Time (in)consistency and real options: Much ado about nothing?

Josef Schosser*

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Abstract

If decision makers have exclusive rights to particular investment projects, they frequently have the opportunity to delay these investments. This paper analyzes the effect of quasi-hyperbolic discounting, i.e. time-inconsistent preferences on the exercise timing of such options to defer an investment. It complements earlier work on this issue covering risk aversion and capital market interaction. The results are as follows: In a number of cases, the capital market environment provides for the irrelevance of quasi-hyperbolic discounting. Besides this, a different behavior of time-inconsistent and time-consistent decision makers occurs only for quite specific parameter conditions. In light of experimental evidence for time-inconsistent behavior, this provokes the following question: Is time-inconsistent behavior really driven by quasi-hyperbolic discounting, but rather by more fundamental irrationality (like a disregard of fairly ubiquitous market opportunities)?

Keywords: Capital market interaction, Quasi-hyperbolic discounting, Real options, Risk attitude, Time inconsistency.

JEL Classification: D81, D91, G13, G31.

*University of Passau, Department of Business Administration and Economics, 94030 Passau, Tel. +49 851 5093274, e-mail: josef.schosser@uni-passau.de.

1 Introduction

Time inconsistency (sometimes called dynamic inconsistency) is among the most extensively discussed behavioral departures of decision theoretic directives. In the words of Lengwiler (2004, p. 145), it can be described as follows: "We say that an ... intertemporal decision is *time consistent* (emphasis in original) if, when the next period comes along, it will be optimal for you to stick to the same planned path. The decision is *time inconsistent* if a plan that is optimal from the point of view of one period is no longer optimal from the point of view of a later period."

Our paper is concerned with so-called real options. This term, coined by Stewart Myers (Myers, 1977) indicates investments that offer options for future action and are thus characterized by several decision points in time, meaning a dynamic decision structure (cf. Dixit and Pindyck, 1994). Related literature refers to different options providing managerial flexibility (e.g. the option to abandon, contract, expand, stage, defer, grow or switch). Our focus is on the option to defer, or delay, an investment. The value of this option depends on the time of its "exercise". Exercising suboptimally (where it is worth waiting or when the optimal exericse time has passed) reduces its value. This is where the problem of time inconsistency arises.

To date, only a few papers have tackled the issue of time inconsistency in the sense of quasi-hyperbolic discounting and options to defer an investment. The most important one, the study by Grenadier and Wang (2007), is concerned with risk-neutral decision makers that do not deal with (or have no access to) capital market interaction. A similar setting has been developed by Quah and Strulovici (2013) in the context of optimal stopping. Clearly, the option to defer an investment can be understood as a problem of optimal stopping, where the flow payoffs are zero and the termination payoff equals the benefits of exercising the option. Yet, the assumption of risk neutrality is quite problematic as experimental and empirical evidence shows that decision makers are risk-averse (for an overview, see Lengwiler, 2004, pp. 68–101). Moreover, following the economic principle, opportunities to trade on the capital market should not be overlooked, as the devaluation of future results occurs not only due to impatience or pure time preference but also as a consequence of (market-based) opportunity costs (Mulligan, 1996; Read, 2004). Additionally, financial instruments permit the decision maker to optimally spread consumption over time and states according to their preferences (Smith and Nau, 1995; Smith, 1998).

Therefore, the paper at hand is structured along two dimensions (see Table 1). The first one refers to the decision maker's risk attitude and includes the assumptions of risk neutrality and risk aversion. The second one covers the extent to which capital market strategies are admissible. Here, distinction is made between the cases of no capital market, only risk-free borrowing and lending, as well as spanning. In contrast to the continuous-time models developed by Grenadier and Wang (2007) and Quah and Strulovici (2013), the framework presented exhibits a time-discrete structure. In a way, it builds on the article of Khan et al. (2013). Their paper deals with related questions, yet mingles market-based and preference-based valuation in a somewhat debatable manner. We will return to Khan et al. (2013) in a later section.

The remainder of the paper progresses as follows. The next section introduces our basic framework and reflects the impact of quasi-hyperbolic dis-

Table 1: Classification of subject-based literature

risk attitude	admissible capital market strategies: no capital market	incomplete market in the sense of (only) risk-free borrowing and lending	spanning
risk neutrality	Grenadier and Wang (2007), Quah and Strulovici (2013)	this paper (scenario I)	this paper (scenario III)
risk aversion	this paper (scenario IV)	this paper (scenario II)	this paper (scenario III)

counting for certain combinations of risk attitude and admissible capital market strategies. In Section three, we present a numerical example and perform a comparative static analysis. Section four summarizes major findings and shows some implications.

2 The model

2.1 Formalization of time (in)consistency

Frequently, the concept of stationarity is used to formalize time consistency (e.g. Fishburn and Rubinstein, 1982, p. 681). It is stated as follows. Provided (x, h^1) denotes an outcome x to be received h^1 periods ahead, the following relation holds.

$$\text{If } (x, h^1) \succ (y, h^2) \text{ then } (x, h^1 + \Delta) \succ (y, h^2 + \Delta)$$
$$(\text{analogous for } \prec \text{ as well as } \sim).$$

That means, a comparison between two time-dependent outcomes depends only on the difference $h^2 - h^1$ between the points in time. If the points in time are advanced or deferred by the same amount Δ, the order of preference will be preserved. Experimental evidence suggests frequent violations of stationarity (e.g. Thaler, 1981; Benzion et al., 1989), i.e. time-inconsistent behavior.

The most common way to model time-inconsistent preferences is to assume that at each point in time, the decision maker applies so-called quasi-hyperbolic discounting in accordance with ($\forall t = 0, 1, \ldots, T$)

$$\mathbb{U}^t(x_t, x_{t+1}, \ldots, x_T) = U(x_t) + \beta \sum_{\tau=t+1}^{T} \delta^{(\tau-t)} U(x_\tau)$$
$$\text{and } 0 < \beta, \delta < 1$$

(Phelps and Pollak, 1968; Laibson, 1997). Here, $(x_t, x_{t+1}, \ldots, x_T)$ denotes a stream of cash flows. $U(x_\tau)$ represent identical functions of periodical cash flows. These utilities are "discounted" by the factors β and δ. In particular, $t = 0$ implies

$$\mathbb{U}^0(x_0, x_1, \ldots, x_T) = U(x_0) + \beta \sum_{\tau=1}^{T} \delta^{\tau} U(x_\tau).$$

It is easy to show that these preferences may lead to dynamic inconsistency. Imagine the choice between deterministic cash flows $\hat{x}_\tau$ ($\tau > 0$) and $\hat{x}_{\tau+1}$. As of time $t = 0$, the decision maker exhibits a preference for $\hat{x}_\tau$ if and only if

$$\beta\delta^\tau U(\hat{x}_\tau) > \beta\delta^{\tau+1}U(\hat{x}_{\tau+1}) \tag{1}$$

or

$$U(\hat{x}_\tau) > \delta U(\hat{x}_{\tau+1}). \tag{2}$$

Yet, from the point of view of $t = \tau$, the decision maker will prefer $\hat{x}_\tau$ if and only if

$$U(\hat{x}_\tau) > \beta\delta U(\hat{x}_{\tau+1}), \tag{3}$$

which clearly contradicts the former.

2.2 Basic framework

In the following, we set up a simple two-period model. The preferences of the decision maker are given by the intertemporal utility function

$$\mathbb{U}^0(x_0, x_1, x_2) = U(x_0) + \beta \sum_{\tau=1}^{2} \delta^\tau U(x_\tau).$$

Where $\beta = 1$, the familiar concept of exponential discounting is obtained. $\beta < 1$ results in quasi-hyperbolic discounting as described above. Concerning periodical utility, we differentiate between the cases of risk neutrality

$$\begin{aligned} U(x_\tau) &= x_\tau \\ \Rightarrow \mathbb{U}^0(x_0, x_1, x_2) &= x_0 + \beta \sum_{\tau=1}^{2} \delta^\tau x_\tau \end{aligned} \tag{RN}$$

and (constant absolute) risk aversion

$$\begin{aligned} U(x_\tau) &= 1 - e^{-\alpha x_\tau} \\ \Rightarrow \mathbb{U}^0(x_0, x_1, x_2) &= 1 - e^{-\alpha x_0} + \beta \sum_{\tau=1}^{2} \delta^\tau \left(1 - e^{-\alpha x_\tau}\right).^1 \end{aligned} \tag{RA}$$

We want to analyze the following option to defer an investment. As shown in Figure 1, the benefits of exercising the option (modeled as a lump-sum payoff) follow a multiplicative binomial process, where b denotes the initial value. u and d represent growth factors that satisfy $u > d$ and $u = {}^1\!/_d$. p indicates the probability of an up-move.

"Exercise" of the option is possible in $t = 1$ or $t = 2$ at cost of investment f. To simplify calculations, $b = f$ is assumed. If the option is held until maturity at date $t = 2$, the decision maker will only exercise if two successive up-moves occur, leading to a cash flow of $u^2 b - f$. A premature exercise at $t = 1$ may be advantageous after one up-move. The resulting cash flow of $ub - f$ has to be weighed up against the consequences of deferral. The latter amount to $u^2 b - f$ (two up-moves) and 0 (one up- and one down-move), respectively.

[1] According to Kirkwood (2004), an appropriately chosen exponential utility function closely approximates general utility functions in most cases.

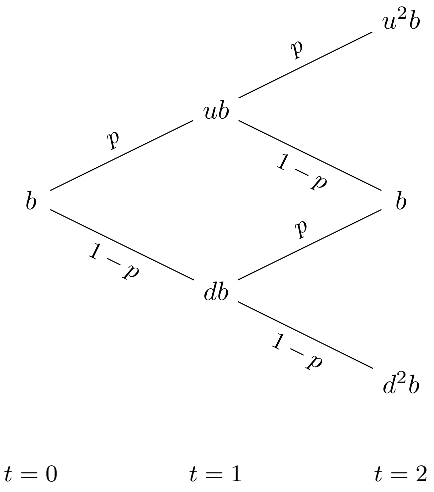

Figure 1: Process of the benefits of exercising the option

2.3 Scenario I: Risk neutrality and risk-free borrowing and lending

In the first scenario, the case of risk neutrality and risk-free borrowing and lending is analyzed. As a basis, we calculate the project value, meaning the multi-period certainty equivalent of the risky stream (given a particular state-contingent exercise strategy) to the decision maker. This is done in two steps. The first step deals with the situation of a single period, the second generalizes. The certainty equivalent is understood as a certain amount at the date of valuation whose utility equals the expected utility of the risky stream when borrowing and lending opportunities are taken into account.[2] Formally, for $t = 0$, we obtain

$$\begin{aligned} &\max_{y}(ce - y) + \beta\delta\left[(1+r)y\right] \\ &= \max_{z}(-z) + \beta\delta E\left[x_1 + (1+r)z\right] \end{aligned}$$

where ce denotes the multi-period certainty equivalent and y as well as z represent risk-free borrowing and lending over one period. r indicates the risk-free market rate. Inserting

$$ce = \frac{E\left[x_1\right]}{1+r}$$

and

$$y = z + ce$$

(and ignoring issues of uniqueness if $\beta\delta \neq {}^{1}/_{(1+r)}$) yields

$$\begin{aligned} &\max_{z}\left(ce - (z + ce)\right) + \beta\delta\left[(1+r)(z+ce)\right] \\ &= \max_{z}(-z) + \beta\delta\left[E\left[x_1\right] + (1+r)z\right] \end{aligned}$$

[2]The literature on option pricing frequently uses the term utility indifference price (cf. Henderson and Hobson, 2009)

or

$$\max_z(-z) + \beta\delta\left[E\left[x_1\right] + (1+r)z\right]$$
$$= \max_z(-z) + \beta\delta\left[E\left[x_1\right] + (1+r)z\right]$$

which shows that valuation (and the determination of the optimal exercise strategy) are preference-free and time inconsistency has no effect. Now the second step is quite trivial. Due to the properties of the expectation operator, i.e. linearity and the law of iterated expectations, this result can be generalized in an obvious way to multiple periods. An alternative derivation of our finding can be found in Hakansson (1969). As a consequence of the valuation result, the following exercise policy is obtained. The decision maker exercises prematurely after one up-move if the net benefits of immediate exercise $ce_{1,e}$ exceed the value associated with waiting to invest $ce_{1,n}$, i.e.

$$ce_{1,e} = ub - f > ce_{1,n} = p\frac{u^2b - f}{1+r}.$$

2.4 Scenario II: Risk aversion and risk-free borrowing and lending

The second scenario covers the combination of risk aversion and risk-free borrowing and lending. It is analyzed with recourse to the work of Smith (1998). To start with, note that the intertemporal utility function (RA) can be written in the form of

$$\mathbb{U}^0(x_0, x_1, x_2) = 1 - e^{-\alpha x_0} + \beta\sum_{\tau=1}^{2}\delta^\tau\left(1 - e^{-\alpha x_\tau}\right)$$
$$= \sum_{\tau=0}^{2}k_\tau - \sum_{\tau=0}^{2}k_\tau e^{-\alpha x_\tau},$$

using utility weights $k_0 = 1$, $k_1 = \beta\delta$ and $k_2 = \beta\delta^2$. Defining risk tolerances $\rho_\tau = \bar{\rho} = {}^1\!/\alpha$ and taking care of the invariance to positive affine transformations, we obtain

$$\mathbb{U}^0(x_0, x_1, x_2) = -\sum_{\tau=0}^{2}k_\tau e^{-x_\tau/\rho_\tau}, \tag{SM}$$

which equals the preference funtion used by Smith (1998, p. 1969). Moreover, the above binomial lattice can be transferred into a decision tree (Figure 2). As is customary, the boxes are called decision nodes and indicate decisions to be made. The circles are called chance nodes and represent the states relevant to the point of decision. Outcomes are written to the tips of the tree.

Smith (1998) introduces the following rollback procedure to solve for the optimal policy (as of date $t = 0$):

(i) Calculate net present values (NPVs) for each endpoint in the tree by discounting all cash flows along the path leading to that endpoint using the risk-free rate

$$ce_2 = \sum_{\tau=1}^{2}\frac{x_\tau}{(1+r)^\tau}.$$

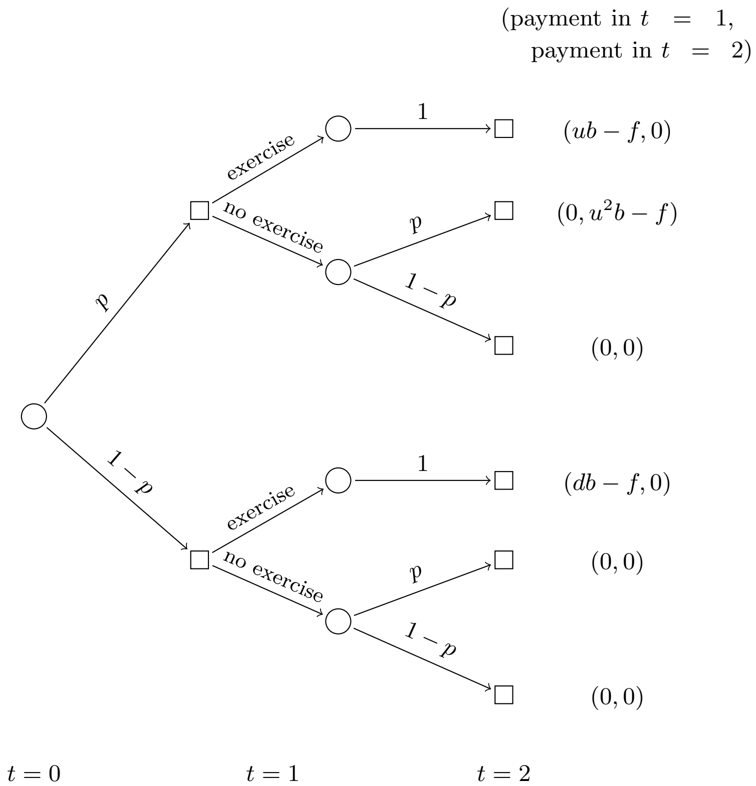

Figure 2: Decision tree of the option exercise problem

(ii) At chance nodes, calculate so-called effective certainty equivalents using the exponential utility function (SM) with effective risk tolerance R_τ. Formally, this involves ($\forall \tau = 1, 2$)

$$ce_{\tau-1} = -R_\tau \ln\left(E\left[e^{-ce_\tau/R_\tau}\right]\right)$$

$$\text{with} \quad R_\tau = \bar{\rho}\sum_{t=\tau}^{2}(1+r)^{-t},$$

where ce_τ denotes possible successor values.

(iii) At decision nodes, choose the maximum value of the available alternatives.

(iv) The value at the root of the tree equals (in our terminology) the project's multi-period certainty equivalent.

As Smith (1998, p. 1697) points out, the multi-period certainty equivalent to the decision maker does not depend on the utility weights k_τ. Consequently, there is no valuation effect of quasi-hyperbolic discounting. After one up-move, the rollback procedure stated translates into the following exercise policy. The decision maker exercises prematurely if

$$ce_{1,e} = ub - f > ce_{1,n} = -\frac{\bar{\rho}}{1+r}\ln\left(pe^{-(u^2b-f)/\bar{\rho}} + (1-p)e^{-0/\bar{\rho}}\right).$$

2.5 Scenario III: Spanning

In the third scenario, we analyze the case of spanning. This means that the cash flow of the project to be valued can be perfectly duplicated on the capital market (DeAngelo, 1981). Again, the project value is understood as multi-period certainty equivalent of the decision maker, i.e. a certain amount at the date of valuation, whose utility equals the expected utility of the risky stream (given a particular state-contingent exercise strategy) when trading opportunities in duplicating securities (including risk-free borrowing and lending) are taken into account. Formally, as of date $t = 0$, this implies

$$\begin{aligned}
&\max_{\psi_0,\psi_1} U\left(ce - \psi_0^T P_0\right) + \beta\delta E\left[U\left((\psi_0 - \psi_1)^T P_1\right)\right] + \beta\delta^2 E\left[U\left(\psi_1^T P_2\right)\right] \\
&= \max_{\xi_0,\xi_1} U\left(-\xi_0^T P_0\right) + \beta\delta E\left[U\left(x_1 + (\xi_0 - \xi_1)^T P_1\right)\right] + \beta\delta^2 E\left[U\left(x_2 + \xi_1^T P_2\right)\right],
\end{aligned}$$

where ψ_τ, ξ_τ ($\forall \tau = 0, 1$) denote state-contingent trading strategies in duplicating securities (including risk-free borrowing and lending). P_τ ($\forall \tau = 0, 1, 2$) represent vectors of state-contingent securities prices. Setting ($\forall \tau = 0, 1$)

$$\xi_\tau = \psi_\tau - \psi_\tau^d,$$

where ψ_τ^d represents the state-contingent duplicating trading strategy, yields

$$\begin{aligned}
&\max_{\psi_0,\psi_1} U\left(ce - \psi_0^T P_0\right) + \beta\delta E\left[U\left((\psi_0 - \psi_1)^T P_1\right)\right] + \beta\delta^2 E\left[U\left(\psi_1^T P_2\right)\right] \\
&= \max_{\psi_0,\psi_1} U\left(-[\psi_0 - \psi_0^d]^T P_0\right) + \beta\delta E\left[U\left(x_1 + ([\psi_0 - \psi_0^d] - [\psi_1 - \psi_1^d])^T P_1\right)\right] \\
&\quad + \beta\delta^2 E\left[U\left(x_2 + [\psi_1 - \psi_1^d]^T P_2\right)\right].
\end{aligned}$$

With the definition of the duplicating trading strategy, we obtain

$$\max_{\psi_0,\psi_1} U\left(ce - \psi_0^T P_0\right) + \beta\delta E\left[U\left((\psi_0 - \psi_1)^T P_1\right)\right] + \beta\delta^2 E\left[U\left(\psi_1^T P_2\right)\right]$$
$$= \max_{\psi_0,\psi_1} U\left(\psi_0^{d^T} P_0 - \psi_0^T P_0\right) + \beta\delta E\left[U\left((\psi_0 - \psi_1)^T P_1\right)\right] + \beta\delta^2 E\left[U\left(\psi_1^T P_2\right)\right].$$

Obviously

$$ce = \psi_0^{d^T} P_0,$$

this means that the multi-period certainty equivalent of the decision maker equals the current value of the duplicating trading strategy irrespective of the shape of periodical utility. The valuation (and the determination of the optimal exercise strategy) are thus preference-free and there is also no effect of quasi-hyperbolic discounting.

A derivation of the optimal exercise policy requires the specification of available capital market instruments. It is assumed that, in addition to the possibility of risk-free borrowing and lending at an interest rate r, a risky market instrument is traded. The price process of this security follows a multiplicative binomial process as shown in Figure 3. $\bar{u}$ and $\bar{d}$ represent growth factors that satisfy $\bar{u} > \bar{d}$, $\bar{u} = 1/\bar{d}$ and $\bar{u} > 1 + r > \bar{d}$.

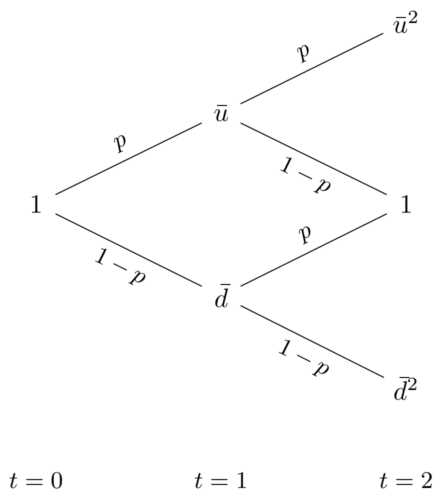

Figure 3: Price process of the risky market instrument

With the help of this specification, the duplication of project cash flows can be condensed in the use of so-called risk-neutral probabilities

$$q = \frac{1 + r - \bar{d}}{\bar{u} - \bar{d}}$$

(for the up-move) and $1 - q$ (for the down-move), respectively. Consequently, the decision maker exercises prematurely after one up-move if

$$ce_{1,e} = ub - f > ce_{1,n} = q\frac{u^2 b - f}{1 + r}.$$

An explicit instruction is obtained, if the "underlying" of the real option is traded or if the benefits of exercising the option exhibit the same dynamic as the risky (market) instrument, i.e. $u = \bar{u}$. In this case, as is well-known (Merton, 1973, p. 144; Cox et al., 1979, pp. 234-235), a premature exercise of the option to defer is never optimal.

2.6 Scenario IV: Risk aversion and no capital market

The fourth scenario covers the case of risk aversion without capital market interaction. In this context, the value of the project equals the multi-period certainty equivalent of the risky stream (given a particular state-contingent exercise strategy) at the date of valuation. From the point of view of $t = 0$, it is defined by

$$\mathbb{U}^0(ce, 0, 0) = E\left[\mathbb{U}^0(x_0, x_1, x_2)\right]$$

or

$$1 - e^{-\alpha ce} = 1 - e^{-\alpha x_0} + \beta \sum_{\tau=1}^{2} \delta^\tau E\left[1 - e^{-\alpha x_\tau}\right].$$

Consequently, after one up-move, we obtain the following exercise policy. A premature exercise results in a cash flow of

$$ce_{1,e} = ub - f.$$

In contrast, the expected utility of waiting to invest amounts to

$$E\left[\mathbb{U}^1(x_1, x_2)\right] = p\beta\delta\left(1 - e^{-\alpha(u^2b-f)}\right).$$

With the condition

$$ce_{1,n} = U^{-1}\left(E\left[\mathbb{U}^1(x_1, x_2)\right]\right),$$

this translates into

$$ce_{1,n} = -\frac{1}{\alpha}\ln\left(1 - p\beta\delta\left(1 - e^{-\alpha(u^2b-f)}\right)\right).$$

Clearly, the decision maker exercises after one up-move if $ce_{1,e} > ce_{1,n}$.

Using this result, it is possible to elaborate on the effect of time inconsistency. A time-inconsistent decision maker will exercise the option earlier than a time-consistent decision maker if

$$\begin{aligned} &-\frac{1}{\alpha}\ln\left(1 - p\beta\delta\left(1 - e^{-\alpha(u^2b-f)}\right)\right) \\ &< ub - f < -\frac{1}{\alpha}\ln\left(1 - p\delta\left(1 - e^{-\alpha(u^2b-f)}\right)\right) \end{aligned}$$

holds. Solving

$$-\frac{1}{\alpha}\ln\left(1 - p\beta\delta\left(1 - e^{-\alpha(u^2b-f)}\right)\right) \stackrel{!}{=} ub - f$$

for β yields

$$\bar{\beta} = \frac{1 - e^{-\alpha(ub-f)}}{p\delta\left(1 - e^{-\alpha(u^2b-f)}\right)}.$$

If $\beta \leq \bar{\beta}$ (and $\bar{\beta} < 1$) a time-inconsistent decision maker will exercise the option earlier (for a similar reasoning, Khan et al., 2013, p. 207). $\bar{\beta}$ is thus called the critical value of the present-bias self-control parameter. As the expression above shows, $\bar{\beta}$ is a function of α, u, b, f, δ and p.

3 Numerical example and comparative static analysis

To illustrate the effect of quasi-hyperbolic discounting, we use an example with the parameter shown in Table 2. To start with, we conduct some comparative

Table 2: Parameter specification

Parameter	Value
u	1.25
b	100
p	0.75
f	100
α	0.02
δ	0.95
r	0.05

static analysis. The partial derivative of the critical self-control parameter $\bar{\beta}$ with respect to the coefficient of risk aversion α equals

$$\frac{\partial \bar{\beta}}{\partial \alpha} = \frac{(ub-f)e^{-\alpha(ub-f)}}{p\delta\left(1-e^{-\alpha(u^2b-f)}\right)} - \frac{(u^2b-f)p\delta e^{-\alpha(u^2b-f)}\left(1-e^{-\alpha(ub-f)}\right)}{\left[p\delta\left(1-e^{-\alpha(u^2b-f)}\right)\right]^2}.$$

As Figure 4(a) shows, it is positive for reasonable values of α. This could be explained in the following way. An increase in risk aversion causes the certainty equivalent of future cash flows to decline. As a result, waiting will be less beneficial. In order to lift the certainty equivalent up to the level of the benefits resulting from immediate exercise, the self-control parameter has to rise. Therefore, a wider range of time-inconsistent decision makers will exercise prematurely.

The sensitivity of the critical self-control parameter $\bar{\beta}$ to the magnitude of an up-move u can be computed as follows

$$\frac{\partial \bar{\beta}}{\partial u} = \frac{\alpha b e^{-\alpha(ub-f)}}{p\delta\left(1-e^{-\alpha(u^2b-f)}\right)} - \frac{2\alpha ubp\delta e^{-\alpha(u^2b-f)}\left(1-e^{-\alpha(ub-f)}\right)}{\left[p\delta\left(1-e^{-\alpha(u^2b-f)}\right)\right]^2}.$$

The expression is positive for conventional values of u (see Figure 4(b)). This may be due to the following mechansism. If u rises, both the benefits resulting from immediate exercise and the certainty equivalent of future cash flows will increase, yet the former effect will dominate the latter. Consequently, waiting will be less beneficial. In order to equalize the certainty equivalent of future cash flows with the level of the benefits associated with immediate exercise, the self-control parameter has to rise. Hence, a wider range of time-inconsistent decision makers will exercise prematurely.

The partial derivative of $\bar{\beta}$ with respect to the probability of an up-move p equals

$$\frac{\partial \bar{\beta}}{\partial p} = -\frac{1-e^{-\alpha(ub-f)}}{p^2\delta\left(1-e^{-\alpha(u^2b-f)}\right)} < 0.$$

Obviously, this expression is negative (see also Figure 4(c)). The result can be explained as follows. An increase in p causes the certainty equivalent of future

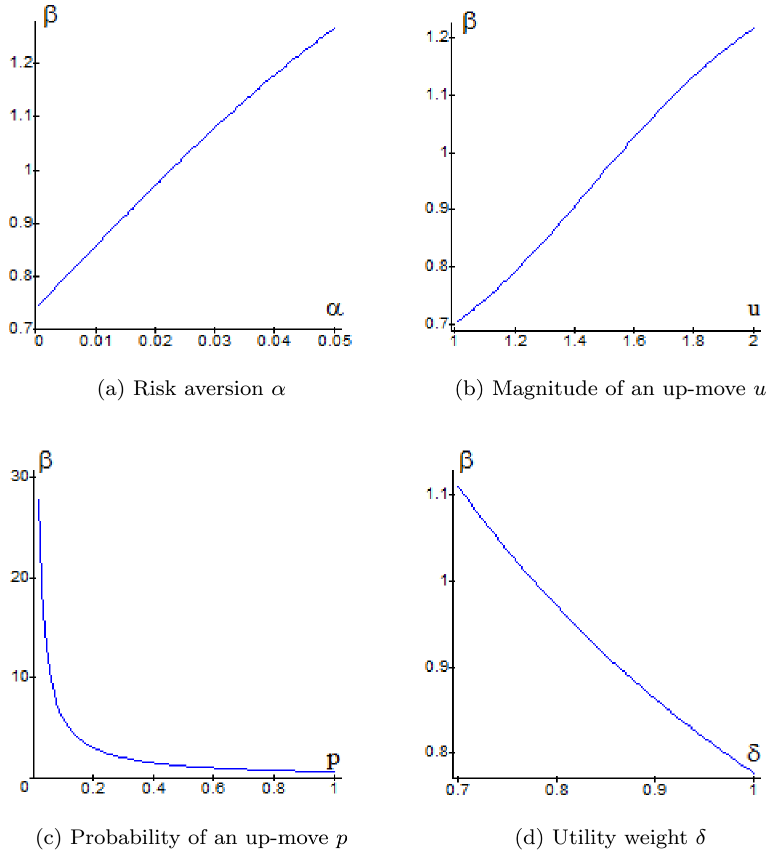

Figure 4: Sensitivity of the critical self-control parameter $\bar{\beta}$

cash flows to rise. This means that waiting will be more beneficial. In order to reduce the certainty equivalent to the level of the benefits resulting from immediate exercise, the self-control parameter has to decrease. Therefore, a narrower range of time-inconsistent decision makers will exercise prematurely.

Finally, we investigate the sensitivity of the critical self-control parameter $\bar{\beta}$ to the utility weight (or impatience) δ. Our computations yield

$$\frac{\partial \bar{\beta}}{\partial \delta} = -\frac{1 - e^{-\alpha(ub-f)}}{p\delta^2 \left(1 - e^{-\alpha(u^2 b - f)}\right)} < 0.$$

Clearly, this expression is negative (see also Figure 4(d)). Our explanation is as follows. If δ increases, the certainty equivalent of future cash flows will rise. Waiting will thus be more beneficial. In order to equalize the certainty equivalent of future cash flows with the level of the benefits associated with immediate exercise, the self-control parameter has to fall. Hence, a narrower range of time-inconsistent decision makers will exercise prematurely.

A numerical analysis of the optimal exercise policy provides further insights. As Table 3 in the appendix shows, in the case of risk neutrality and risk-free borrowing and lending, waiting to invest is preferred for reasonable parameter combinations. Due to preference-free valuation, individual components (e.g. the utility weight δ) have no impact on the decision to be made.

The results concerning our second scenario are less explicit (see Table 4 in the appendix). In case of risk aversion and risk-free borrowing and lending, both premature exercise and waiting to invest occur. This means that an immediate exercise of the option is possible irrespective of the particular values of β and δ that characterize the decision maker's time preference.

Our third scenario considers the case of spanning. Once again, valuation is preference-free and individual components do not influence the decision to be made. Instead, optimal exercise depends on the properties of the stochastic processes governing the benefits of exercising the option and the market price of the risky financial instrument. As Table 5 in the appendix shows, a premature exercise is more likely to occur for higher volatilities of the market instrument, provided the stochastic processes mentioned above are possitively correlated, i.e. $u > 1$ and $\bar{u} > 1$.[3]

In the case of risk aversion and no capital market interaction, the decision maker's preferences exert influence on the exercise policy of the option to defer the investment. As one would expect, exercise happens earlier for smaller values of the utility weight δ and the self-control parameter β (see Table 6 in the appendix). Yet, immediate exercise in all parameter combinations considered requires the self-control parameter to be unrealistically low.[4]

4 Major findings and implications

This paper is concerned with the valuation of options to defer an investment in the presence of quasi-hyperbolic discounting. It derives the following results. Project characteristics and capital market environment constitute important

[3] Quah and Strulovici (2013) obtain similar results in the context of optimal stopping with unspecified (and possibly individualistic) stochastic discounting.

[4] The specification of $\beta = 0.7$ and $\delta = 0.95$ implies a discount rate of no less than 50.38 % over one year.

determinants of the decision maker's exercise policy. This means that both premature exercise and waiting to invest are possible irrespective of the values of the self-control parameter β. Moreover, different behavior of investors exhibiting exponential and quasi-hyperbolic discounting occurs only for quite specific parameter conditions.

As mentioned in the review of literature earlier on, our study partially shares the object of investigation with the work of Khan et al. (2013). Yet their findings clearly differ from ours. This is mostly due to an inconsistent combination of market-based and preference-based valuation (see Khan et al., 2013, p. 212). If there is a *traded* "underlying" (or at best, spanning is possible), so-called *risk-neutral* probabilities can be calculated and used for valuation (see our discussion of scenario III). In this case, individual preferences (i.e. quasi-hyperbolic discounting) play no role. On the other hand, if there is no such market instrument (incomplete market), individual preferences have to be considered via the stated preference function. Here, probabilities are the result of *subjective* estimation and cannot be calculated in the sense of option pricing theory.[5]

The paper at hand documents the limited effect of quasi-hyperbolic discounting in off-the-shelf economic settings. In light of empirical and experimental evidence for time-inconsistent behavior, the following question merits examination. Is time-inconsistent behavior really driven by quasi-hyperbolic disounting, or rather by more fundamental irrationality (such as a disregard of fairly ubiquitous market opportunities)?

[5] A more detailed analysis of Khan et al. (2013) is available from the author upon request.

A Numerical example and optimal exercise policy

The figures in parentheses are to be understood as follows. The first one refers to the net benefits of immediate exercise $ce_{1,e}$. The second number reflects the certainty equivalent of future cash flows associated with deferring the investment $ce_{1,n}$. $(ce_{1,e}\star, ce_{1,n})$ denotes the case of $ce_{1,e} > ce_{1,n}$, i.e. premature exercise after one up-move. $(ce_{1,e}, ce_{1,n}\star)$ is to be interpreted as $ce_{1,e} < ce_{1,n}$, in which case waiting to invest becomes optimal.

A.1 Scenario I: Risk neutrality and risk-free borrowing and lending

Table 3: Optimal exercise in scenario I

	δ		
u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 15\star)$	$(10, 15\star)$	$(10, 15\star)$
$u = 1.25$	$(25, 40.18\star)$	$(25, 40.18\star)$	$(25, 40.18\star)$
$u = 1.5$	$(50, 89.29\star)$	$(50, 89.29\star)$	$(50, 89.29\star)$
$u = 1.8$	$(80, 160\star)$	$(80, 160\star)$	$(80, 160\star)$

A.2 Scenario II: Risk aversion and risk-free borrowing and lending

Table 4: Optimal exercise in scenario II

	δ		
u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 14.16\star)$	$(10, 14.16\star)$	$(10, 14.16\star)$
$u = 1.25$	$(25, 33.36\star)$	$(25, 33.36\star)$	$(25, 33.36\star)$
$u = 1.5$	$(50, 55.53\star)$	$(50, 55.53\star)$	$(50, 55.53\star)$
$u = 1.8$	$(80\star, 64.42)$	$(80\star, 64.42)$	$(80\star, 64.42)$

A.3 Scenario III: Spanning

Table 5: Optimal exercise in scenario III

(a) $\bar{u} = 0.6$

u	δ: $\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 11.56\star)$	$(10, 11.56\star)$	$(10, 11.56\star)$
$u = 1.25$	$(25, 30.97\star)$	$(25, 30.97\star)$	$(25, 30.97\star)$
$u = 1.5$	$(50, 68.82\star)$	$(50, 68.82\star)$	$(50, 68.82\star)$
$u = 1.8$	$(80, 123.33\star)$	$(80, 123.33\star)$	$(80, 123.33\star)$

(b) $\bar{u} = 1.2$

u	δ: $\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 11.82\star)$	$(10, 11.82\star)$	$(10, 11.82\star)$
$u = 1.25$	$(25, 31.66\star)$	$(25, 31.66\star)$	$(25, 31.66\star)$
$u = 1.5$	$(50, 70.35\star)$	$(50, 70.35\star)$	$(50, 70.35\star)$
$u = 1.8$	$(80, 126.06\star)$	$(80, 126.06\star)$	$(80, 126.06\star)$

(c) $\bar{u} = 1.8$

u	δ: $\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10\star, 7.95)$	$(10\star, 7.95)$	$(10\star, 7.95)$
$u = 1.25$	$(25\star, 21.28)$	$(25\star, 21.28)$	$(25\star, 21.28)$
$u = 1.5$	$(50\star, 47.30)$	$(50\star, 47.30)$	$(50\star, 47.30)$
$u = 1.8$	$(80, 84.74\star)$	$(80, 84.76\star)$	$(80, 84.76\star)$

A.4 Scenario IV: Risk aversion and no capital market

Table 6: Optimal exercise in scenario IV

(a) $\bar{\beta} = 0.7$

u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10\star, 9.05)$	$(10\star, 9.38)$	$(10\star, 9.71)$
$u = 1.25$	$(25\star, 19.74)$	$(25\star, 20.53)$	$(25\star, 21.34)$
$u = 1.5$	$(50\star, 29.29)$	$(50\star, 30.61)$	$(50\star, 31.96)$
$u = 1.8$	$(80\star, 32.46)$	$(80\star, 33.97)$	$(80\star, 35.53)$

(b) $\bar{\beta} = 0.8$

u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 10.49\star)$	$(10, 10.88\star)$	$(10, 11.26\star)$
$u = 1.25$	$(25\star, 23.32)$	$(25\star, 24.30)$	$(25\star, 25.30)$
$u = 1.5$	$(50\star, 35.33)$	$(50\star, 37.03)$	$(50\star, 38.80)$
$u = 1.8$	$(80\star, 39.45)$	$(80\star, 41.45)$	$(80\star, 43.53)$

(c) $\bar{\beta} = 0.9$

u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 11.97\star)$	$(10, 12.42\star)$	$(10, 12.86\star)$
$u = 1.25$	$(25, 27.18\star)$	$(25, 28.38\star)$	$(25, 29.60\star)$
$u = 1.5$	$(50\star, 42.20)$	$(50\star, 44.41)$	$(50\star, 46.72)$
$u = 1.8$	$(80\star, 47.59)$	$(80\star, 50.25)$	$(80\star, 53.07)$

(d) $\bar{\beta} = 1.0$

u	$\delta = 0.92$	$\delta = 0.95$	$\delta = 0.98$
$u = 1.1$	$(10, 13.50\star)$	$(10, 14.01\star)$	$(10, 14.52\star)$
$u = 1.25$	$(25, 31.37\star)$	$(25, 32.81\star)$	$(25, 34.30\star)$
$u = 1.5$	$(50, 50.17\star)$	$(50, 53.06\star)$	$(50, 56.14\star)$
$u = 1.8$	$(80\star, 57.31)$	$(80\star, 60.94)$	$(80\star, 64.85)$

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