, if the lifetime of the derivative is relatively short, such as with financial options which generally have a maturity of only a few months, exhaustive investigation of best stochastic process is not such a relevant issue. In such cases, the choice of an adequate stochastic process can be driven by ease of parameter estimation and option price valuation model. Dixit & Pindyck (1994) argue that, in short periods of time, price processes of GBM type are mainly guided by the stochastic shocks, whereas as time passes the drift component becomes more relevant. Thus, since many models are based on Wiener increments similarly to GBM, the search for an appropriate process can be considered an "expensive" task. On the other hand, when dealing with long lifetime assets, searching for the "best" stochastic process can be crucial to its valuation and the investment rule decision. Bastian-Pinto, Brandão & Hahn (2009) show that the switch-option value for Brazilian sugar-ethanol industry, can change from 20% to 70% increment of the base case, when the price uncertainties (in this case sugar and ethanol) are modeled as an MRM or as an GBM, respectively. Kerr, Martin, Pereira, Kimura & Lima (2009) estimate the optimum trees cutting time in the forest products investments considering uncertainties modeled with MRM and GBM. They conclude that the critical prices level for the cutting decision relative to the waiting time change substantially with different types of process. In the case studied by the authors, the use of an MRM would anticipate the exercise decision of the cutting time option as when the results are compared to GBM.

Although the mixing of different stochastic processes might yield more realistic models, this implies in using multifactor models and certainly implies in greater difficulty regarding parameters estimation. Usually, multiple factor models require future price series of the assets for parameter calibration, such as in Schwartz (1997) and Schwartz & Smith (2000). These authors use the state space/Kalman filter approach to estimate the models parameters. Nevertheless, for several commodities and other variables, future prices are not available, such as is the case with ethanol prices and the volume of traffic on a toll road. In these cases, despite the advantages of using multiple factor models, the choice of stochastic process might be influenced by limitations related to the database availability.

Regarding the quality of the available databases for parameter estimation, it is important to consider both the length and the frequency of the price series. As a general rule, the use of long time series is recommended for estimating the drift parameters. Taking into account that the variance of the drift estimator is proportional to time, the longer the series, the more efficient will be the estimator. Also, the frequency is relevant to calibrate the volatility parameters: the higher the frequency of the series the better the volatility estimator quality.

Finally, an issue that should be considered in the choice of the stochastic process is its applicability of close form solutions and/or numerical solutions used in real options valuation. Comparing GBM with other models such as MRM, one of its biggest advantages is the small number of parameters needed for calibration and the ease in obtaining analytical solutions, which are huge incentives to its use. Generally, the use of MRM does not permit the use of analytical solutions to value to real options at hand, which implies in the use of numerical solutions such as Monte Carlo Simulation (MCS) and Binomial Lattices[1]. Usually, it is possible to obtain solutions for multiple factor models using MCS or when there is more than one uncertainty to be considered in the analysis. It is important to observe that before Longstaff & Schwartz (2001) MCS was only used in the solution of European options and since then, with the development of optimization methods pluggable to MCS.

[1] Nelson & Ramaswamy (1990) and Bastian-Pinto, Brandão & Hahn (2010) present alternative approaches for binomial lattice to the MRM.

6 Conclusions

This work focused on discussing some approaches for investigating the appropriateness of using MRM stochastic processes in real options valuation problems, and tested these with several times series. Results indicate that traditional approaches such as ADF and even Variance Ratio measures may be insufficient to clearly determine whether a time series bears resemblance with a MRM diffusion process and that measuring their parameters may be a more informative way to determine which process to choose. In many practical situations – mainly in projects with long lifetime – the choice of a stochastic process can be relevant for real options valuation, as its dictates both option valuation and optimal investment rule.

A suggestion for future research in this area is a formal derivation of asymptotic properties of parameter estimators for the some models considered in this paper, given that the uncertainty come from plugging estimates on the conditional expectation formulae might change results in option price valuation.

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