Optimal switching decisions under stochastic volatility with fast mean reversion☆

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Abstract

We study infinite–horizon, optimal switching problems under a general class of stochastic volatility models that exhibit "fast" mean–reversion by using techniques from homogenisation theory. This leads to perturbation theory, providing closed–form approximations to the full switching problem which is often intractable, both analytically and numerically. We apply our general results to certain, well–known switching problems and volatility models, providing qualitative information on the effect of multi–scale stochastic volatility on optimal switching decisions and hysteresis. Our results indicate that multi–scale stochastic volatility strongly affects the frequency and the optimal timing of switching between modes. The proposed methodology is of interest to a number of applied problems involving switching flexibility, for example optimal production management of natural resources or foreign direct investment in the face of fluctuating exchange rates.

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1. Introduction

An important class of problems arising in operations research are the so–called optimal switching problems, in which the objective is to find the optimal time to initiate/terminate a production process or enter/exit a market. Such problems find numerous applications in production management and capacity choice (see, e.g. Dixit, 1989; Trigeorgis, 1993; Pindyck, 1988; McDonald and Siegel, 1985), but also in fields such as natural resource economics (see, e.g. Brennan and Schwartz, 1985; Paddock et al., 1988), maritime economics, etc. (Sødal et al., 2008; Kavussanos and Tsekrekos, 2011)

In an important paper, which has generated a lot of discussion, Brekke and Øksendal (1994) have formulated a general optimal switching problem under uncertainty as an impulse control problem (see also Bensoussan and Lions, 1984, for a thorough examination of impulse control problems), and have solved it using methods from stochastic analysis.

All aforementioned contributions consider optimal switching problems where the uncertainty in the economic system is represented by one or more stochastic processes with constant (or deterministically time–varying) volatilities. However, the data do not always support this assumption and there is evidence that, for instance, commodity prices display stochastic volatility effects, which may develop on different time scales (see for example the evidence and stylised facts in Hikspoor and Jaimungal, 2008; Eydeland and Wolyniec, 2003)

Stochastic volatility models (see Taylor, 1994, for a review) have gained much attention by academics and practitioners alike, in the valuation and hedging of financial derivatives, not only as an alternative to the Black and Scholes (1973) framework, but also as a powerful tool that can better model and explain economic variables and systems. As Fouque et al. (2003a) point out, one characteristic feature of volatility is that its mean–reversion rate is quite "fast", as compared to the time scale of evolution of other economic state variables. This feature is referred to as *fast mean–reverting volatility*, and there is significance empirical evidence of its presence in equity prices, exchange rates and commodity prices (see for example Alizadeh et al., 2002; Fouque et al., 2003b; Hikspoor and Jaimungal, 2008). The above observations have led Fouque, Papanicolaou, Sircar and co–workers to study the dynamics of such "fast" volatility processes and their net effect on the prices of financial options, an endeavour that led to important qualitative and quantitative findings. (see also the results in Zhu and Chen, 2011a,b; Chen and Zhu, 2012; Souza and Zubelli, 2011).

To the best of our knowledge, and despite the fact that asset and commodity prices have been documented to exhibit fast mean–reverting volatility, the study of its effects on optimal switching problems related to economic decision–making has been overlooked. Theory and intuition offer little guidance, a priori, as to whether fast mean–reverting stochastic volatility should lead to more or less frequent switches between the admissible modes of a production process. It is the aim of this paper to examine optimal switching decisions under multi–scale stochastic volatility.

Motivated by the important qualitative findings of Fouque et al. (2003a,b), concerning the effects of fast mean–reverting volatility on the pricing and hedging of financial derivatives, it is natural to raise the question of whether the "fast" varying stochastic volatility features of e.g. commodity time series, may affect the solution of optimal switching problems, and this is the main object of this paper.

Following the seminal work of Fouque et al. (2003a) on multi–scale volatility and using the perturbation method as in Fouque et al. (2000), we formulate and solve an infinite–horizon, optimal switching problem under uncertainty, in the spirit of Duckworth and Zervos (2001), but driven by a general class of stochastic volatility models that exhibit fast mean–reversion. The perturbation method allows us to approximate the optimal switching problem with a sequence of simplified valuation systems, each one offering a "correction" of different order to the constant–volatility solution that has been documented in the literature. These corrections, that are the effect of fast stochastic volatility, are derived in closed–form, allowing one to analytically approximate the solution of the general switching problem under fast mean–reverting stochastic volatility up to the desired order. Our analytic approach is important, as the full multi–scale optimal stopping problem is difficult and tricky to handle by numerical methods, and thus our analytic results offer useful benchmarks for the numerical analysis of the full problem.

Our closed–form solution are of interest to decision–makers dealing with processes or projects that can be switched from and to an idle/active mode, contingent on the evolution of economic variables that are documented to exhibit fast mean–reverting stochastic volatility, such as exchange rates and energy and commodity prices. To this end, we apply our general results to a number of benchmark, mean–reverting stochastic volatility models, and we explicitly derive the "correction" terms due to multi–scale effects in a simple entry/exit problem. Assessing their effects on the qualitative features of the solution, we find that when the uncertainty in an economic system exhibits fast mean–reverting stochastic volatility: (a) optimal switching between modes will be *more frequent*, (b) agents will be more willing to activate earlier and will endure higher losses before deciding to optimally suspend operations and (c) findings (a) and (b) are more pronounced for lower (more negative) levels of correlation between price and volatility uncertainty, faster volatility mean–reversion speeds and higher effective volatility levels.

The rest of the paper is organized as follows: Section 2 presents the basic setting of our optimal switching problem under fast mean–reverting stochastic volatility. In Section 3 we derive analytical approximations of the optimal switching policy and value functions via homogenisation theory arguments for the general switching problem. In Section 5 we apply our results to a number of benchmark models related to decision–making and stochastic volatility, and provide explicit expressions for the effects of multi–scale stochastic volatility on optimal switching, and comment extensively on the qualitative implications. Finally, Section 6 concludes the paper.

2. Basic setting

The problem we consider in this paper is that of finding the optimal sequence of switching times (i.e. times of opening and closing) of a multi–mode production process, given the costs of opening, closing and operating in a certain mode and assuming that the economic state is governed by a system of stochastic processes with stochastic volatility that exhibits fast mean reversion.

In order to fix ideas, consider a production process or an investment project that can produce a single commodity or product whose price (and price change volatility) are varying as a system of stochastic processes (to be specified below). The process/project can operate in two modes, say open and closed (or active and idle). In the open/active mode, the project yields a flow payoff that depends on the commodity/product price. In the closed/idle mode, the project incurs a constant flow loss. Transition from one mode to the other can take place instantaneously and an unlimited number of times, but at constant fixed costs that are incurred each time. The risk–free interest rate r is constant and the owner of the process/project (e.g. a firm) is risk–neutral and a price–taker, in that its decisions do not affect the price and price volatility dynamics.[1]

In order to avoid confusion, in the remainder of the paper we will use the terms production process, resource price, active mode and idle mode instead of the equivalent terms investment project, product or commodity price, open mode and closed mode.

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying the filtration $\{\mathcal{F}_t\} = \mathbb{F}$ satisfying the usual conditions of right continuity and augmentation by P–negligible sets and carrying a standard two–dimensional $\mathbb{F}$–adapted Brownian motion $\{\mathbf{W}_t\}$.

Assume that the resource price P is modeled by the following latent factor stochastic volatility model (Fouque et al., 2003a)

$$dP_t = \mu P_t dt + f(Y_t) P_t dW_t^P, \quad P_0 = p_0 \tag{1}$$

$$dY_t = \delta^{-2}(m - Y_t)\, dt + \frac{\nu\sqrt{2}}{\delta} dW_t^Y, \quad Y_0 = y_0 \tag{2}$$

where the Wiener process $\left[W_t^P W_t^Y\right]'$ is correlated to $\mathbf{W}_t$ by

$$\begin{bmatrix} W_t^P \\ W_t^Y \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \rho & \sqrt{1-\rho^2} \end{bmatrix} \mathbf{W}_t,$$

with $|\rho| < 1$ constant. In the above, the volatility of P_t is $\sigma_t = f(Y_t)$, driven by a "fast" mean–reverting latent stochastic factor Y_t, and $f : I \subset \mathbb{R} \to \mathbb{R}$ is a smooth bounded function on the compact set I. In the above, δ is a small positive number that governs the degree of mean reversion and corresponds to the "fast" time scale of this process.

The decisions to switch from one mode of operation to the other can be modeled by a stochastic process $Q = \{Q_t\} \in \mathcal{Q}$, where $\mathcal{Q}$ denotes the family of all $\{\mathcal{F}_t\}$–adapted, finite variation, càglàd processes Q with values in $\{0, 1\}$, with $Q_t = 0$ or 1 denoting whether the production process is idle or active at time t. Let q_0 denote the mode of the production process at $t = 0$.

[1] The assumption of risk–neutrality is not crucial for the solution and it is made only for simplicity. The extension to a risk–averse process/project owner is straightforward and we make it available upon request from the authors. Equally non–crucial is the assumption of instantaneous transition from one mode of operation to the other. Switches between modes that take time to implement could be easily be accomodated, only at the cost of extra notation.

We can associate with each starting value triplet (q_0, p_0, y_0) and sequence of switching decisions Q, the (total, infinite horizon) present value function:

$$J_{(q_0,p_0,y_0)}(Q) = \mathbb{E}\left[\int_0^\infty e^{-rs}\left[R_1(P_s)\,Q_s + R_0(P_s)\,(1-Q_s)\right]ds \right.$$
$$\left. - \sum_{s\geq 0} e^{-rs}\left[K_0\,(\Delta Q_s)^+ + K_1\,(\Delta Q_s)^-\right]\right] \quad (3)$$

which takes into account the switching costs, with $\Delta Q_t = Q_{t+} - Q_t$ and $(\Delta Q_t)^\pm = \max(\pm\Delta Q_t, 0)$. Here, the sub–linear function $R_q : \mathbb{R}_+ \to \mathbb{R}$ is the payoff flow in mode q, and K_q, are the costs from switching *from* (i.e. leaving) mode q, with $q \in \{0,1\}$. Naturally, we require that $K_0 + K_1 > 0$ so that one cannot earn arbitrarily high profits simply by constantly changing the production process' operating mode back and forth. Possible choices for R_q used in the literature are $R_q(P_t) = (P_t - c) \times \mathbf{1}_{\{q=1\}}$ (see Dixit, 1989) or $R_q(P_t) = h(P_t) \times \mathbf{1}_{\{q=1\}} - C \times \mathbf{1}_{\{q=0\}}$ (see Duckworth and Zervos, 2001), with $\mathbf{1}_x$ the indicator function that takes the value of one if condition x holds, and zero otherswise.

The objective is to maximise the functional $J : \mathcal{Q} \to \mathbb{R}$, as provided by equation (3) over all possible switching choices Q. Define the value function V^δ by

$$V_q^\delta(p,y) = \sup_{Q\in\mathcal{Q}} J_{(q_0,p_0,y_0)}(Q) \quad (4)$$

so that $V_0^\delta(p,y)$ (respectively $V_1^\delta(p,y)$) denotes the maximum net present value obtained when starting at (p_0, y_0) in the idle (active) state and following optimal switching policies. The superscript δ is used to emphasise the dependence of the value function on the small parameter δ.

Using standard results on impulse control theory (see e.g. Duckworth and Zervos, 2001; Bensoussan and Lions, 1984), we see that the value functions V_q^δ satisfy the following Hamilton–Bellman–Jacobi equation that takes the form of the quasivariational inequality

$$\max\left\{\mathcal{L}^\delta V_q^\delta(p,y) + R_q(p),\, V_{1-q}^\delta(p,y) - V_q^\delta(p,y) - K_q\right\} = 0, q = \{0,1\}\,. \quad (5)$$

where the operator $\mathcal{L}^\delta$ is the generator of the process (P,Y), defined by

$$\mathcal{L}^\delta = \delta^{-2}\mathcal{L}_0 + \delta^{-1}\mathcal{L}_1 + \mathcal{L}_2$$

with

$$\mathcal{L}_0 = (m-y)\frac{\partial}{\partial y} + \nu^2 \frac{\partial^2}{\partial y^2}$$
$$\mathcal{L}_1 = \sqrt{2}\nu\rho f(y)\, p \frac{\partial^2}{\partial p \partial y}$$
$$\mathcal{L}_2 = \frac{1}{2} f^2(y)\, p^2 \frac{\partial^2}{\partial p^2} + \mu p \frac{\partial}{\partial p} - rI,$$

where I is the identity operator.

The variational inequality in (5) is a free boundary value problem, the solution of which defines two price levels, $P_0^\delta(y) > P_1^\delta(y)$, that allow us to specify the optimal switching times. In particular, for times t such that $P_t \geq P_0^\delta(y)$ the production process should be turned (or remain) active, whereas for times t such that $P_t \leq P_1^\delta(y)$ the production process should be turned (or remain) idle. Thus, unlike the constant volatility case in Merton (1973), where the price levels that "trigger" optimal switching are *unknown constants*, here $P_0^\delta(y), P_1^\delta(y)$ are *unknown functions* of the starting state of the latent variable driving volatility that need to be specified as part of the solution. In the remainder, it is helpful to remember that in our notation, for $q \in \{0,1\}$, K_q (respectively $P_q^\delta(y)$) is the cost to be incurred (respectively the price process level that needs to be reached) for optimally switching *from* mode q.

3. Optimal switching policy under fast mean–reverting stochastic volatility

3.1. Notation

The solution of the problem described in the previous section depends on the choice of the small parameter δ. Assuming analytic dependence of $V_q^\delta(p,y)$ and $P_q^\delta(y)$ on δ, by Taylor's theorem these functions are determined by the sequences $V_q^\delta := \{V_q^n\}$ and $P_q^\delta := \{P_q^n\}$ for $n = 0, 1, 2, \ldots$, through equations

$$V_q^\delta(p,y) = \sum_{n=0}^{\infty} \delta^n V_q^n(p,y), \text{ and } P_q^\delta(y) = \sum_{n=0}^{\infty} \delta^n P_q^n(y). \tag{6}$$

Observe that $V_q^n(p,y) = \frac{\partial^n}{\partial \delta^n} V_q^\delta(p,y)\big|_{\delta=0}$ and $P_q^n(y) = \frac{\partial^n}{\partial \delta^n} P_q^\delta(y)\big|_{\delta=0}$, i.e. we will reserve the use of the superscript n on a function to denote the action of the operator $\frac{\partial^n}{\partial\delta^n}\cdot\big|_{\delta=0}$ on it, or equivalently on a sequence to denote the n–th term of the sequence. In the proof of our main proposition, the following simple re–labelling of the $n+1$ term of the resource price sequence will prove helpful

$$\overline{P}_q^n = P_q^{n+1}(y) \,. \tag{7}$$

Moreover, we will use the notation $*$ for the convolution of two sequences, i.e.

$$P * Z = \{(P * Z)^n\}, \quad \text{where} \quad (P * Z)^n = \textstyle\sum_{i=1}^n P^{n-i} Z^i,$$

and employ superscript (k) to denote the k–times convolution, i.e.

$$P^{(k)} = \Bigg(\underbrace{P * \cdots * P}_{k \text{ terms}}\Bigg), \quad \text{so that} \quad P^{(k),\ell} = \Bigg(\underbrace{P * \cdots * P}_{k \text{ terms}}\Bigg)^{\ell}$$

is to be understood as the ℓ–th term of the sequence $P^{(k)}$.

Finally, it will prove notationally convenient to define the quantities

$$C(k,\ell) = \tfrac{1}{\ell!}\tfrac{\partial^\ell}{\partial p^\ell}\left[V_{1-q}^k(p,y) - V_q^k(p,y)\right]\Big|_{p=P_q^0(y)}, \qquad (x)^+ = \max(x,0)\,,$$

$$L_q = \mathbb{E}\left[\textstyle\int_0^\infty e^{-rs} R_q(p_s)\,ds\right], \qquad I_q = \begin{cases} 0, & \text{if } q = 0 \\ \infty, & \text{if } q = 1 \end{cases},$$

for $q \in \{0,1\}$.

3.2. Asymptotic formulation of the optimal switching problem

The main result of the paper is summarised in the following proposition that can be used in order to obtain approximate solutions to the variational inequality in (5).

Proposition 1 *Assuming the expansions in equation* (6)*, the value functions V_q^n and the price thresholds P_q^n, $n \in \{0,1,2,\ldots\}$ and $q \in \{0,1\}$, satisfy the systems of equations*

$$\sum_{j=(n-2)^+}^{n} \mathcal{L}_{n-j} V_q^j(p,y) = -R_q(p) \times \mathbf{1}_{\{n-2=0\}} \tag{8}$$

$$\mathbf{1}_{\{n-2\geq 0\}} \times \sum_{\substack{k,\ell,m\in\mathbb{N}:\\ k+\ell+m=n-2}} C\left(k,\ell\right) \overline{P}_q^{(\ell),m} = K_q \times \mathbf{1}_{\{n-2=0\}} \tag{9}$$

$$\mathbf{1}_{\{n-2\geq 0\}} \times \sum_{\substack{k,\ell,m\in\mathbb{N}:\\ k+\ell+m=n-2}} \left(\ell+1\right) C\left(k,\ell+1\right) \overline{P}_q^{(\ell),m} = 0 \tag{10}$$

$$\mathbf{1}_{\{n-2\geq 0\}} \times \lim_{p\to I_q} V_q^{n-2}\left(p,y\right) = L_q \times \mathbf{1}_{\{n-2=0\}} \text{ for any } y \tag{11}$$

where $\mathbf{1}_{\{x\}}$ *is the indicator function, taking the value of one if condition* x *holds and zero otherwise, and* $\overline{P}_q^{(\ell),m}$ *is the* m*–th term of the* ℓ*–times self–convolution of* $\overline{P}_q^{m}$ *as the latter is defined by equation* (7)*.*

Proof of Proposition 1. The proof of the proposition is provided in Appendix A. ■

Observe that equation (8) is a second–order differential equation connecting the unknown functions V_q^j for $j = n-2, n-1, n$. Equations (9)–(11) are boundary conditions that can be used to calculate the corrections to the switching boundaries and are activated only for $n \geq 2$, i.e. for orders $\mathcal{O}(\delta^0) \equiv \mathcal{O}(1)$ and above.

A significant benefit of the above proposition is that it can be used sequentially, according to a step–by–step procedure outlined in the next section, through which the value function and price threshold corrections due to "fast" mean–reverting stochastic volatility can be worked out analytically.

4. Step–by–step procedure for analytic derivation of optimal solution asymptotic terms

The procedure to sequentially determine $\left\{V_q^n\left(p\right), P_q^n\left(y\right)\right\}$ for $n \in \{0,1,\ldots\}$, $q \in \{0,1\}$ is as follows (where the explicit dependence on p and y is occasionally suppressed for convenience):

Step 0

For $n = 0$ the system (8)–(11) reduces to the simple second–order differential equation $\mathcal{L}_0 V_q^0\left(p,y\right) = 0$. Since the operator $\mathcal{L}_0$ involves differentiations with respect to the y variable only, it is easy to see that V_q^0 is a function of p only. Note that the exact dependence of V_q^0 on p will only be completely specified at a later step (*Step 2*).

Step 1

For $n = 1$ the system (8)–(11) reduces to the second–order differential equation $\mathcal{L}_0 V_q^1 (p, y) = -\mathcal{L}_1 V_q^0 (p)$ which, having obtained V_q^0 from the previous step, is treated as an equation for the unknown function V_q^1. Since V_q^0 is a function of p only, and $\mathcal{L}_1$ involves differentiations with respect to y, the equation for V_q^1 reduces to $\mathcal{L}_0 V_q^1 (p, y) = 0$, from which, using a similar argument as in step 0 above, we deduce that V_q^1 is a function of p only. Again the dependence of on p will only be exactly specified at a later step (*Step 3*). Furthermore note that the equation form for V_q^1 is the exactly the same as for V_q^0, but with a different right–hand side. This is a recurring pattern for all subsequent steps.

Step 2

For $n = 2$, equation (8) becomes

$$\mathcal{L}_0 V_q^2 + \mathcal{L}_1 V_q^1 + \mathcal{L}_2 V_q^0 + R_q (p) = \mathcal{L}_0 V_q^2 + \mathcal{L}_2 V_q^0 + R_q (p) = 0. \tag{12}$$

However, from this step onwards, the boundary conditions are activated, so that equation (12) is complemented by

$$C(0,0) = V_{1-q}^0 \left(P_q^0\right) - V_q^0 \left(P_q^0\right) = K_q \tag{13}$$

$$C(0,1) = \frac{\partial}{\partial p} \left[V_{1-q}^0 (p) - V_q^0 (p)\right]\Big|_{p=P_q^0} = 0 \tag{14}$$

$$\lim_{p \to I_q} V_q^0(p) = L_q \tag{15}$$

This step will completely determine V_q^0 and P_q^0, and will also determine V_q^2 up to a function of p. Given what is known from *Steps 0* and *1*, one can observe that equation (12) is a non–homogeneous linear equation of the form $\mathcal{L}_0 V_q^2 = F(p)$ for given F. This equation has non–trivial solutions for specific choices of F, which by the Fredholm alternative are specified by those F that belong to the orthogonal complement of the null space of the adjoint operator $\mathcal{L}_0$. This means that equation (12) has a solution if and only if F satisfies the condition $\langle F, \overline{\varphi} \rangle = 0$, where $\overline{\varphi}$ is the solution of equation $\mathcal{L}_0^* \overline{\varphi} = 0$, and $\mathcal{L}_0^*$ is the adjoint operator of $\mathcal{L}_0$, defined by $\langle \mathcal{L}_0 z, w \rangle = \langle z, \mathcal{L}_0^* w \rangle$, with $\langle \cdot, \cdot \rangle$ the L^2 inner product $\langle g, h \rangle = \int_{-\infty}^{+\infty} g(x)\, h(x)\, dx$, where g and h are

Lebesgue square–integrable functions. Note that $\overline{\varphi}$ is the density of the invariant distribution of the latent process Y_t, which easily verified to be $\overline{\varphi} = \frac{1}{\sqrt{2\pi}\nu} e^{-\frac{(m-y)^2}{2\nu^2}}$. If the solvability condition is satisfied, the solution of (12) is of the form $V_q^2 = \Phi_q^0(p,y) + X_q^0(p)$.

The solvability condition $\langle F, \overline{\varphi} \rangle = 0$ yields,

$$-\langle \mathcal{L}_1 V_q^1, \overline{\varphi} \rangle - \langle \mathcal{L}_2 V_q^0, \overline{\varphi} \rangle - R_q = -\langle \mathcal{L}_2 V_q^0, \overline{\varphi} \rangle - R_q = 0 \tag{16}$$

which reduces to

$$\lceil \mathcal{L}_2 V_q^0 \rceil \equiv \lceil \mathcal{L}_2 \rceil V_q^0 = \frac{1}{2} \overline{f}^2 p^2 \frac{\partial^2 V_q^0}{\partial p^2} + \mu p \frac{\partial V_q^0}{\partial p} - r V_q^0 = -R_q \tag{17}$$

where

$$\overline{f}^2 = \lceil f^2(y) \rceil = \int_{-\infty}^{+\infty} f^2(y)\, \overline{\varphi} dy,$$

is the average value of $f^2(y)$ with respect to the invariant distribution of Y_t. Equation (17) is a second–order differential equation for V_q^0, the solution of which provides the exact dependence of V_q^0 on p that was unspecified in *Step 0*. It is interesting to note that equation (17) is the Black–Scholes–Merton equation with a constant volatility, $\overline{f}^2$, the so–called effective volatility, which is an average over the invariant distribution of the "fast" latent process.

Equations (17) and (13)–(15) collectively, uniquely determine V_q^0 and P_q^0, $q \in \{0, 1\}$. The zero–order value functions V_q^0 are given by

$$V_q^0(x) = (-1)^q \int_{I_q}^{P_q^0} G_q(x, \overline{x}) R_q(\overline{x}) d\overline{x} \tag{18}$$

for $q \in \{0, 1\}$, with $G_q(x, \overline{x})$ the Green's functions of equation (17) subject to (13)–(15). To conserve space, these, along with the procedure to determine the zero–order resource price thresholds P_q^0, are relegated in an appendix (Appendix C) that is submitted as supplementary material to the manuscript.

Once V_q^0 and P_q^0 are completely determined, this step also determines V_q^2, up to a constant of p. By substituting the solvability condition (16) into the right–hand side of equation (12) and rearranging, one gets

$$\mathcal{L}_0 V_q^2 = (\lceil \mathcal{L}_2 \rceil - \mathcal{L}_2) V_q^0 = \Delta f(y) \Theta^0(p), \tag{19}$$

with

$$\Delta f(y) = \tfrac{1}{2}\left(\overline{f}^2 - f^2\left(y\right)\right) \quad \text{and} \quad \Theta_q^0\left(p\right) = p^2 \tfrac{\partial^2 V_q^0}{\partial p^2}. \tag{20}$$

Since the operator $\mathcal{L}_0$ involves differentiations with respect to the y variable only, and V_q^0 only depends on p, one can write

$$V_q^2\left(p, y\right) = \Phi_q^0(p, y) + X_q^0(p) \tag{21}$$

with $\Phi_q^0(p, y)$ the solution of

$$\mathcal{L}_0 \Phi_q^0(p, y) = \Delta f(y) \Theta^0(p), \tag{22}$$

and $X_q^0(p)$ a constant with respect to y that will be specified at the $n = 4$ order.[2]

Thus, in step $n = 2$, V_q^0 and P_q^0 are completely specified, and V_q^2 is determined up to the function $X_q^0(p)$.

Step 3

The procedure is similar for each $n > 2$, where value function and resource price threshold "corrections" V_q^{n-2} and P_q^{n-2} are completely specified, and V_q^n is obtained up to an unknown function of p (to be obtained at order $n + 2$).

Equations (8)–(11) now become

$$\mathcal{L}_0 V_q^n + \mathcal{L}_1 V_q^{n-1} + \mathcal{L}_2 V_q^{n-2} = 0 \tag{23}$$

$$\sum_{\substack{k,\ell,m: \\ k+\ell+m=n-2}} C\left(k, \ell\right) \overline{P}_q^{(\ell),m} = 0 \tag{24}$$

$$\sum_{\substack{k,\ell,m: \\ k+\ell+m=n-2}} \left(\ell + 1\right) C\left(k, \ell + 1\right) \overline{P}_q^{(\ell),m} = 0 \tag{25}$$

$$\lim_{p \to I_q} V_q^{n-2}(p) = 0 \tag{26}$$

[2]More accurately, one can write $V_q^2\left(p, y\right) = \Psi_q^0(y)\Theta_q^0(p) + X_q^0(p)$, with $\Psi_q^0(y)$ the solution of $\mathcal{L}_0\Psi_q^0(y) = \Delta f(y)$ in this instance. However, for higher orders of n, it will be notationally more convenient to write $\Psi_q^0(y)\Theta_q^0(p)$ as $\Phi_q^0(p, y)$.

From the Fredholm alternative, the solvability condition is

$$-\langle \mathcal{L}_1 V_q^{n-1}, \overline{\varphi} \rangle - \langle \mathcal{L}_2 V_q^{n-2}, \overline{\varphi} \rangle = 0, \tag{27}$$

which reduces to $\lceil \mathcal{L}_2 V_q^{n-2} \rceil = \lceil \mathcal{L}_2 \rceil V_q^{n-2} = -\lceil \mathcal{L}_1 V_q^{n-1} \rceil$ and

$$\frac{1}{2} \overline{f}^2 p^2 \frac{\partial^2 V_q^{n-2}}{\partial p^2} + \mu p \frac{\partial V_q^{n-2}}{\partial p} - r V_q^{n-2} = -\omega p \Omega_q^{n-2}(p), \tag{28}$$

with ω and $\Omega_q^{n-2}(p)$ defined in Appendix B, for all $n > 2$. Equation (28), subject to (24)–(26) completely specifies V_q^{n-2} and P_q^{n-2}. Two notes are in order here:

(a) Given the definitions in Appendix B, it is easy to verify that the right–hand side of (28) is always a known (albeit involved) expression of value functions $V_q^0, \ldots, V_q^{n-3}$ which are already determined at previous orders of n.

(b) For $n > 2$, the system (28)–(26) is no longer a free boundary problem. Once V_q^{n-2} are determined from (28) and (26), equations (24) and (25) can be solved analytically for the price threshold "corrections", P_q^{n-2}.

Both of these notes will become apparent in the next section, where our procedure is applied to a known problem.

Finally, substituting the solvability condition (27) into the right–hand side of equation (23) and rearranging, yields

$$\mathcal{L}_0 V_q^n = \left(\lceil \mathcal{L}_2 \rceil - \mathcal{L}_2 \right) V_q^{n-2} - \mathcal{L}_1 V^{n-1} + \lceil \mathcal{L}_1 V^{n-1} \rceil.$$

With V_q^{n-2} completely specified in this step, and V_q^{n-1} known (up to a function of p) from the previous step, the solution of V_q^n, up to a function of p, is

$$V_q^n = \Phi_q^{n-2}(p, y) + X_q^{n-2}(p) \tag{29}$$

with $\Phi_q^{n-2}(p, y)$ defined in Appendix B for all $n > 2$.

Step 4

Step 3 is repeated for all orders $\mathcal{O}(\delta^{n-2})$, $n > 2$ for which the asymptotic terms of the optimal switching solution are required.

Step 5

Determine the solutions $V_q^{\delta}(p,y)$ and $P_q^{\delta}(y)$ by adding up the asymptotic terms, as in equation (6).

In the next section we demonstrate the procedure by deriving asymptotic terms of the optimal solution under "fast" mean–reverting stochastic volatility, for a simple switching problem that is highly–cited in the literature.

5. Effects of "fast" mean–reverting stochastic volatility on optimal switching decisions

In this section, we illustrate the effects of "fast" mean–reverting stochastic volatility on the optimal switching strategy, using a general 'fast" mean–reverting stochastic volatility model in the context of the Dixit (1989) entry/exit problem. In the context of this entry/exit problem, the starting point for the analysis is the quasi–variational inequality in (5), with the choice of $R_q(P_t) = (P_t - c) \times \mathbf{1}_{q=1}$, for $q = 0, 1$, with c a given constant.

In applying *Steps 0* and *1* of the procedure, we observe that V_q^0 and V_q^1, $q = \{0, 1\}$ are independent of y.

In *Step 2*, for $n = 2$ the value functions $V_q^0(P)$ for $q \in \{0, 1\}$, are uniquely specified via equation (18), whose solution yields,

$$V_q^0(P) = (1-q)\,AP^{\alpha} + q\left(BP^{\beta} + \frac{P}{r-\mu} - \frac{c}{r}\right), \quad q \in \{0,1\},$$

with

$$\alpha, \beta = \frac{-\mu + \frac{1}{2}\overline{f}^2 \pm \sqrt{2\overline{f}^2 r + \left(\mu - \frac{1}{2}\overline{f}^2\right)^2}}{\overline{f}^2} \tag{30}$$

Moreover, the system

$$A\left[P_q^0(y)\right]^{\alpha} + (-1)^q K_q = B\left[P_q^0(y)\right]^{\beta} + \frac{P_0^q(y)}{r-\mu} - \frac{c}{r} \tag{31}$$

$$A\alpha\left[P_q^0(y)\right]^{\alpha-1} = B\beta\left[P_q^0(y)\right]^{\beta-1} + \frac{1}{r-\mu} \tag{32}$$

for $q \in \{0, 1\}$ is a system of four algebraic euations, the solution of which determines the constants A, B and the zeroth–order product price switching thresholds $P_0^0(y), P_1^0(y)$.

Note that this is formally identical to the constant–volatility solution in Dixit (1989, eq. 6–7 and 12–15), with the important difference that the constant variance σ^2 has to be replaced by $\overline{f}^2$, the so–called *effective variance* which is the average of the volatility function $f^2(y)$ over the invariant measure of the process Y. Therefore, the leading term is equivalent to the solution given by a constant volatility model, but the constant volatility depends on the choice of the volatility function $f(y)$ employed in the original multi–scale stochastic volatility model. All following orders, $n = 3, 4, \ldots$ are essentially corrections to this constant, effective volatility case, corrections that are the effect of fast mean–reverting stochastic volatility.

Moreover, from (20)–(22), V_q^2 is

$$V_q^2(p, y) = \Gamma P^2 \frac{\partial^2 V_q^0(p)}{\partial p^2} + X_q^0(p), \quad \Gamma = \mathcal{L}_0^{-1}\left[\Delta f(y)\right]$$

with $X_q^0(p)$ to be determined at *Step 4*. Note that the exact value of the constant Γ depends on the choice of $f(y)$.

Step 3, for $n = 3$, uniquely specifies the value functions $V_q^1(P)$ and price thresholds $P_q^1(Y)$, for $q = \{0, 1\}$, via equation (28) that becomes

$$\lceil \mathcal{L}_2 \rceil V_q^1 = -\lceil \mathcal{L}_1 V_q^2 \rceil = -\omega P \Omega_q^1(P). \tag{33}$$

Given the definition of Ω_q^1 from Appendix B, one can verify that the right–hand side is just a function of P only. From (24)–(26), the relevant boundary conditions are

$$C(1,0) + C(0,1) P_q^1(y) = 0 \tag{34}$$

$$C(1,1) + C(0,2) P_q^1(y) = 0 \tag{35}$$

$$\lim_{p \to I_q} V_q^1(p) = 0 \tag{36}$$

Solving (33)–(36) yields the order $\mathcal{O}(\delta)$ value function "corrections" that are due to fast mean–reverting stochastic volatility

$$V_q^1(P) = \omega\Gamma\left[(1-q)P^\alpha + qP^\beta\right]\left[\frac{(-1)^q D_q \ln P}{\overline{f}^2(\alpha - \beta)}\right.$$

$$
+\frac{\left(D_0P_0^{\alpha}+D_1P_0^{\beta}\right)P_1^{(1-q)\beta+q\alpha}\ln P_0-\left(D_0P_1^{\alpha}+D_1P_1^{\beta}\right)P_0^{(1-q)\beta+q\alpha}\ln P_1}{\overline{f}^2\left(\alpha-\beta\right)\left(P_0^{\beta}P_1^{\alpha}-P_0^{\alpha}P_1^{\beta}\right)}\Bigg] \tag{37}
$$

and the order $\mathcal{O}\left(\delta\right)$ "corrections" to the switching thresholds,

$$
\begin{aligned}
&P_q^1\left(y\right)=\omega\Gamma \\
&\times\frac{\left(D_0P_q^{\alpha+1}+D_1P_q^{\beta+1}\right)\left(P_0^{\alpha}P_1^{\beta}-P_0^{\beta}P_1^{\alpha}\right)+P_q^{\alpha+\beta+1}\left(D_0P_{1-q}^{\alpha}+D_1P_{1-q}^{\beta}\right)\left(\alpha-\beta\right)\ln\frac{P_1}{P_0}}{\overline{f}^2\left(P_0^{\alpha}P_1^{\beta}-P_0^{\beta}P_1^{\alpha}\right)\left(\alpha-\beta\right)\left[B\beta\left(\beta-1\right)P_q^{\beta}-A\alpha\left(\alpha-1\right)P_q^{\alpha}\right]}
\end{aligned} \tag{38}
$$

In the above, $D_q=\begin{cases} D_0=A\alpha^2\left(\alpha-1\right), & \text{if } q=0 \\ D_1=B\beta^2\left(\beta-1\right), & \text{if } q=1\end{cases}$, α,β are as in (30) and the constants A,B are as determined in equations (31)–(32). Moreover, in order to simplify notation, the zeroth–order product price switching thresholds $P_q^0, q=\{0,1\}$ are denoted as $P_q\equiv P_q^0, q=\{0,1\}$ in equations (37)–(38) above (i.e. the zero of the order in the superscript is dropped for notational convenience).

This step also provides information regarding $V_q^3\left(P\right)$, through equation (19) that becomes

$$
\mathcal{L}_0V_q^3\left(P\right)=\Delta f\left(y\right)P^2\frac{\partial^2V_q^1\left(P\right)}{\partial P^2}-\omega f\left(y\right)P\frac{\partial^2V_q^2\left(P\right)}{\partial P\partial y}+\omega\Gamma D_qP^{(1-q)\alpha+q\beta}
$$

This determines $V_q^3\left(P\right)$, up to a constant of P, which is used in the next order to uniquely determine the second–order corrections $V_q^2\left(P\right)$ and $P_q^2\left(y\right)$.

After the asymptotic terms are estimated (by repeating this step of the procedure for all needed orders of δ, in *Step 5* the solution is approximated by equation (6).

The above results clearly show that the effect of "fast" mean–reverting stiochastic volatility on the optimal switching decisions are quantified through the statistical average $\overline{f}$, its variance and its correlation ρ with the product price process, as well as the constant Γ. All these quantities can be explicitly calculated once a specific volatility model is chosen. The next two examples demonstrate this.

Example 1: The stochastic volatility model in Fouque et al. (2003c)

As a first example, we choose the stochastic volatility model employed by Fouque et al. (2003c), where the product price change volatility σ_t is related to the latent "fast" mean–reverting factor y via $\sigma_t = f(y) = \exp(y)$, restricted on a compact subset so as not to affect the calculations within the accuracy of our comparisons. This example has also been employed by Zhu and Chen (2011a) in their numerical investigation.

It can be shown that for this specific stochastic volatility model, the parameters $\overline{f}$ and Γ involved in (37)–(38) are given by

$$\overline{f}^2 = e^{2\left(m+\nu^2\right)} \quad \text{and} \quad \Gamma = \frac{e^{5\nu+3m}\left(1-e^{2\nu^2}\right)}{2\nu^2}$$

To investigate how "fast" mean–reverting stochastic volatility affects optimal switching and hysteresis, define for $q = \{0, 1\}$

$$V_q^{FPSS}(P, y) - V_q^D(P) \approx \delta V_q^1(P) \quad \text{and} \quad P_q^{FPSS}(y) - P_q^D \approx \delta P_q^1(y) \tag{39}$$

as the value function and the optimal switching threshold differences for the idle ($q = 0$) and active ($q = 1$) modes under the stochastic volatility model in Fouque et al. (2003c, FPSS) and the constant–volatility problem in Dixit (1989, D), with the constant σ set equal to $\overline{f}$. For simplicity, only the first–order δ differences are considered.

The effect (of the first order corrections) due to "fast" mean reverting stochastic volatility is quantitatively assessed in Figures 1–3. What is noticeable immediately in Figure 1 is that the value differences due to "fast" mean–reverting stochastic volatility are not monotone with respect to the resource price P. An equally interesting aspect of Figure 1 is that the value differences in (39) change sign (from positive to negative or vice–versa). Both the position of the maximum, as well as the zero crossing point can be obtained analytically in terms of the parameters of the problem (and we make their exact formulae available upon request).

Figure 2 focuses on the effect of "fast" mean–reverting stochastic volatility on $P_q^{FPSS}(y)$ and P_q^D, the product price thresholds that warrant optimal switching from the idle ($q = 0$) and the active ($q = 1$) modes under stochastic and constant volatility.

Panel (a) of Figure 2 plots (bold line) the constant $\sigma = \overline{f}$ switching thresholds in Dixit (1989), as a function of the switching costs $K_0 = K_1 = K$, which are restricted to be symmetric. Also plotted are the "corrected"

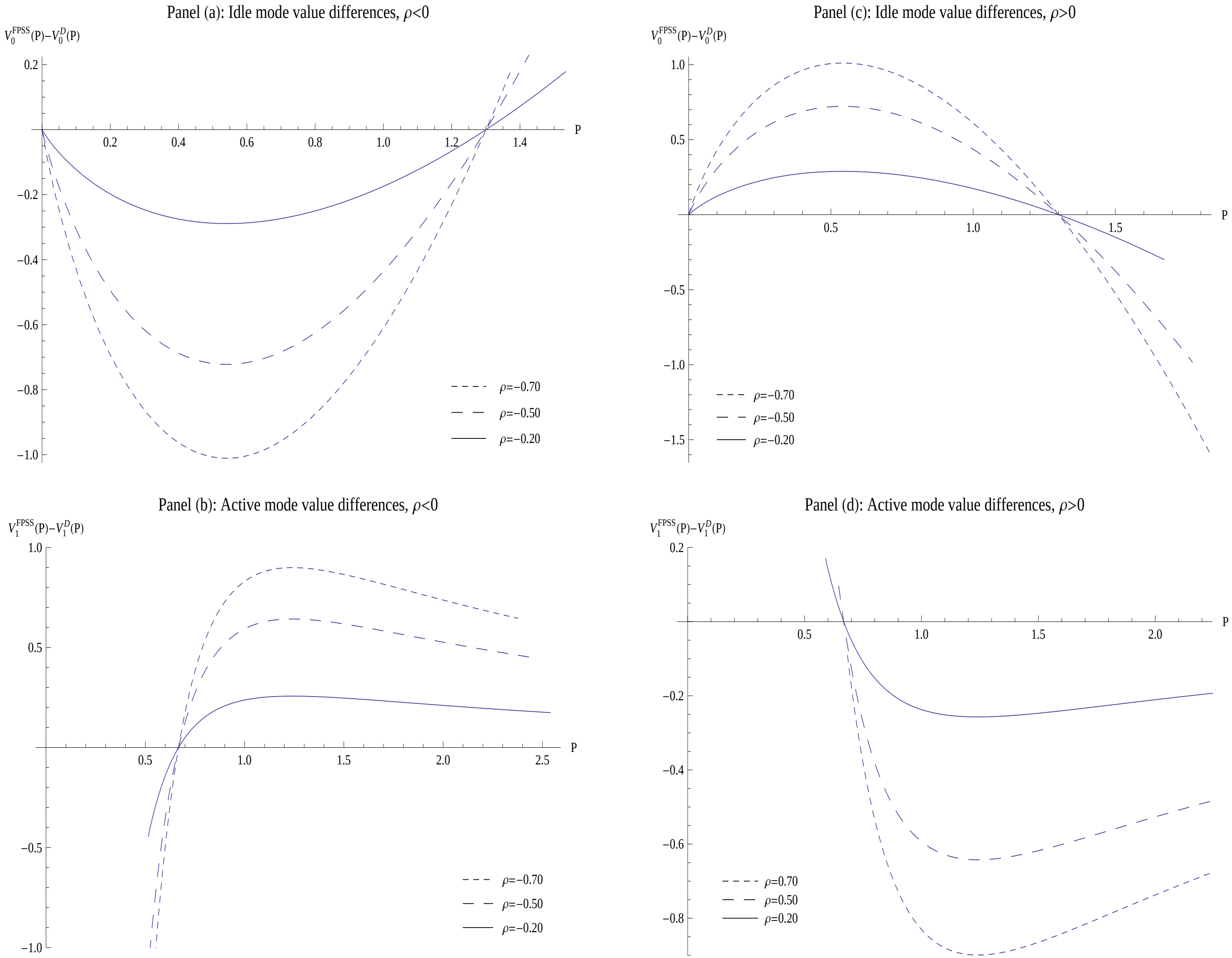

Figure 1: The Figure plots, for different values of the correlation coefficient ρ, the value function differences $V_q^{FPSS}(P) - V_q^{D}(P)$ for the idle ($q = 0$, Panels (a)–(c)) and active ($q = 1$, Panels (b)–(d)) modes under the stochastic volatility model in Fouque et al. (2003c, FPSS) and the constant–volatility problem in Dixit (1989, D), with the constant σ set equal to $\overline{f}$. In Panels (a)–(b) (respectively (c)–(d)), the correlation coefficient ρ is negative (respectively positive). In Panels (a)–(c) (respectively (b)–(d)) the product price P takes values in the "idle region" $\left[0, P_0^{\delta}\right]$ (respectively in the "active region" $\left[P_1^{\delta}, +\infty\right]$). In all panels the rest of the parameters are $r = 0.025$, $\mu = 0.02$, $K_0 = 4$, $K_1 = 2$, $c = 1$, $m = \ln 0.1$, $\nu = 1/\sqrt{2}$ and $\delta = 1/\sqrt{200}$.

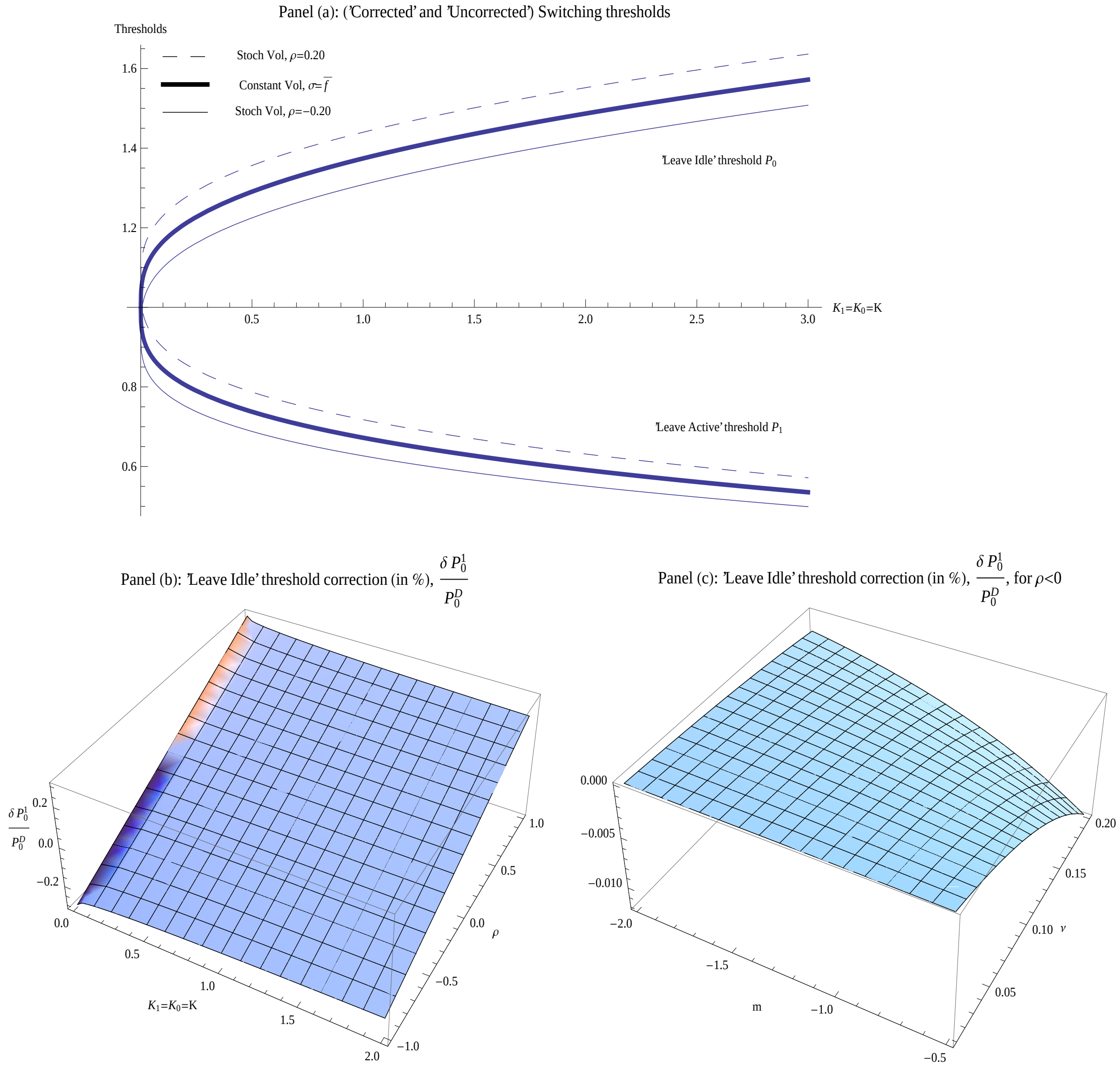

Figure 2: The Figure plots in Panel (a) P_q^{FPSS} and P_q^D, i.e. the optimal switching product price thresholds under the stochastic volatility model in Fouque et al. (2003c, FPSS) and the constant–volatility problem in Dixit (1989, D), with the constant σ set equal to $\overline{f}$. The thresholds are plotted as a function of the switching costs K_0 and K_1, which are set to equal values, and for different values of the correlation coefficient ρ. Panels (b) and (c) plot $\left(P_q^{FPSS} - P_q^D\right)/P_q^D \approx \delta P_q^1 / P_q^D$, i.e. the optimal switching threshold "corrections" due to "fast" stochastic volatility δP_q^1, as a percentage of P_q^D, the optimal switching product price thresholds under the constant–volatility problem in Dixit (1989, D). Panels (b) and (d) only plot the $q = 0$ (i.e. leave–idle–mode) case in order to save space; the $q = 1$ (i.e. leave–active–mode) percentage "corrections" are qualitatively identical. In Panel (c), the correlation between changes in the product price P and in the latent volatility factor Y is $\rho = -0.20$. In all Panels, the rest of the parameters are $r = 0.025$, $\mu = 0.02$, $m = \ln 0.1$, $\nu = 1/\sqrt{2}$, $K_0 = 4$, $K_1 = 2$, $c = 1$ and $\delta = 1/\sqrt{200}$.

switching thresholds $P_q^0 + \delta P_q^1$, for $\rho = \pm 0.20$. Depending on the correlation coefficient, the "fast corrected" switching thresholds can be above or below the constant–volatility ones. For $\rho < 0$ which is the most usual case in financial variables (see for example Fouque et al., 2000), the owner of the process/project that can be switched from and to an idle/active mode will optimally decide to leave the current mode at lower product prices: Lower prices (than those in the constant volatility case with $\sigma = \overline{f}$) lead to project activation and the owner is willing to incur more losses before activation is suspended on the down side.

The degree by which $P_q^{FPSS}(y) < P_q^D$ for $\rho < 0$ is found to be positively related to the "vol of vol" level ν and the long–run volatility level m, and negatively related to the correlation level ρ. These relationships can be visualized with the aid of Panels (b) and (c) in Figure 2 that plot the percentage changes in the thresholds due to "fast" mean–reverting stochastic volatility. In Panel (b), as $\rho \to \pm 1$ the "corrections" δP_q^1 can be as significant as $\pm 20\%$ of the constant–vol benchmarks P_q^D. In Panel (c) it can be seen that this percentage correction appears convex in m and ν, that positively determine the effective volatility level $\overline{f}$.

"Fast" mean–reverting stochastic volatility not only affects the position of the switching thresholds, but also the likelihood of reaching them in finite time as Figure 3 demonstrates. To accomplish this we first demonstrate how one can calculate the conditional probability of switching from the current mode by a finite time horizon under "fast" mean–reverting stochastic volatility (and then compare it with the constant volatility case $\sigma = \overline{f}$ in Figure 3). To conserve space, this is summarised in an appendix (Appendix D) that is submitted as supplementary material to the manuscript.

Panels (a) and (b) plot the conditional probability in equation (D.5) of Appendix D for $q = 0$, as a function of m, δ and ν. The probability is scaled by the corresponding conditional probability under the constant–volatility setting in Dixit (1989, D). Ratios above unity indicate that switching from the idle mode and activating the investment project is significantly *more probable* under "fast" mean–reverting stochastic volatility than the constant volatility benchmark. Ratios are below unity for $q = 1$ (not plotted to save space), suggesting that deactivating the investment project is significantly *less probable* under "fast" mean–reverting stochastic volatility.

Since activating (deactivating) an investment project becomes more (less) probable, a natural question that arises is whether one would observe more or less frequent mode switches under "fast" mean–reverting stochastic volatil-

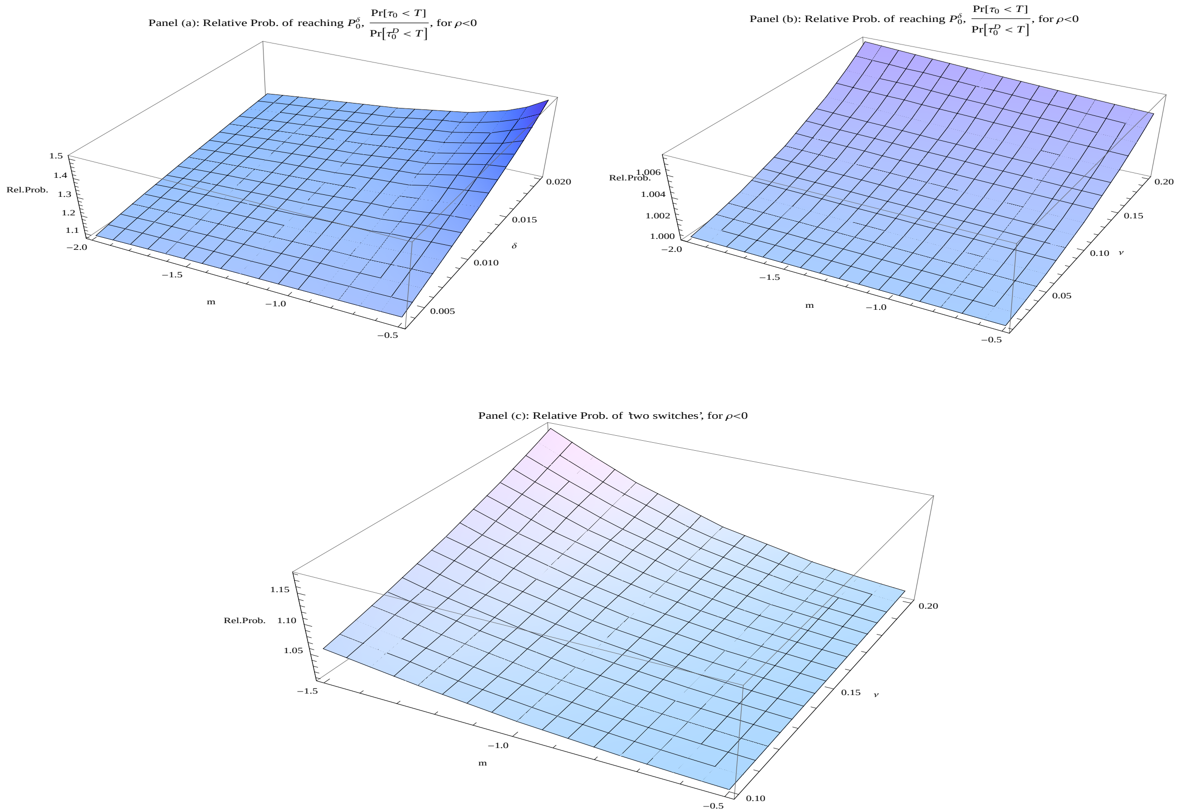

Figure 3: The Figure plots in Panels (a) and (b) the ratio $Pr(\tau_q < T)/Pr(\tau_q^D < T)$, i.e. the conditional probability of reaching the optimal switching product price threshold P_q^δ until time T (starting at $P_s \equiv p = \frac{1}{2}\left(P_0^\delta + P_1^\delta\right)$) under the stochastic volatility model in Fouque et al. (2003c, FPSS) as a percentage of the same conditional probability but under the constant–volatility problem in Dixit (1989, D). Panel (c) plots the ratio $Pr(P_t \leq P_1^\delta, m_t^P \geq P_0^\delta)/Pr(P_t \leq P_1, m_t^P \geq P_0)$. The numerator is the conditional probability of starting at $P_s \equiv p = \frac{1}{2}\left(P_0^\delta + P_1^\delta\right)$ in mode $q = 0$, switching to mode $q = 1$ and subsequently back to mode $q = 0$ again until time T under the stochastic volatility model in Fouque et al. (2003c, FPSS). The denominator is the same conditional probability but under the constant–volatility problem in Dixit (1989, D). Panels (a) and (b) only plot the $q = 0$ (i.e. leave–idle–mode) case in order to save space; the $q = 1$ (i.e. leave–active–mode) relative probabilities are qualitatively identical. In all Panels, the rest of the parameters are $r = 0.025$, $\mu = 0.02$, $m = \ln 0.1$, $\nu = 1/\sqrt{2}$, $\rho = -0.20$, $K_0 = 4$, $K_1 = 2$, $c = 1$ and $\delta = 1/\sqrt{200}$.

ity, compared to the constant volatility case. The answer, for reasonable parameter values (namely a negative correlation ρ between volatility and price), is that *more frequent* "switches" are to be expected. An indication of this is provided in Panel (c) of Figure 4 that plots the conditional probability in equation (D.6) of Appendix D. This is the probability of switching from idle ($q = 0$) to active ($q = 1$), to idle ($q = 0$) again in finite time T, and it is again scaled by the corresponding conditional probability under the constant–volatility case for the purposes of the Figure. Ratios are above unity for any parameter values (not only the ones plotted), suggesting that "fast" mean–reverting stochastic volatility makes an agent switch between operational modes more frequently when compared to the constant volatility case.

Overall, the numerical investigation in this subsection suggests that under "fast" mean–reverting stochastic volatility that is negatively correlated with the product price process (as is usually the case), the owner of a process/project that can be switched from and to an idle/active mode will be more willing to switch to the active state, will endure higher losses before deciding to suspend operations, and is likely to make more frequent switches in a given investment horizon. This willingness and increased frequency appears more pronounced for lower (more negative) correlation levels, faster volatility mean–reversion speeds and higher effective volatility levels.

Example 2: The stochastic volatility model in Renault and Touzi (1996)

We provide an additional example of the approach by employing a correlated version of the stochastic volatility model in Renault and Touzi (1996), which is based on the model by Hull and White (1987). In these models the product price change volatility σ_t is related to the latent "fast" mean–reverting factor y via $\sigma_t = f(y) = \sqrt{y}$. It can be shown that for this specific stochastic volatility model,

$$\overline{f}^2 = m$$

and

$$\Gamma = -\frac{1}{2^{5/4}\sqrt{\pi\nu}}\left[\frac{\nu}{\sqrt{2}}\Gamma\left(\frac{3}{4}\right)M\left(-\frac{1}{4},\frac{1}{2},-\frac{m^2}{2\nu^2}\right)+m\Gamma\left(\frac{5}{4}\right)M\left(\frac{1}{4},\frac{3}{2},-\frac{m^2}{2\nu^2}\right)\right]$$

where $\Gamma(z) = \int_0^{+\infty} t^{z-1}e^{-t}dt$ is the Gamma function and $M(\theta,\vartheta,x)$ is Kummer's confluent hypergeometric function (see Abramowitz and Stegun, 1972).

Choosing a different specific stochastic volatility model changes the mag-

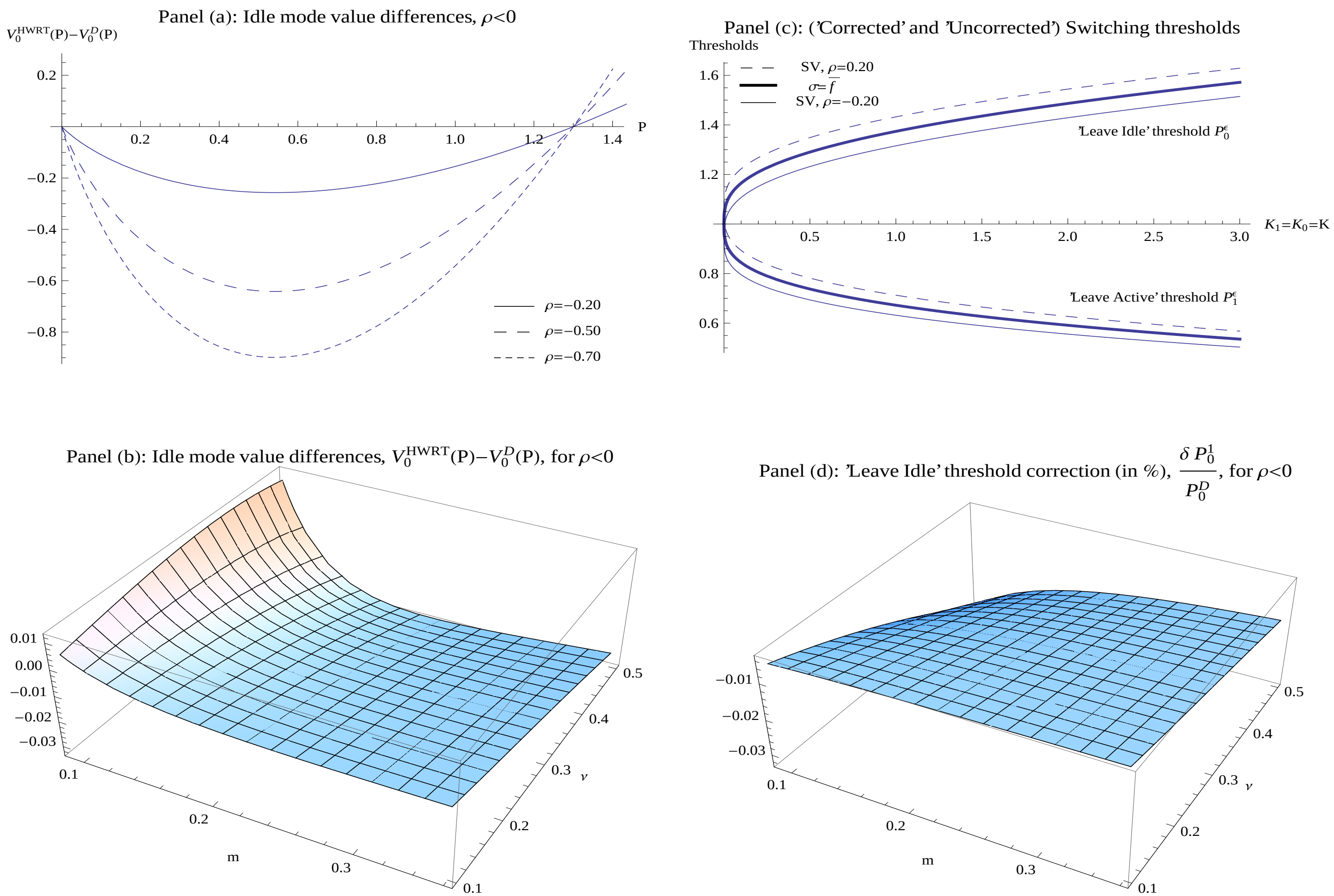

Figure 4: The Figure plots in Panel (a), for different values of the correlation coefficient ρ, the value function difference $V_q^{HWRT}(P,y) - V_q^D(P)$ for the idle ($q=0$) mode over the "idle region" $P \in \left[0, P_0^\delta\right]$, under the stochastic volatility model in Hull and White (1987) and Renault and Touzi (1996) and the constant–volatility problem in Dixit (1989), with the constant σ set equal to $\overline{f}$. Panel (b) plots the same value difference as a function of ν and m, i.e. the volatility and the long–run mean of the latent "fast" stochastic volatility factor Y in equation (2) for $\rho = -0.20$. Panel (c) plots P_q^{HWRT} and P_q^D, i.e. the optimal switching product price thresholds under the two models, as functions of the switching costs K_0 and K_1, which are set to equal values, for different values of the correlation coefficient ρ. Panel (d) plots, for the idle ($q=0$) mode as a function of ν and m and for $\rho = -0.20$, $\left(P_q^{HWRT} - P_q^D\right)/P_q^D = \delta P_q^1/P_q^D$, i.e. the optimal switching threshold "corrections" due to "fast" stochastic volatility $\sqrt{\epsilon}P_q^1$, as a percentage of P_q^D, the optimal switching product price thresholds under the constant–volatility problem in Dixit (1989, D). In all Panels, the rest of the parameters are $r = 0.025$, $\mu = 0.02$, $K_0 = 4$, $K_1 = 2$, $c = 1$, $m = 0.027$, $\nu = 0.30$ and $\delta = 1/\sqrt{200}$.

nitudes of value function and switching threshold differences, but does not affect qualitatively the impact of "fast" mean–reverting stochastic volatility on optimal switching decisions, as Figure 4 demonstrates. Value differences $V_q^{HWRT}(P,y) - V_q^{D}(P)$ in Panels (a) and (b) are again non–monotone with respect to P and their sign can be positive or negative depending on parameter values (ρ, m and ν). Moreover, in Panels (c) and (d) the "fast corrected" switching thresholds are again found to be above or below the constant–volatility ones depending on the correlation coefficient sign, and the percentage threshold correction due to "fast" stochastic volatility are non–linear in m and ν, that positively determine the effective volatility level $\overline{f}$. Similarly, the conditional probabilities of reaching the switching thresholds (not plotted to save space; available upon request) are qualitatively similar to the ones in the previous subsection.

Overall, it should be stressed that the intention of the specific stochastic volatility examples presented in this section, is simply to demonstrate the ability of the perturbation method in Fouque et al. (2003a) to "correct" the well–known solution of the general optimal switching problem under constant volatility. The analytic results that were presented for the stochastic volatility models in Fouque et al. (2003c), Hull and White (1987) and Renault and Touzi (1996), for the optimal switching problem in Dixit (1989) are merely intended as indications of this ability.

6. Conclusions

Optimal switching problems are at the heart of many economic decisions: from the personnel hiring and firing policies of firms and universities, to the optimal production management of natural resource investments under fixed costs, to the entry and exit from foreign direct investments in the face of fluctuating exchange rates.

Empirical research has provided ample evidence that many economic variables of interest—such as commodity prices, equity prices and exchange rates—exhibit volatility that is (partly or wholly) governed by a quickly mean–reverting stochastic factor (see for example the empirical evidence in Alizadeh et al., 2002; Fouque et al., 2003b; Hikspoor and Jaimungal, 2008).

Multi–scale stochastic volatility models (Fouque et al., 2003a) that allow for latent volatility factors to quickly mean–revert and decorrelate exponentially fast have gained much attention in the option pricing literature. However, it is not clear whether the presence of a "fast" mean–reverting stochastic volatility factor in the dynamics of commodity prices or exchange rates has any implications for the optimal policies of switching problems that are contingent on such dynamics. More importantly, the direction and nature of any such implications is still in our view an open issue worth investigating.

This paper formulates and solves an infinite–horizon, optimal switching problem under uncertainty and a general class of stochastic volatility models that exhibit "fast" mean–reversion. Using the perturbation method (as in Fouque et al., 2000), the general optimal switching problem is divided into a sequence of simplified valuation systems, each one offering a "correction" of different order to the constant–volatility solution that has been documented in the literature. These corrections, that are the effect of "fast" stochastic volatility, allow one to analytically approximate the solution of the general switching problem under fast mean–reverting stochastic volatility up to the desired order.

As a demonstration of the approximation, we explicitly derive the "correction terms" due to fast mean–reverting stochastic volatility for the highly–cited entry and exit switching model of Dixit (1989) under a number of alternative "fast" volatility dynamics. There we show that when the uncertainty in an economic system exhibits fast mean–reverting stochastic volatility: (a) optimal switching between modes will be *more frequent*, (b) agents will be more willing to activate earlier and will endure higher losses before deciding to optimally suspend operations and (c) findings (a) and (b) are more pronounced for lower (more negative) levels of correlation between price and volatility uncertainty, faster volatility mean–reversion speeds and higher effective volatility levels. We believe these findings should be of interest to the managers of processes/projects that can be switched from and to an idle/active mode, contingent on the evolution of economic variables that are documented to exhibit fast mean–reverting stochastic volatility such as exchange rates and commodity prices.

A. Appendix: Proof of Proposition 1

With the Hamilton–Bellman–Jacobi equation in the quasivariational inequality form of equation (5), it is easy to see that the problem can be written as the following system of partial differential equations

$$\mathcal{L}^{\delta} V_q^{\delta}(p, y) = -R_q(p) \tag{A.1}$$

$$V_{1-q}^{\delta}\left(P_q^{\delta}(y), y\right) - V_q^{\delta}\left(P_q^{\delta}(y), y\right) = K_q \tag{A.2}$$

$$\frac{\partial}{\partial p}\left[V_{1-q}^{\delta}(p,y)-V_{q}^{\delta}(p,y)\right]\Big|_{p=P_q^{\delta}(y)} \;=\; 0 \tag{A.3}$$

$$\lim_{p\to I_q} V_q^{\delta}(p,y) \;=\; L_q \tag{A.4}$$

for $q=\{0,1\}$.

Equations (A.2) and (A.3) describe what happens at the optimal resource price levels $P_q^{\delta}(y)$, and are sometimes referred to as the "value–matching" and "smooth–pasting" conditions (see Dumas, 1991; Shackleton and Sødal, 2005). The limits in equation (A.4) suggest that the values $V_q^{\delta}(p,y)$, for extremely high or low product prices where switching is essentially "worthless", should converge to the present value of operating in mode q forever.

Substituting in equation (A.1) the asymptotic expansion of the value functions and product price thresholds in (6), it should be easy to verify that at each order $\mathcal{O}(\delta^{n-2})$, for $n\in\{0,1,2,\ldots\}$, the left–hand side is always of the form $\mathcal{L}_0V^n+\mathcal{L}_1V^{n-1}+\mathcal{L}_2V^{n-2}$, or equivalently $\sum_{j=(n-2)^+}^{n}\mathcal{L}_{n-j}V_q^j$ with $j\in\{(n-2)^+,\ldots,n\}$. The right–hand side is $-R_q(p)$ at order $\mathcal{O}(\delta^0)=\mathcal{O}(1)$ for $n=2$ and zero for any other n. Hence equation (8) in the text.

In order to get equation (9) from (A.2), write the latter, by using the expansion in (6) as

$$\sum_{k=0}^{+\infty}\delta^k V_{1-q}^k\left(\sum_{k=0}^{+\infty}\delta^k P_q^k\right)-\sum_{k=0}^{+\infty}\delta^k V_q^k\left(\sum_{k=0}^{+\infty}\delta^k P_q^k\right)=K_q$$

where k is used as the counter and the dependence of P_q^k on y is dropped for notational convenience. Take each term $V_{\cdot}^k\left(\sum_{k=0}^{+\infty}\delta^k P_q^k\right)$ in the summations above and expand them around P_q^0 to get

$$\sum_{k=0}^{+\infty}\delta^k\left\{\sum_{\ell=0}^{+\infty}\left[\frac{1}{\ell!}\left.\frac{\partial^{\ell}V_{1-q}^k}{\partial P^{\ell}}\right|_{P=P_q^0}\times\left(\sum_{k=1}^{+\infty}\delta^k P_q^k\right)^{\ell}\right]\right\}$$
$$-\sum_{k=0}^{+\infty}\delta^k\left\{\sum_{\ell=0}^{+\infty}\left[\frac{1}{\ell!}\left.\frac{\partial^{\ell}V_{q}^k}{\partial P^{\ell}}\right|_{P=P_q^0}\times\left(\sum_{k=1}^{+\infty}\delta^k P_q^k\right)^{\ell}\right]\right\}=K_q \tag{A.5}$$

Observe that

$$\sum_{k=1}^{+\infty} \delta^k P_q^k = \delta \sum_{m=0}^{+\infty} \delta^m P_q^{m+1} = \delta \sum_{m=0}^{+\infty} \delta^m \overline{P}_q^m$$

where the last equality comes from the definition of the convenient re–labelling we applied in equation (7). Define

$$\psi\left(\delta\right) = \textstyle\sum_{m=0}^{+\infty} \delta^m \overline{P}_q^m \quad \text{and} \quad C\left(k,\ell\right) = \frac{1}{\ell!}\frac{\partial^\ell}{\partial p^\ell}\left[V_{1-q}^k\left(p\right) - V_q^k\left(p\right)\right]\Big|_{p=P_q^0(y)}$$

and substitute in (A.5) to get

$$\sum_{k=0}^{+\infty} \delta^k \left\{ \sum_{\ell=0}^{+\infty} C\left(k,\ell\right)\left[\delta\psi\left(\delta\right)\right]^\ell \right\} = K_q \tag{A.6}$$

Observe that

$$\begin{aligned}\psi\left(\delta\right)^2 &= \left(\sum_{m=0}^{+\infty} \delta^m \overline{P}_q^m\right)\left(\sum_{m=0}^{+\infty} \delta^m \overline{P}_q^m\right) = \sum_{m=0}^{+\infty} \delta^m \sum_{i=0}^{m} \overline{P}_q^{m-i}\overline{P}_q^i \\ &= \sum_{m=0}^{+\infty} \delta^m \overline{P}_q^{(2),m},\end{aligned}$$

so that upon iteration

$$\begin{aligned}\psi\left(\delta\right)^\ell &= \left(\sum_{m=0}^{+\infty} \delta^m \overline{P}_q^m\right)\left(\sum_{m=0}^{+\infty} \delta^m \overline{P}_q^{(\ell-1),m}\right) = \sum_{m=0}^{+\infty} \delta^m \sum_{i=0}^{m} \overline{P}_q^{(\ell-1),m-i}\overline{P}_q^i \\ &= \sum_{m=0}^{+\infty} \delta^m \overline{P}_q^{(\ell),m}\end{aligned}$$

with $\overline{P}_q^{(\ell),m}$ defined in the proposition as the m–term of the ℓ–times self–convolution of $\overline{P}_q^m$. This can be recursively calculated via

$$\overline{P}_q^{(\ell),m} = \sum_{i=0}^{\ell} \overline{P}_q^{(\ell-1),m-i}\overline{P}_q^i,$$

with $\overline{P}_q^{(0),m} = \mathbf{1}_{\{m=0\}} = \begin{cases} 1, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}$ by definition.

Substitute this in equation (A.6) to get

$$\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=0}^{+\infty}\delta^{k+\ell+m}C\left(k,\ell\right)\overline{P}_q^{(\ell),m} = K_q.$$

At each order of δ^{n-2} with $n \geq 2$, only the terms $k+\ell+m = n-2$ apply, thus equation (9) in the proposition. The remaining conditions in equations (A.3)–(A.4) can be treated similarly. For future reference, one can write the full problem (for *all* n, and $q \in \{0,1\}$) as

$$\sum_{n=0}^{+\infty}\delta^{n-2}\sum_{j=(n-2)^+}^{n}\mathcal{L}_{n-j}V_q^j\left(p\right) = -R_q\left(p\right) \tag{A.7}$$

$$\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=0}^{+\infty}\delta^{k+\ell+m}C\left(k,\ell\right)\overline{P}_q^{(\ell),m} = K_q \tag{A.8}$$

$$\sum_{k=0}^{+\infty}\sum_{\ell=0}^{+\infty}\sum_{m=0}^{+\infty}\left(\ell+1\right)\delta^{k+\ell+m}C\left(k,\ell+1\right)\overline{P}_q^{(\ell),m} = 0 \tag{A.9}$$

$$\sum_{n-2=0}^{+\infty}\lim_{p\to I_q}\delta^{n-2}V_q^{n-2}\left(p\right) = L_q \tag{A.10}$$

from which it is not difficult to verify that for *each* $n = \{0,1,\ldots\}$, the value functions V_q^n and the price thresholds P_q^n are determined by the sub–problems of (A.7)–(A.10) that are summarised in Proposition 1 of the text.[3]

[3]Note that n defines the order $\mathcal{O}\left(\delta^{n-2}\right)$ of the asymptotic expansion *but also* uniquely determines the values of j,k,ℓ,m and i. The summation index j is simply equal to the non–negative integers in $\{n-2,n-1,n\}$. The non–negative integers k,ℓ and m are uniquely defined by n, as only the permutations of k,ℓ,m for which $k+\ell+m = n-2$ are relevant for the δ^{n-2} order in (A.8)–(A.9). Finally, i in the self–convolution of the price thresholds is uniquely determined by the value of ℓ.

B. Appendix: Definitions for equations (28) and (29)

Recall that $\Delta f(y) = \frac{1}{2}\left(\overline{f}^2 - f^2(y)\right)$, and define $\Theta_q^n(p) = p^2 \frac{\partial^2 V_q^n}{\partial p^2}$ and $\omega = \sqrt{2}\nu\rho$. Then, for all $n > 2$ equation (29) is

$$V_q^n = \Phi_q^{n-2}(p, y) + X_q^{n-2}(p)$$

with $\Phi_q^{n-2}(p, y)$ recursively calculated via

$$\Phi_q^n(p, y) = \mathcal{L}_0^{-1}\left\{\Delta f(y)\,\Theta_q^n(p) + \omega p\left[\Omega_q^n(p) - f(y)\frac{\partial^2 \Phi_q^{n-1}}{\partial p \partial y}\right]\right\},$$

for $q = \{0, 1\}$, with

$$\Omega_q^n(p) = \lceil f(y)\frac{\partial^2 \Phi_q^{n-1}}{\partial p \partial y}\rceil,$$

starting from $\Phi_q^0(p, y) = \mathcal{L}_0^{-1}\left[\Delta f(y)\,\Theta_q^0(p)\right]$ in equation (21) and $\Omega_q^0(p) \equiv 0$.

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