Now Or Never
End-game effects in time limited real R&D option investment

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First version: 2015/01/25
Current version: 2015/01/26

Keywords: Research and development, R&D, real options, time limit, finite-lived
Classification: G13 option pricing; C61 dynamic programming, optimization

Abstract

This paper investigates the valuation of an option to invest in an R&D project of uncertain cost on a finite time horizon. The common body of knowledge attributes a positive value to flexibility and the possibility to postpone an investment in order to wait for the arrival of new information. From this point of view a decision maker will assume that as time passes and the R&D projects terminal date approaches the value of the option to invest in the project will gradually vanish. Hence it appears logical that the willingness to invest in a project declines as time is running out. In my work I show that under certain conditions the opposite is the case i.e. as the deadline for completion approaches, decision makers increase their willingness to invest in the progress of the R&D project as a last effort to claim the reward and avoid failure. What appears as non-rational behavior of an individual who seeks to avoid sunk costs, is actually a fully rational investment strategy.

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1. Introduction

Consider a firm that has a monopoly on the right but not the obligation to carry out a risky R&D project. In this context risk means that the ultimate cost to completion and time-to-build is only known with certainty once the project is finished. Due to technical or financial capacity restrictions it is not possible to complete the project instantaneously. Rather it requires a continuing effort in the form of R&D spending by the firm that is pursuing the endeavor. Suppose further that invested capital is put to very specific use and can therefore not be recovered if the development should fail or be abandoned half-way through the project. The investment's payoff which is assumed readily quantifiable is only distributed when the project is finished successfully i.e. the remaining costs to completion equal zero. How should the firm or a decision maker value such a real option?

After the paper of Pindyck (1993), that presented a novel approach to study irreversible investment decisions that take time to complete, several authors including Dixit & Pindyck (1994), Abel *et al.* (1996), Kort (1998), Schwartz & Moon (2000) and Paxson (2003) have investigated this topic. Common assumptions are that the remaining cost to completion and/or the value of the finished project are subject to random shocks modeled via Wiener and/or Poisson processes and that a control variable, denoted I for investment, linearly affects the trend of cost to completion over time. These conditions typically lead to an optimum investment strategy in the form of a bang-bang solution i.e. the firm either engages fully in the project or defers/abandons it. With the exception of Mölls & Schild (2012), who are the first to consider a minimal investment rate unless the project is abandoned, all existing studies can be classified by the set of alternatives of the decision maker into one of two categories. Existing models may be identified as either pure-exit models, where the firm has to fully engage in the project or abandon it irreversibly, or pure-switching models, where the firm may choose between different levels of R&D spending but cannot retreat entirely if the project turns out to be a failure.

Suppose now the firm has a venture capitalist aboard who forces it to enter a contract specifying a fixed deadline for the project[2]. Therefore the payoff becomes time-dependent and in turn the optimum decision rule should also exhibit some time-dependency. Published literature does not consider real R&D options' payoffs to be explicitly time dependent. Notable exceptions are the works of Messica & David (2000) and Armada *et al.* (2007). Yet these cannot answer the question for the Pindyck model of investment under uncertainty as Messica & David (2000) use a revisited Lucas (1971) model for the valuation where payoffs start to accrue as soon as the project is kicked off and Armada *et al.* (2007) use only two discrete points in time in their corrected Carr (1988) model to valuate the real option.
Therefore the question about the influence of a finite time horizon in the Pindyck real options model for R&D investment remains unanswered. My paper focuses on what hap-

[2]Consider milestones, contracts or other requirements of the venture capitalist as possible reasons for the terminal date.

pens to the firm's investment strategy as time passes and the terminal date approaches. Furthermore, the relation between the optimum decision rule and Tobin's marginal q will be investigated.

The remainder of this paper is structured as follows. Section 2 explains the basic intuition about a finite-lived real R&D option. Furthermore, it develops an alternative understanding of technical uncertainty involved in the R&D process and applies it to derive a decision rule for the maximization of the real R&D options value. Afterwards the model is extended to three states of investment and the optimum strategy for the deterministic case is formulated. Section 3 elaborates on the application of a discrete binomial grid and the Bellman method to solve the real option valuation problem when uncertainty about cost to completion is present. Section 4 presents a numerical example with results and observations including the relation of the value function to Tobin's marginal q. Section 5 puts the results into context with existing real R&D options literature. Section 6 concludes with a short summary of the papers main findings and the methodological approach.

2. Model

2.1. Intuition and general properties

Intuition suggests that as the terminal date, i.e., the point in time when the project has to be finished approaches the value of waiting declines. This is due to the depletion of options to begin the development as time goes by and therefore the positive value of flexibility dwindles.

To test this hypothesis, let us assume an R&D project with uncertain costs to completion K. To determine the value of the real option to pursue this project, the optimal control, i.e., optimal investment strategy $I(K, t)$ has to be determined.
In the published literature, that does not consider time-dependence of payoffs, such a solution is typically characterized by two states. One state is considered full engagement of the firm in the R&D project i.e. $I(K) = I_{\max}$. The other state is, depending on the particular model, either abandoning the project or reducing the R&D spending to some value $I(K) = I_{\min}$. If $I_{\min} = 0$ the model permits the firm to wait for the arrival of new information i.e. random shocks without any cost. The optimal investment strategy is usually characterized by a single parameter K^* denoted the critical cost level. Thus, any value of expected cost to completion satisfying $K \leq K^*$ entices the firm to complete the R&D project as fast as possible. If, on the other hand, $K > K^*$ the firm will withdraw from the endeavor at least temporarily by reducing its effort or even shutting down the development entirely. This property, being a consequence of the linearity of the first-order condition of optimality, is often referred to as bang-bang solution.
Assuming time-dependence of the payoffs I expect a decline of the option value towards the terminal date and suppose that the critical cost level K^* follows the decline monotonically as well.

Another standard result of real option valuation problems is a graph that visualizes the value of the real option as function of expected cost to completion. Due to the positive value of flexibility of the decision maker to stop or defer the project at any instant in time, the value of the option subject to uncertainty will always exceed the net-present-value (NPV) of the deterministic case. Figure 1 illustrates the option values calculated by a NPV decision maker and an individual that is aware of uncertainty. Furthermore the critical cost level K^* is indicated.

2.2. Re-introducing technical uncertainty

The left side of Equation (1) indicates the stochastic process denoted *technical uncertainty* in the Pindyck (1993) publication. The core property of this representation of uncertainty is that *'.. as long as the firm does not invest in the project no additional information about this project becomes available, so nothing changes about the firm's expectation concerning future costs.'* Kort (1998). Therefore there is no value to waiting. Provided that the bang-bang property of the solution is already known in advance

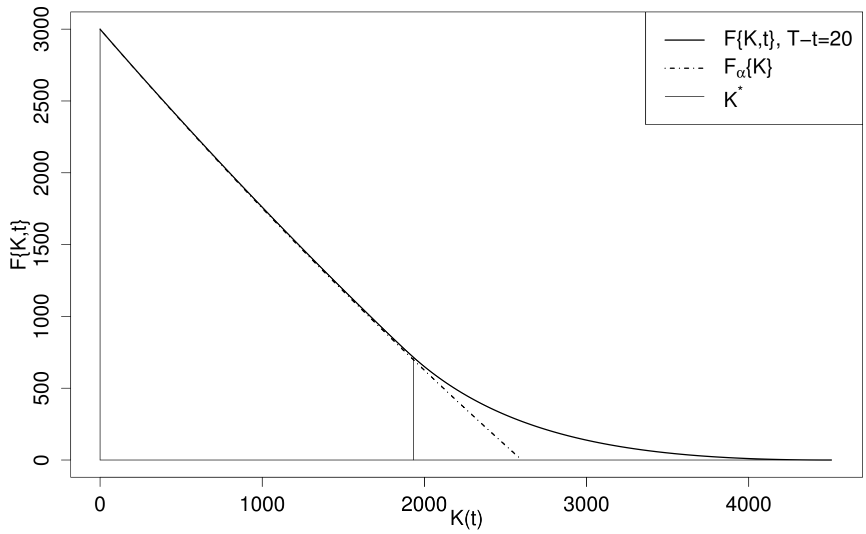

Figure 1: Value function curves in a deterministic (NPV) environment and assuming uncertainty over cost to completion. For the uncertain case the critical cost level K^* is indicated.

due to the linearity of dK in I, the process governing the evolution of future cost to completion can be remodeled to the formulation shown on the right side of Equation (1) without loss of generality. Though it has to be ensured that no new information is revealed unless the firm is investing in the development of the project at another point in the solution process.

$$dK = -Idt + \beta I^{\frac{1}{2}} K^{\frac{1}{2}} dW_t \rightarrow dK = -Idt + \beta I_{\max}^{\frac{1}{2}} K^{\frac{1}{2}} dW_t = -Idt + \sigma K^{\frac{1}{2}} dW_t \tag{1}$$

2.3. Solution in continuous time

The value of the real R&D option in continuous time is given by Equation (2). This formulation shows that option value $F(t)$ of the opportunity to engage in the R&D endeavor at some point in time t and an unknown time of completion T^* is equal to the option value at t plus its future expectation discounted over a marginal dt minus the R&D spending on that instant in time.

$$F(t) = E\left(\int_t^{T^*} -Ie^{rs} ds\right) = -Idt + e^{-rdt}(F(t) + E(dF)) \tag{2}$$

The insertion of the redefined technical uncertainty from Equation (1) and the application of Ito's Lemma lead to the Hamilton-Bellman-Jacobi equation given by Equation

(3) which has to be determined under proper boundary conditions. Essentially this equation compares the immediate cash-flows against the expected future performance of the asset. Whereby a decision maker's primary concern is the supremum of this relationship i.e. the optimal contingent investment strategy that implements the highest attainable value of the investment opportunity. The complete derivation is indicated in Section A.1.

$$rF = \sup_{I \in \mathbb{I}+} \left\{ -I + \frac{E(dF)}{dt} \right\} \tag{3}$$

The Hamilton-Bellman-Jacobi equation is a second-order elliptic partial differential equation. This class of problems is already difficult to solve by itself yet the boundary conditions imposed by time-dependence of the payoffs complicate the solution process even further. Suppose the firm that holds the real R&D option receives a payoff of P if it finishes the project ahead of the terminal date and no payoff at all if it reaches the terminal date without completing the development. Therefore the option value of the R&D opportunity is subject to the boundary conditions given by Equation (4).

$$F(K,t) = \begin{cases} F(0,t) = P & ,\forall t \in [0,T) \\ F(K,T) = 0 & ,\forall K \\ F(\lim_{K\to\infty} K, t) = 0 & ,\forall t \end{cases} \tag{4}$$

Confronted with the problem specified in Equation (3) and the decision whether to invest in the project or not how should the firm decide? Calculating the partial derivative of Equation (3) with respect to the control variable I leads to the decision rule given by Equation (5). An investment strategy based on this formula inevitably leads to a bang-bang decision pattern. Either $dF/dI < 0$ an the firm retreats from the project or $dF/dI > 0$ and the firm will engage at its maximum capacity. Should the case that $dF/dI = 0$ occur the decision maker will be indifferent whether to allocate another marginal monetary unit to the project or not. Note that $-\partial F/\partial K$ denotes Tobin's marginal q within the decision rule.

$$\frac{dF}{dI} = \frac{1}{r}\left(-1 - \frac{\partial F}{\partial K}\right) = 0 \tag{5}$$

Therefore, an investment strategy maximizing the real R&D option's value based on Tobin's marginal q is given by Equation (6).

$$q = -\frac{\partial F}{\partial K} \begin{cases} \leq 1 & \text{, retreat from R\&D} \\ = 1 & \text{, indifference} \\ \geq 1 & \text{, engage in R\&D} \end{cases} \tag{6}$$

2.4. Extending the decision space - The three state model

I define the decision maker's opportunity set as follows. At any point in time she has to decide whether to keep the option to invest and successfully finish the project or abandon it.
If she decides to shut down, the control variable is set to $I(K,t) = 0$ and the associated option value immediately and irreversibly becomes $F(K,t) = 0$.
If, on the other hand, the firm wants to keep the opportunity to successfully complete the project, an investment of at least I_κ for the acquiring of new information about the remaining cost of the project is required[3]. The minimal R&D spending I_κ is denoted the periodic *keep-alive-cost* of the project. When the project is kept in such a waiting position, I call it a *passive project*. This minimal funding does not contribute to the projects progress, yet it allows the decision maker to observe the next random-shock to estimated cost to completion $K(t)$ and thereby face a new decision of the same kind. This is economically comparable to the periodic maintenance cost mentioned in Brennan & Schwartz (1985) or the minimal investment rate required in Mölls & Schild (2012) and ensures that no information is revealed unless the firm costly maintains the project.

Any allocation that exceeds I_κ (up to a limit of $I_{\max}$) directly reduces the projects remaining costs to completion as shown in Equation (7). The according state is referred to as *active project* with I_π denoting R&D spending beyond the maintenance cost I_κ.

$$E[K(t+dt)] = K(t) - (I - I_\kappa)dt = K(t) - I_\pi dt \tag{7}$$

Equation (8) illustrates the opportunity set for the investment variable I for the decision maker in a mathematical form.

$$I = \begin{cases} 0 & \text{, abandon project} \\ I_\kappa & \text{, passive project (wait)} \\ I_\kappa + I_\pi = (I_\kappa, I_{\max}] & \text{, active project (invest)} \end{cases} \tag{8}$$

Under these assumptions the control variable I in the technical uncertainty formulation from Section 2.2 is replaced by the control variable I_π denoting active engagement in the development (Equation (9)). Furthermore, the R&D project's cost to completion is subject to random shocks as information unfolds governed by $\sigma K^{\frac{1}{2}} dW_t$. Thus, the evolution of the R&D projects cost to completion can be referred to as controlled[4] Cox-Ross-Ingersoll (CIR) diffusion process.

$$dK = -I_\pi dt + \sigma K^{\frac{1}{2}} dW_t \tag{9}$$

Resulting from the above definition I expect to find three distinct regions within the vector space spanned by the time and cost-to-completion axes. These regions, corre-

[3] One could think of the minimal investment rate as spending on periodic maintenance cost to retain trained staff, laboratory equipment et cetera.
[4] The description refers to the replacement of the CIR process' auto-regressive drift by a control variable.

sponding to abandonment of the project, passive observation of new information and active engagement in the project, are separated by two critical thresholds, which in general are functions of time. I denote these thresholds the *activation boundary*, i.e. the curve where an investor is indifferent whether to wait or engage in the project for another increment on the time-axis given by $K_\pi(t)$, and the *shutdown boundary*, i.e. the curve where an investor is indifferent whether to wait for another random-shock along the time-axis or immediately abandon the R&D project given by $K_\kappa(t)$.
According to Mölls & Schild (2012) this three-state model leads to a system of coupled partial differential equation that is complex to solve. Section 3 briefly sketches a method based on a binomial grid that is easy to comprehend and may provide new insights on real R&D options with time-dependent payoffs.

2.5. The deterministic case

Before heading on to the solution method in Section 3, a brief look at the deterministic case is tempting. Suppose no uncertainty is present and the costs to finish the project are quantifiable in advance. Then the development will take $\Delta T^* = K/I_\pi^*$ to complete the project, where $I_\pi^* = I_{\max} - I_\kappa$ is the maximum productive capacity of overall R&D spending and I_κ is the non-productive maintenance cost introduced in the previous section 2.2. Remodeling the deterministic solution proposed by Pindyck (1993) to accord with the assumptions of active and passive engagement results in the option value given by Equation (10).

$$F_\alpha(K) = \max\left[Pe^{-rK/I_\pi} - \int_0^{K/I_\pi^*} I_\pi^* e^{-rt} dt - \int_0^{K/I_\pi^*} I_\kappa e^{-rt} dt, 0\right] \tag{10}$$

Deriving a decision rule for the firm holding the real R&D option is straight forward. Actively engage in the project whenever the option value $F(K) > 0$ otherwise abandon the option. In a deterministic setting there is no value of waiting. Utilizing the indifference of the NPV decision maker about whether to invest in the project or abandon it at $F(K_\alpha) = 0$, the critical cost threshold K_α can easily be obtained as seen in Equation (11).

$$K_\alpha = \frac{I_\pi^*}{r} \ln\left(1 + \frac{rP}{I_\pi^* + I_\kappa}\right) \tag{11}$$

Figure 2 depicts the decision field of the firm when no uncertainty is present. It advises the firm to actively engage in the project whenever the remaining cost to complete the development $K(t) \leq K_\omega(t)$ with $K_\omega(t) = \min[K_\alpha, (T-t)I_\pi^*]$. In the white areas of the decision field the value of the real R&D option is $F(K,t) = 0$. In the gray field it follows the dot-dashed curve indicated in Figure 1.

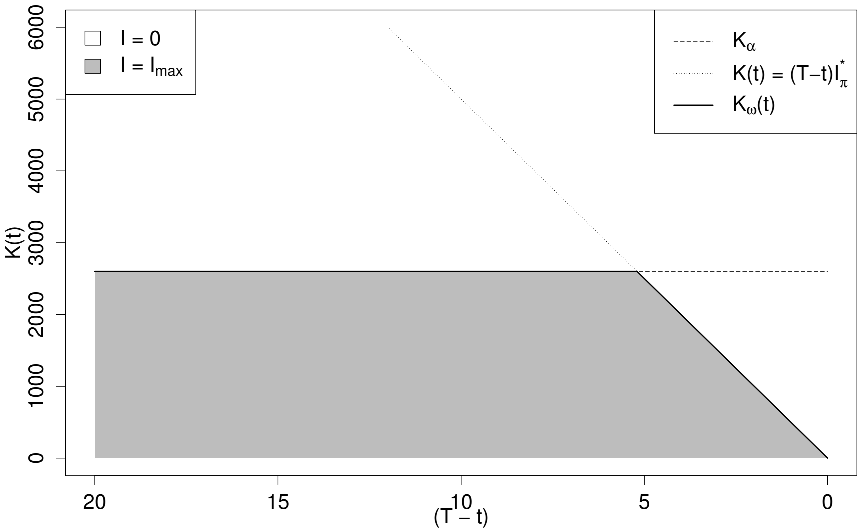

Figure 2: Decision field of the firm applying a net-present-value rule. Active engagement in the project is advised for all value of $K(t) \leq K_\omega(t)$.

3. Method

The proposed method consist of two major steps. The first one is the construction of a discrete grid to computationally evaluate the real R&D option, the second step is the optimization of the project value by selection of the control variable I. The problem is implemented in the R language (R Core Team (2014)) for statistical computing.

3.1. Generating the grid

The controlled CIR process governing the evolution of cost to completion $K(t)$ is modeled by computing and storing a representative number of possible values as nodes in a discrete grid. An individual node is addressed as $K(n, s)$ with indexes $n \in [0, \ldots, N]$ for the computation of the actual cost to completion $K(t)$ and $s \in [0, \ldots, t/\Delta t, \ldots, T/\Delta t]$ for the elapsed time t. However, an unmodified CIR process does not lead to a recombining grid.

In Nelson & Ramaswamy (1990) a method that maps the CIR process into a diffusion with constant volatility which can be modeled in a recombining tree is proposed. For some value K of remaining cost the x-transform of K is equal to $x(K) = (2\sqrt{K})/\sigma$ with zero as lower boundary for K. The backward transformation $k(X)$ to a value of cost to completion K is given by Equation (12).

$$k(X) = \begin{cases} (\sigma^2 X^2)/4 & , x > 0 \\ 0 & , otherwise \end{cases} \tag{12}$$

Let $X = n\sqrt{\Delta t}$ be a scheme that constitutes the cost-to-completion values of the nodes within the grid via backward transformation i.e. $K(n, s) = k(n\sqrt{\Delta t})$. Assigning discrete points in time s to the cost-to-completion values a binomial grid of discrete nodes $K(n, s)$ is fabricated. Since cost to completion has a lower bound of $K_{\min} = 0$ and the time axis is bounded on both sides, only the upper border on the axis indicating cost to completion needs further specification.

This upper boundary value is characterized by N given by $N = \lceil \{\frac{2}{\sigma}\sqrt{\frac{K_{\max}}{\Delta t}}\} \rceil$. The value of $K_{\max}$ has to be determined by the decision maker so that the fixed point of the *shutdown boundary* lies within the interval of investigated values of cost to completion i.e. $\max[K_\kappa(t)] \in [0, K_{\max}]$. If the *shutdown boundary* hits the upper border of the cost axis $K_{\max}$ has been chosen to small an the decision maker suffers a loss of information since the K-t-plane is fashioned to narrow. During the experiments a value of $K_{\max} = 3P$ proved itself a suitable dimension for the upper boundary. Note that the fixed point of the *shutdown boundary* is of course affected by the R&D project risk measure and therefore $K_{\max}$ has to be adjusted accordingly if σ is changed.

Nelson and Ramaswamy also allow the model to deal with diffusion processes whose drifts are large in comparison to their variance. They achieve this by incorporating the possibility of jumps in their model instead of specifying a fixed pair of possible successors for every node. I use this measure to specify a jump modifier J for every node in the grid based on the node's cost-to-completion value and the investment capacity restriction I_{max}.

Once the grid for the valuation of the real R&D option is available in the form of a large matrix, its bordering nodes can be set to meet the boundary conditions of the problem. These are directly related to the economic properties of the R&D project and specified in Section 2.3 as well as Equation (13). The last line represents the valuation function for all non-bordering nodes.

$$F\{K,t\} = \begin{cases} 0 & ,t = T \wedge K(n, T/\Delta t) > 0 \\ 0 & ,t \leq T \wedge K(N,s) = K_{\max} \\ P & ,t \leq T \wedge K(n,s) = 0 \\ V(I,K,t) & ,t < T \wedge K(n,s) > 0 \end{cases} \tag{13}$$

The boundary conditions determine that the value of any node, whose cost to completion $K(n,s)$ at the terminal date exceeds zero, equals zero. Furthermore they dictate the option value of any cost node $K(N,s) = K_{\max}$ equals zero at any point in time. Thereby the optimization problem obtains an upper boundary and higher remaining costs need not be modeled in the grid. Eventually the option value of all nodes where the remaining cost equals zero is set to the payoff P. For all other nodes the following investment strategy optimization procedure is applied.

3.2. Optimization of the investment strategy

The optimization of the investment strategy applies the Bellman principle i.e. backwards induction to the remainder of nodes. Utilizing the nodes at the terminal date, where payoffs are known for certain, the option value of the nodes in the penultimate time step is maximized by application of Equation (15). That is, the control variable I is set to maximize the option value as the sum of the current cash-flows i.e. R&D spending and the discounted future value of the real R&D option.

In the binomial grid that models the diffusion process every node has two possible successor nodes. Thus, the expected costs to completion on time-step into the future are calculated as sum of the probability weighted costs to completion of the successors. Since the cost-to-completion value is already known for every node by construction, Equation (7) can be utilized to derive the probability of an up-move in a pair of successor nodes as function of the *active investment* and the remaining cost value of the current node. The resulting probability function is given in Equation (14)

$$
\begin{aligned}
E[K(t+\Delta t)] &= K(n,s) - I_\pi \Delta t = \dots \\
\dots &= \Pi(I_\pi)K(n+J+1,s+1) + (1-\Pi(I_\pi))K(n+J-1,s+1) \\
\rightarrow \Pi(I_\pi) &= \frac{K(n,s) - I_\pi \Delta t - K(n+J-1,s+1)}{K(n+J+1,s+1) - K(n+J-1,S+1)}
\end{aligned}
\tag{14}
$$

However, determining the optimum choice of I is not straight forward as the set of alternatives contains discrete as well as continuous choices. The method applied in this paper is to compute and compare the option values of all three possible decisions i.e. *active project*, *passive project* and *abandon project*. The opportunity that maximizes the R&D project's value is chosen and the control variable $I(K,t)$ and the option value $F\{K,t\}$ are stored in their appropriate matrices for later evaluation steps.

$$
V(I,K,t) = \max_{I \in \mathbb{I}+} \left\{ -I\Delta t + \frac{1}{(1+r)^{\Delta t}} * F\{E[K(t+\Delta t)]\} \right\} =
$$
$$
= \max \begin{cases}
0 & , I = 0 \\
 & \\
-I_\kappa \Delta t + \frac{1}{(1+r)^{\Delta t}}[\dots & \\
\Pi(0)F\{K(n+1,s+1)\} + \dots & \\
(1-\Pi(0))F\{K(n-1,s+1)\}] & , I = I_\kappa \\
 & \\
-I_\kappa \Delta t - I_\pi \Delta t + \frac{1}{(1+r)^{\Delta t}}[\dots & \\
\Pi(I_\pi)F\{K(n+J+1,s+1)\} + \dots & \\
(1-\Pi(I_\pi))F\{K(n+J-1,s+1)\}] & , I = (I_\kappa, I_{max}]
\end{cases}
\tag{15}
$$

Alternatively the optimum decision can be derived directly from the Bellman equation's derivative with respect to I_π Equation (17) or Tobin's marginal q specified by Equation (18). The required calculation of the derivative of the probability measure $\Pi(I_\pi)$ with respect to I_π is given in Equation (16).

$$
\frac{d\Pi}{dI_\pi} = \frac{-\Delta t}{K(n+j+1,s+1) - K(n+j-1,s+1)}
\tag{16}
$$

If the method is modified in such a way, only the option value for the case specified by the decision rule given in Equation (19) needs to be calculated. Note how Tobin's marginal q for the discrete grid in Equation (18) adopts an expression familiar to the $-dV/dK$ given in Equation (5).

$$
\frac{dV}{dI_\pi} = \Delta t \left\{ \frac{1}{(1+r)^{\Delta t}} \frac{F\{K(n+J-1,s+1)\} - F\{K(n+J+1,s+1)\}}{K(n+J+1,s+1) - K(n+J-1,s+1)} - 1 \right\} = \Delta t\{q-1\}
\tag{17}
$$

$$q = \frac{1}{(1+r)^{\Delta t}} \frac{F\{K(n+J-1,s+1)\} - F\{K(n+J+1,s+1)\}}{K(n+J+1,s+1) - K(n+J-1,s+1)} \tag{18}$$

$$I = \begin{cases} 0 & , q \leq 0 \\ I_\kappa & , 0 < q < 1 \\ I_\kappa + I_\pi & , q \geq 1 \end{cases} \tag{19}$$

Although the solution is expected to exhibit some form of bang-bang behavior in the R&D spending there could theoretically exist some internal maximum in the option value of an arbitrary point on the K-t-plane. The numerical method also allows to determine any internal maximum relative to R&D spending I at the cost of additional computation time. Therefore the option value of *active investment* needs to be calculated for every pair of successor nodes given by $j \in [0, -2, \ldots, -2i, \ldots, J]$. After identification, the value of this internal maximum of investment would be compared to the option values of waiting and abandoning the R&D project. Note that I_π needs to be reversely computed for every pair of nodes evaluated in order to avoid overspending. Also note that no internal maxima were identified during the experimentation phase.

4. Results

Throughout this paper a numerical example is used to demonstrate the proposed method and results and develop the arguments. $P = 3000$ is the payoff in monetary units the firm receives if it finishes the project in time. Should the firm fail it receives nothing. The time horizon to complete the development amounts to $T = 20$ time units with a spacing of $\Delta t = 0.001$ time units for the binomial grid. The firm is able to allocate $I_{\max} = 505$ monetary units per unit time towards the R&D endeavor. The nonproductive maintenance cost that was introduced in subsection 2.4 amounts to $I_\kappa = 5$ monetary units per unit time. The risk-less rate of interest by which future earnings and expenses are discounted equals $r = 0.05$. Eventually the parameter σ that comes from the projects risk assessment and acts as the variance parameter of the Wiener process equals $\sigma = 5.5$.

The numerical procedure returns two matrices that contain the option value F(K,t) and the optimum choice for the control variable I for every (discrete) contingency in the K-t-plane. These matrices are used to sketch the typical option value graph as a function of cost to completion and the decision field of the option holder.
The option value, as a function of expected cost to completion, accords with existing literature if the distance to the terminal date is large (Figure 1). This implies that uncertainty over future cost to completion raises the value of the option to invest beyond the conventional net-present-value (NPV). The curve exhibits a linear behavior for small values of K(t) and a smooth pasting arc as seen in Dixit & Pindyck (1994) towards a value of zero as remaining cost is increased.
Returning to the numerical example introduced at the beginning of this section I obtain the following results. A NPV decision maker with unlimited time is indifferent whether to pursue the R&D project at full pace or abandon the option at a remaining cost value $K_\alpha = 2600.77$. Once uncertainty is introduced to the real option valuation problem, the decision rule becomes more complex. The critical thresholds at the beginning of the time horizon i.e. $T - t = 20$ amount to the following values. The *shutdown boundary* where the option value equals $F\{K, t\} = 0.00$ is found at $K_\kappa(t) = 4481.01$. The *activation boundary* is located at $K_\pi(t) = 1934.43$ with a corresponding option value of $F\{K, t\} = 713.36$. These values and thresholds should be equivalent to the fixed points obtained from suitably modified solution methods in the literature. A possible starting point for such an approach could be the paper of Mölls & Schild (2012).

But how does the optimum investment rule look like as the window of opportunity is about to close? A first glance at the firm's decision field (Figure 3) is startling. I cannot only observe a definite time dependence in both thresholds but also a surprisingly counter intuitive behavior of the *activation boundary* $K_\pi(t)$.
If the remaining time to the terminal date is large, the cost to completion a decision maker is willing to accept to engage in R&D tends toward a fixed point, i.e. $\lim_{t\to\infty} K_\pi(t) = K^*$.
However, on a short horizon the *activation boundary* does, in contrast to intuition, not

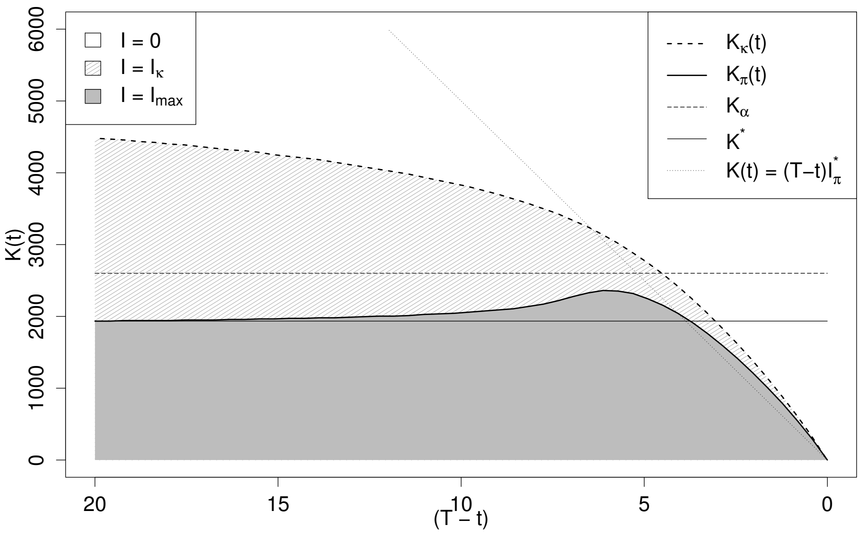

Figure 3: Decision field indicating the optimum investment rule as colored fields with their respective thresholds $K_\kappa(t)$ and $K_\pi(t)$. Also shown are the indifference cost of a NPV decision maker K_α, the fixed point in critical cost K^* and the R&D investment capacity restriction given by $K(t) = (T - t)I^*_\pi$.

decrease monotonically but exhibits a distinct maximum in the investor's willingness to fund the R&D project instead. In the constructed example the maximum acceptable cost to completion to actively pursue the project amounts to $K^\dagger = \max[K_\pi(t)] = 2361.11$. Such a maximum will always be observable, provided that uncertainty over remaining cost $\sigma > 0$ and the considered option's time frame is sufficiently large compared to the investment capacity restriction. All the more uncertainty about future cost is present in the stochastic process the more pronounced the maximum in the *activation boundary* $K^\dagger$ will be. I denote this observation the *Now Or Never effect* for the remainder of this paper.

If $\sigma = 0$ the question for the optimum investment strategy becomes deterministic. As stated earlier there is no value to waiting if no uncertainty is present and therefore the *activation boundary* and the *shutdown boundary* are congruent. In this case the critical cost to completion is given by $K_\omega(t) = \min[K_\alpha, (T - t)I^*_\pi]$. This case is illustrated in Figure 2.

The decision matrix agrees with previous studies as the observed optimum investment rule is indeed bang-bang. That is, the investing firm will always pursue the R&D project at full pace if the marginal $q \geq 1$ and retreat to waiting at $0 < q < 1$. Should the marginal q drop to zero or below the project will be abandoned. This is consistent with intuition as well as the results from Mölls & Schild (2012). Once the active investment capacity restriction given by $K(t) = (T - t)I^*_\pi$ is added to Figure 3 another surprising result becomes apparent. The decision maker is willing to accept a gamble in order to maintain her chances to claim the payoff if the terminal date is approaching. That means she is willing to fully engage in the R&D endeavor at a point in time where it can only be finished through the aid of chance.

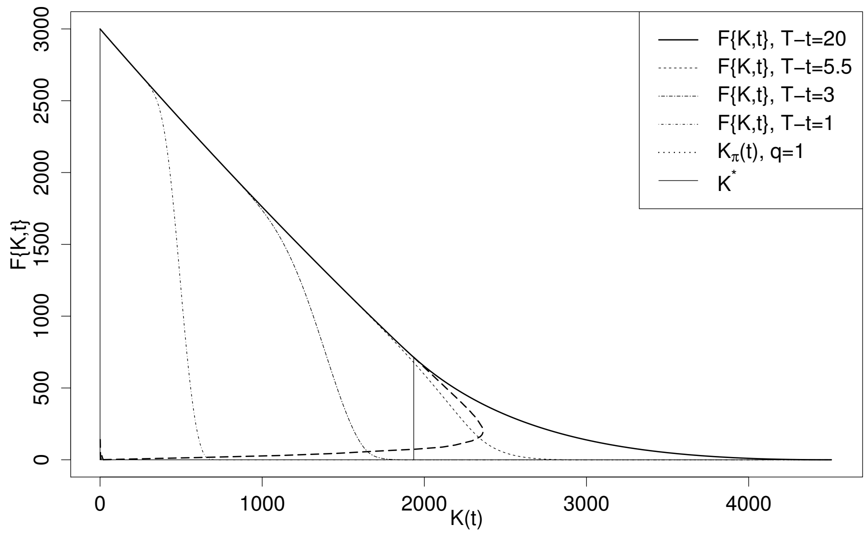

Figure 4: Option value F as a function of K for three different distances to the terminal date $(T-t) \in [20, 5.5, 3.0, 1.0]$. $K_\pi(t)$ indicates the *activation boundary* where Tobin's marginal q is equal or slightly larger than 1. This curve exhibits a distinct maximum before settling in the fixed point K^*.

Figure 4 shows that the *Now Or Never effect* can also be depicted in the standard option value graph found in most works dedicated to real R&D options valuation. Left of the boundary full engagement is the optimum choice while to the right it is waiting until the process either hits the *activation* or the *shutdown boundary.* Information regarding the point in time when the effect reaches its maximum is however not contained in this visualization style unless additional value graphs are plotted. In figure 4 the value function curves for $T-t \in [1.0, 3.0, 5.5]$ are added to demonstrate the optimum strategy indication based on the position on the curve relative to the $K_\pi(t)$ boundary.
A general observation of all experiments is that with an increase in uncertainty the *shutdown boundary* rises in $K(t)$ while the *activation boundary* declines. Thus it seems that the value of waiting increases with the level of uncertainty.

As short summary of this section the following key points are of importance. The critical cost level K^* computed by scientific standard methods leads to a long-term fixed point in acceptable cost to completion. On a finite horizon the firm's willingness to fund an uncertain R&D project depends on the expected cost to completion and the distance to the terminal date. Thus the well known K^* specifying the critical cost for activation of the project becomes $K^*(t) = K_\pi(t)$. This critical cost value has a distinct maximum that is typically located somewhere close to $T - t = K_\alpha / I^*_\pi$. This is denoted *Now Or Never effect*. It is also noteworthy that a decision maker is willing to accept gambles in order to claim the payoff as the window of opportunity is about to close.

5. Discussion

Certainly the *Now Or Never effect* requires some discussion. This section discusses the reasons for the distinct maximum in decision maker's willingness to pursue a risky R&D project shortly before time to do so runs out. Furthermore, I try to briefly explain why other authors fail to observe the effect and what its discovery implies for the valuation of real R&D options.

5.1. The Now Or Never effect

At the activation boundary the decision maker considers the trade-off between investing another marginal monetary unit or just keeping the option to invest later alive. Since the value of waiting deteriorates faster than the value of the ongoing project does, this trade-off moves in favor of an *active project* before the decision maker gradually gives up due to the time constraint. Another explanation is that the further away from the terminal date the more potential call options (see Abel *et al.* (1996)) to begin the R&D project are available to the firm. As these options deplete the incentive to invest grows and thereby the willingness to accept higher remaining costs to completion rises.

If the firm has abundant time to finish the project the decision maker will tend to wait and thereby receive 'cheap' random shocks if the value of cost to completion is above K^*. Thereby the firm can hope to increase its net payoff if the stochastic process governing future costs moves in favor of the R&D project.

Actually the *activation boundary* appears as if some irrational sunk cost considerations were involved i.e. the decision maker wants to avoid a loss and therefore is willing to accept higher cost in order to finish the project. However, this argument is not valid as the evolution of cost to completion is a Markovian process and the Bellman method does only consider future events. Thus the sunk cost argument must be rejected.

5.2. Relation to published literature

While many authors consider real options with several stages of investment, only two of them consider a distinct time-dependency of payoffs. So why are they unable to observe the *Now Or Never effect*?

It turns out that Armada *et al.* (2007) apply a valuation approach that they refer to as corrected Carr (1988) method which utilizes only two discrete points in time. Due to the lack of sufficiently many assessment points on the time axis, this approach certainly fails to provide any insight on time dependency of the optimum investment rule.

Regarding the work of Messica & David (2000) the situation is somewhat different. The paper supplements time dependent returns into the Lucas (1971) model of risky R&D project valuation. However their assumptions about the nature of payoffs are entirely different as (small) returns start to trickle as soon as the project is launched. Also there is no time limited up to when the project has to be completed.

5.3. Implications for the valuation of R&D projects

The unveiling of the *Now Or Never effect* has several implications. It appears counter-intuitive that it may be beneficial to keep a risky real option alive even if this decision implies a maintenance or *keep-alive cost* and the option is currently out of the money.
The presence of the effect also suggests that a real option even if not actively pursued should be re-evaluated from time to time because the decision rule itself is time-dependent.
Most important are the consequences for short-term, high risk R&D projects. Since the *activation boundary* has no time to settle on the fixed point K^* the critical cost level for entry in the project is more likely to be found around $K^\dagger$. Thus the decision maker's optimum willingness to pursue the project could be underestimated which would in turn result in foregone investment opportunities.

6. Conclusion

In this paper I show that an evaluation of investments subject to uncertainty, e.g. large R&D projects, by means of a binomial method can shed new light on the optimum investor behavior for projects of uncertain cost with a specified terminal date. While the optimum decision rule on a per node level turns out to be bang-bang as many authors before have pointed out before, some insights are hidden in the overall decision field.

Actually, an investment strategy that seems to be not fully rational can be observed in the results. This leads to the notion of an individual that wishes to avoid sunk costs. However, since sunk cost considerations are ruled out by the definition of the stochastic process and the solution method, something else has to be responsible for the witnessed endgame effect in a rational decision maker's behavior. I denote this the *Now Or Never effect*.

Despite the prior intuition of a time-dependent investment rule, this effect characterized by an increasing willingness to pursue the R&D project i.e. a maximum in acceptable cost to completion before the window of opportunity closes is an unexpected discovery. Its governing parameters appear to be the uncertainty over future cost to completion σ and the capacity restriction for R&D spending $I_{\max}$.

The findings suggest that a fully rational decision maker whose capacity restriction leaves her with sufficient time to complete the project before the terminal date tends to wait for a long time unless the option is already deep in the money. Thereby the project receives cheap random shocks in the form of new information over cost to completion that may increase the option's returns. Eventually as the value of waiting depletes, the end-game effect kicks in. Thus, the decision maker will lose her patience and accept a higher critical cost level to engage in the project. She is even willing to go beyond the point where the project can be finished in a deterministic environment and accept gambles in order to claim the payoff as time runs out.

Furthermore I relate the dynamic programming results with existing marginal q-theory and watch the firm willing to pursue the project as soon as $q \geq 1$. The curve that connects the points where Tobin's marginal $q = 1$ characterizes the *activation boundary* $K_\pi(t)$. This curve can be depicted in the traditional value function graph as well as in the more recent decision field.

Possible topics for future research include the investigation of applicability of a trinomial grid to represent Pindyck's model of technical uncertainty or the extension of the grid to a third dimension. This allows to model competing R&D projects of the same firm[5] with maintenance costs to study portfolio diversification effects or projects of different firms competing for the same development in a leader-follower setting.

[5] Often referred to as horse races.

A. Appendix

A.1. Continuous time

The starting point for the valuation of the real R&D option is given by Equation 20.

$$F = E\left(\int_t^{T^*} -Ie^{rs}ds\right) = -Idt + e^{-rdt}(F + E(dF)) \tag{20}$$

Application of Ito's Lemma and insertion of the revised technical uncertainty lead to Equation 21.

$$\begin{aligned} dF &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial K}dK + \frac{1}{2}\frac{\partial^2 F}{\partial K^2}dK^2 + \ldots = \\ &= \frac{\partial F}{\partial t}dt + \frac{\partial F}{\partial K}[-Idt + \sigma K^{\frac{1}{2}}dW_t] + \frac{1}{2}\frac{\partial^2 F}{\partial K^2}(\sigma^2 K dt) + \ldots \end{aligned} \tag{21}$$

Omitting terms that contain powers of dt larger than 1 leads to Equation 22.

$$E(dF) \cong \frac{\partial F}{\partial t}dt - I\frac{\partial F}{\partial K}dt + \frac{1}{2}\sigma^2 K\frac{\partial^2 F}{\partial K^2}dt \tag{22}$$

Reinserting the expected change in option value to the initial Equation 20 leads to Equation 23. Note that e^{-rdt} is replaced by a Taylor series expansion aborted after the second argument.

$$F \cong -Idt + (1 - rdt)\left(F + \left[\frac{\partial F}{\partial t} - I\frac{\partial F}{\partial K} + \frac{1}{2}\sigma^2 K\frac{\partial^2 F}{\partial K^2}\right]dt\right) \tag{23}$$

Relocation of the equation's terms leads to the Hamilton-Bellman-Jacobi equation given in 24 or 25 respectively. Equation 25 indicates the immediate cash-flows of the option and the expected future changes to it's value.

$$rF = \sup_{I \in \mathbb{I}}\left\{-I + \frac{\partial F}{\partial t} - I\frac{\partial F}{\partial K} + \frac{1}{2}\sigma^2 K\frac{\partial^2 F}{\partial K^2}\right\} \tag{24}$$

$$rF = \sup_{I \in \mathbb{I}}\left\{-I + \frac{E(dF)}{dt}\right\} \tag{25}$$

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