The investment-uncertainty relationship:

A critical discussion

Andreas Welling [a], Elmar Lukas [a,*], Stefan Kupfer [a]

[a] Faculty of Economics and Management, Otto-von-Guericke University Magdeburg, Germany

Abstract:

Only recently the general assumption of a negative investment-uncertainty relationship has been questioned in the literature and some examples of a non-monotonic investment-uncertainty relationship have been found. By analyzing the influence of uncertainty on the probability to invest in a given time we show that the non-monotonicity of this relationship is not an exemption but the rule. Furthermore, we analyze and discuss the strengths and weaknesses of the different interpretations of the investment-uncertainty relationship and verify by an example that the investment-uncertainty relationship may stay ambiguous as its sign critically depends on the interpretation as well as on time.

Keywords: Real Option; Investment; Uncertainty; Timing

* Corresponding author, Faculty of Economics and Management, Chair in Financial Management and Innovation Finance, Otto-von-Guericke-University Magdeburg, Universitätsplatz 2, D-39106 Magdeburg, Tel: +49 (0) 391 67 - 189 34; Fax: +49 (0) 391 67 - 180 07, e-mail address: elmar.lukas@ovgu.de, andreas.welling@ovgu.de, stefan.kupfer@ovgu.de

1. Introduction

The key concept of real options theory is that under uncertainty it pays to wait with an irreversible investment. Over time additional information becomes available and, thus, facilitates a superior investment decision at a later date. Consequently, a decision-maker that exploits this flexibility value will in expectancy generate higher profits (see e.g. (McDonald & Siegel, 1986), (Dixit & Pindyck, 1994)). Therefore, the influence of uncertainty on a company's propensity to invest in a certain project was hardly questioned in the real options literature for some time due to the plausible argument that higher uncertainty leads to a higher value of the option to wait with the investment. Specifically, it was usually taken for granted that higher uncertainty delays investment (see e.g. (Mauer & Ott, 1995), (Metcalf & Hassett, 1995)).

Recently, however, a strain of literature is developing that examines the influence of uncertainty on investment in more detail (see e.g. (Sarkar, 2000),(Lund, 2005) (Wong, 2007), (Gutiérrez, 2007), (Gryglewicz, Huisman, & Kort, 2008), (Lukas & Welling, 2013)). So far, two main results have been obtained. Firstly, under several assumptions it has been proven or it has been shown by example that the influence of uncertainty on the propensity to invest is non-necessary monotonic. Secondly, it has become apparent that the so called investment-uncertainty relationship can be viewed from various angles. In particular, the influence of uncertainty on the propensity to invest in a certain project can be interpreted as the influence of uncertainty on the probability to invest within a given time ((Sarkar, 2000)), as the influence of uncertainty on the expected time until investment ((Wong, 2007)), or the influence of uncertainty on the optimal investment threshold ((Gryglewicz et al., 2008)). (Gutiérrez, 2007) proposes a fourth interpretation of the investment-uncertainty relationship by combining the first two interpretations. Particularly, he measures the propensity to invest as the expected time until investment if the probability to invest in finite time is equal to one; otherwise he measures it as the probability to invest in finite time. Interestingly, the different four interpretations do not always correspond. That is, depending on the angle from which it is viewed the sign of the investment-uncertainty relationship may differ. However, mostly the literature creates the impression as if the non-monotonicity of the investment-uncertainty relationship and the dissent of the three angles is just an infrequent exception while the negative investment-uncertainty relationship usually prevails.[1] Thus, it is not surprising that a significant proportion of today's real options literature still generally assumes a negative investment-uncertainty relationship.[2] In this context, the major results of this paper should encourage to rethink. Firstly, our results show that in a time-continuous real option setting a non-monotonic investment-uncertainty relationship is not an exemption but the rule. Secondly, we show that the investment-uncertainty relationship is very complex and can only be reasonably defined in a context of time.

The rest of the paper is structured as follows. Section 2 provides a literature review on the investment-uncertainty relationship. In Section 3 the differences of the three different angles, its strengths and weaknesses as well as their practical applicability are discussed. In Section 4 we show that in a canonical real options model the investment-uncertainty relationship is generally non-monotonic.

[1] (Sarkar, 2000) and (Lund, 2005) directly state that they only give examples, while (Wong, 2007), (Gryglewicz et al., 2008) and (Lukas & Welling, 2013) only gain their results under several strict model assumptions.

[2] See for example (Whalley, 2011), (Ting, Ewald, & Wang, 2013).

Section 5 discusses the complexity of the investment-uncertainty relationship by the example of a time-continuous real options model. Finally, Section 6 concludes and lays out directions for future research.

2. Literature review

As already mentioned above, real options literature traditionally postulated a negative investment-uncertainty relationship. This view was first questioned by (Sarkar, 2000). With the help of a simple real options model he "demonstrates that the notion of a negative investment-uncertainty relationship is not always correct". Specifically, an investment opportunity is viewed as a perpetual American option. The project value at time t equals $x(t)/\delta(\sigma) - 1$, whereby the earnings $x(t)$ follow a geometric Brownian motion and the rate-of-return shortfall $\delta(\sigma) = r + \lambda\rho\sigma - \mu$ takes systematical risks into account. It is proven that the optimal investment threshold x^* is indeed monotonically increasing with increasing uncertainty σ. However, by using a numerical example it is shown that an increase in uncertainty might speed up investment. In particular, in the given example the probability to invest in a given time is increasing with increasing uncertainty for low values of uncertainty, though, for high values of uncertainty the negative investment-uncertainty relation prevails. (Sarkar, 2000) explains this result by stating that higher uncertainty not only increases the investment threshold but also increases the probability to reach a certain threshold. Consequently, the combined effect of uncertainty on investment consists of two effects a negative and a positive. While the main result of the paper is true, the explanation has been corrected by (Lund, 2005). Notably, the effect of uncertainty on the probability to reach a certain threshold in a given time is not always positive but its sign depends on the values of the other model parameters. On one hand a higher uncertainty gives a higher probability for very high as well as for very low outcomes on the other hand it reduces the probability of outcomes that are close to the expected value.

Furthermore, (Lund, 2005) criticizes two assumptions made by (Sarkar, 2000). Firstly, in (Sarkar, 2000) increasing uncertainty simply means to increase σ while holding all other parameters fixed. This procedure corresponds to the comparative-static analysis which is quite common in real options literature. Besides some general criticism on the concept of a comparative-static-analysis (Lund, 2005) points out that it is not obvious which values in the model are parameters and which are functions depending on σ. For example in the equation $\delta = r + \lambda\rho\sigma - \mu$ instead of the growth rate μ of the geometric Brownian motion also the rate-of-return shortfall δ could be constant in σ (see also (McDonald & Siegel, 1986)).

(Wong, 2007) suggests and analyzes another interpretation of the investment-uncertainty relationship, namely the influence of uncertainty on the expected time until investment, i.e. the expected time to reach the investment threshold. He builds on the model used in (Sarkar, 2000) but differs in one important aspect. Particularly, he assumes that the project value V and not the earnings follow a geometric Brownian motion. His main contribution is the proof that in his setting the expected time until investment always shows a U-shape pattern against uncertainty if $\rho > 0$. For low levels of uncertainty an increase in σ decreases the expected time until investment while for a high level of uncertainty an increase in σ increases the expected time until investment. This is due to two opposing effects. On one hand uncertainty increases the discount rate $r + \lambda\rho\sigma$ and thus makes waiting more costly which leads to an earlier investment while on the other hand higher uncertainty enhances the value of the option to wait with the investment and thus leads to a later investment. While the first effect, the so called return effect, dominates for low values of uncertainty the latter effect, the so called risk effect, dominates for high values of uncertainty. Interestingly, (Wong, 2007) is also able to prove that the influence of uncertainty on the investment threshold has a U-shape for the same reason. Thus, it is a crucial question whether the earnings or the project value are following a geometrical Brownian motion.

(Gryglewicz et al., 2008) give a third interpretation of the investment-uncertainty relationship by focusing on the influence of uncertainty on the investment threshold. Building on the model of (Sarkar, 2000) they merely drop the assumption of an infinite-project life. Thereby, they can prove that the influence of uncertainty on the investment threshold is non-monotonic if the project has a fixed project-life of $T < \infty$. In particular, the investment-uncertainty relationship is U-shaped, i.e., the investment-threshold is decreasing with uncertainty if the level of uncertainty is low while it is increasing with uncertainty while the level of uncertainty is high. This is due to three effects that act in combination, namely, the discount effect, the volatility effect and the convenience yield effect. Firstly, increasing uncertainty raises the discount rate $r + \lambda\rho\sigma$, thereby erodes the value of future cash flows and, thus, leads to a higher investment threshold. Secondly, higher uncertainty increases the value of the option to wait and, thus, leads to a higher investment threshold. Thirdly, a higher discount rate $r + \lambda\rho\sigma$ also mitigates the value of waiting and, thus, leads to a lower investment threshold. Under the assumption that the project-life is finite it is shown that the convenience-yield-effect dominates the discount-effect and the volatility-effect if the level of uncertainty is low while the discount-effect and the volatility-effect dominate the convenience-yield-effect if the level of uncertainty is high. However, in accordance with the result of (Sarkar, 2000) the discount-effect and the volatility-effect always dominate the convenience-yield-effect if the project life is infinite. While the volatility-effect and the convenience-yield-effect exactly match with the risk-effect and return-effect of (Wong, 2007), respectively, the discount-effect is not observed by (Wong, 2007). Technically, he does not have to consider that future cash flows would have to be discounted properly, because he assumes that instead of the future cash flow it is directly the project value that follows a geometric Brownian motion. In several variations of their model (Gryglewicz et al., 2008) furthermore show that their results still hold if either the length of the project life is assumed to be stochastic or $\delta(\sigma)$ is assumed to be non-linear but concave or that the future cash flows follow a mean-reverted process instead of a geometric-Brownian motion. If additionally to the finite-project life a finite option-life is considered the results vary substantially. In particular, under a finite option-life the investment threshold $x^*(t)$ depends on the remaining life of the option. Specifically, it decreases as the option-life approaches its end. Interestingly, some curves $x^*(t)$ for different values of uncertainty intersect. Thus, the influence of uncertainty on the investment threshold also depends on time. In the given example the investment-uncertainty relationship shows a U-shape pattern if the remaining option-life is high, while even a strictly positive investment-uncertainty relationship is observed if the option life is approaching its end.

(Gutiérrez, 2007) proposes to interpret the investment-uncertainty relationship as the influence of uncertainty on the expected time until investment if the probability to invest in finite time is equal to one and to interpret the investment-uncertainty relationship as the influence of uncertainty on the probability to invest within finite time if this probability is strictly lower than one. He is able to show according to this interpretation a negative investment-uncertainty relationship can only be observed in the canonical real option model if the growth rate μ of the earnings $x(t)$ is lower than zero.

(Lukas & Welling, 2013) deviate from the model of (Gryglewicz et al., 2008) by assuming that the timing of the investment is the outcome of a sequential bargaining process of two parties that both have to bear part of the investment costs. While one party offers the other party a fraction of the surplus of the future investment, the other party decides on the timing of the investment, whereby it already takes the offered fraction into account.[3] In contrast to (Gryglewicz et al., 2008) a U-shaped influence of uncertainty on the investment threshold also prevails in the game-theoretic setting for an infinite project-length. Likewise, it is shown that the influence of uncertainty on the expected time until investment is U-shaped, too. Due to the sequential bargaining a fourth effect of uncertainty on the investment threshold arises which is called the bargaining effect. Particularly, the bargaining effect deters the overall effect of the three effects discussed in (Gryglewicz et al., 2008) in disadvantage of the convenience yield effect. With the help of a numerical example (Lukas & Welling, 2013) moreover show that in their game-theoretical setting uncertainty has also a non-monotonic influence on the probability to invest within a given time.

[3] This setting is similar to the hostile takeover setting in (Lambrecht, 2004).

3. A comparison of the different interpretations of the investment-uncertainty relationship

In this section we analyze how suitable the four different interpretations of the investment-uncertainty relationship identified in the literature are in answering the question if rising uncertainty accelerates or delays investment. Hereby, the term investment-uncertainty relationship is restricted to the influence of uncertainty on the propensity to invest a lump sum in a single project whereby uncertainty can only be increased in exactly one way. In particular, we examine for each of the four interpretations if it

1. allows measuring the propensity to invest
2. allows determining the sign of the investment-uncertainty relationship
3. allows calculating the partial derivative of the propensity to invest with respect to uncertainty

for every given amount of uncertainty.

INSERT TABLE 1 HERE

Table 1 briefly depicts the findings of this section. Thereby, + means that the interpretation of the investment-uncertainty relationship fully meets the criteria, ⊙ means that the interpretation of the investment-uncertainty relationship meets the criteria at least in the canonical real option model, and – means that the interpretation of the investment-uncertainty relationship does not meet the criteria. Summarized, the findings reveal that the investment-uncertainty relationship should not be interpreted as the influence of uncertainty on the investment threshold. Noteworthy, (Gryglewicz et al., 2008) who introduced this interpretation stated that they mainly use it as it is easier to analyze mathematically. The expected time until investment is a reasonable measure of the propensity to invest but only allows determining the sign of the investment uncertainty relationship as long as it is not infinite. Combining the expected time until investment with the probability to invest within finite time as proposed by (Gutiérrez, 2007) allows determining the sign of the investment uncertainty relationship. However, the partial derivative of the propensity to invest with respect to uncertainty is not well defined. The only possible interpretation of the investment-uncertainty relationship that allows calculating the partial derivative of the propensity to invest with respect to uncertainty for every given amount of uncertainty is the probability to invest in a given time τ. Though, in this interpretation the sign of the investment-uncertainty relationship crucially depends on the parameter τ. The detailed analysis is given below in subsections 3.2-3.5 but at first subsection 3.1 gives a definition of the canonical real option model as discussed in (Sarkar, 2000).

3.1 The canonical real option model

The canonical real options model considers a company that at time t_0 can invest in a project which generates a cash flow $x(t)$ per unit of time. The investments costs are $I > 0$ and the cash flow $x(t)$ evolves stochastically over time and follows the geometric Brownian motion (GBM)

$$dx(t) = \mu x(t)dt + \sigma x(t)dW(t), \qquad x(0) = x_0 \geq 0, \tag{1}$$

whereby $\mu \in \mathbb{R}$ is the drift rate, $\sigma > 0$ is the uncertainty parameter and $dW(t)$ is the increment of a Wiener process with zero mean and variance equal to one. It is assumed that the company is using a discount factor following the CAPM formula, i.e., that the project is correlated with the market by factor $\rho \in \mathbb{R}$, the market price of risk is $\lambda \geq 0$ and the risk-free interest rate equals $r \geq 0$.[4] Therefore, the difference between the expected return of the project and the drift rate of the GBM, the so called convenience yield, is given by

$$\delta = r - \mu + \lambda\rho\sigma. \tag{2}$$

As the company does not have to invest immediately but can wait with the investment, the possibility to invest represents a real option. More precisely, it is similar to a perpetual American call. At time t of the investment the company gets

$$\pi(t) = -I + \frac{x(t)}{r - \mu + \lambda\rho\sigma} \tag{3}$$

It can be easily obtained (see also (Dixit & Pindyck, 1994), (Sarkar, 2000)) that the optimal investment time t^*equals

$$t^* = inf\{t \geq t_0 | x(t) \geq x^*\}, \tag{4}$$

i.e. it is the first time the cash flow reaches the optimal investment threshold x^* defined by

$$x^* = \frac{\beta}{\beta - 1}(r + \lambda\rho\sigma - \mu)I, \tag{5}$$

with

$$\beta = \frac{1}{2} - \frac{\mu - \lambda\rho\sigma}{\sigma^2} + \sqrt{\left(\frac{1}{2} - \frac{\mu - \lambda\rho\sigma}{\sigma^2}\right)^2 + \frac{2r}{\sigma^2}}. \tag{6}$$

In the canonical real options model the expected time until investment equals[5]

---

[4] See (Merton, 1973)

[5] See (Wong, 2007)

$$\mathbb{E}t^* = \begin{cases} 0 & x^* \le x_0 \\ \dfrac{ln\left(\frac{x^*}{x_0}\right)}{\mu - \frac{1}{2}\sigma^2} & x^* > x_0 \ \wedge \ \mu - \frac{1}{2}\sigma^2 > 0 \\ \infty & x^* > x_0 \ \wedge \ \mu - \frac{1}{2}\sigma^2 \le 0 \end{cases} \tag{7}$$

and the probability to invest in a given time $\tau > 0$ equals

$$\begin{aligned} \mathbb{P}(t^* \le t_0 + \tau) = max\Bigg\{1, \Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) \\ + \left(\frac{x^*(\sigma)}{x_0}\right)^{\frac{\mu}{\sigma^2}-1} \Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right) - \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)\Bigg\}, \end{aligned} \tag{8}$$

whereby $\Phi()$ is the area under the standard normal distribution.[6] As can be seen in equation (8) this formula is not simple even in the canonical real options case. Furthermore, it has no analytical solution as the area under the standard normal distribution can only be determined numerically. The probability to invest at any point in time can be calculated by

$$\mathbb{P}(t^* < \infty) = \lim_{\tau \to \infty} \mathbb{P}(t^* \le t_0 + \tau). \tag{9}$$

3.2 The influence of uncertainty on the probability to invest in a given time

If the probability to invest in a given time $\tau > 0$ increases with uncertainty it can be concluded that rising uncertainty accelerates investment. From the perspective of probability theory investment is an event that can only take place once, thus the probability that this event occurs before the expiry of a time-period of length τ is well defined. For this reason and as it always lies in the closed interval from zero to one the probability to invest in a given time $\tau > 0$ always exists and is

[6] See also (Sarkar, 2000), (Harrison, 1985), p.11-15.

necessarily quantifiable. Thus, this interpretation allows measuring the propensity to invest for every given amount of uncertainty, allows determining the sign of the investment-uncertainty relationship for every given amount of uncertainty and as the probability is scaled cardinally allows calculating the partial derivative of the propensity to invest with respect to uncertainty for every given amount of uncertainty.

However, the influence of uncertainty on the probability to invest in a given time $\tau > 0$ depends on an additional parameter, notably τ. This fact would be unproblematic if the sign of the investment-uncertainly relationship would not depend on τ. Though, it can be easily verified that in every non-trivial real options model the shape of the investment-uncertainty relationship critically depends on the time τ. In particular, we consider a decision-maker that discounts with the riskless interest rate $r > 0$ and at beginning in time t_0 has the possibility to invest in a specific project. The investment costs are $I > 0$ and $x(t)$ denotes the cash flow of the project for all $t \geq t_0$. As the decision-maker can wait with the investment the investment possibility represents a real option. In absence of uncertainty it is optimal for the decision-maker to invest at $\bar{t}^*$ which is defined by the equation

$$\begin{aligned} & e^{-r\bar{t}^*}\left(\int_{\bar{t}^*}^{\infty} x(s)e^{-r(s-\bar{t}^*)}ds - I\right) \\ & \qquad = \max_{t\in[t_0,\infty]}\left\{e^{-rt}\left(\int_{t}^{\infty} x(s)e^{-r(s-t)}ds - I\right)\right\} \end{aligned} \tag{10}$$

Under the assumption $\bar{t}^* > t_0$, i.e. the real option is non-trivial, the probability to invest before a given time $\tau > t_0$ is equal to zero for all $\tau < \bar{t}^*$ and equal to one for all $\tau > \bar{t}^*$. Now, we assume that the cash flows are uncertain, i.e. they follow a certain stochastic process. From real options literature we know that under uncertainty it is optimal to invest as soon as $x(t)$ reaches a certain threshold which generally is time-dependent. Thus, under uncertainty the optimal investment time equals

$$t^* = inf\{t \geq t_0 | x(t) \geq x^*(t)\}. \tag{11}$$

The probability to invest in a given time of $\tau > 0$ can be written as $\mathbb{P}(t^* \leq t_0 + \tau)$. Necessarily it is $0 \leq \mathbb{P}(t^* \leq t_0 + \tau) \leq 1$ for all $\tau > 0$. Hence, the presence of any uncertainty about the cash flows can only increase the probability to invest before a given time $\tau < \bar{t}^*$ while it can only decrease the probability to invest before a given time $\tau > \bar{t}^*$. A more detailed analysis of this phenomenon with respect to the canonical real option model is given in section 4.

3.3 The influence of uncertainty on the expected time until investment

If the expected time until investment increases with uncertainty it can be concluded that uncertainty delays investment. If we allow infinity as a possible value the expected time until investment is well defined from the perspective of probability theory. Thus, this interpretation allows measuring the propensity to invest for every given amount of uncertainty. However, if the expected time until investment is infinite it does not facilitate quantifiable statements. More specially, it can be stated that this interpretation of the investment-uncertainty relationship is not applicable if a positive probability exists that the investment will never take place. For example in the canonical real option model the influence of increasing uncertainty on the propensity to invest cannot be analyzed for higher uncertainties, i.e. for $\sigma > \sqrt{2\mu}$. Especially, for negative drift rates μ the interpretation of the investment-uncertainty relationship as the influence of uncertainty on the expected time until investment is completely useless. Thus, the interpretation does not allow determining the sign of the investment-uncertainty relationship for every amount of uncertainty and thus does not allow calculating the partial derivative of the propensity to invest with respect to uncertainty for every amount of uncertainty, either. However, an advantage of the interpretation is that it does not depend on additional parameters.

3.4 The influence of uncertainty on the investment trigger

If increasing uncertainty increases the investment trigger this does not directly impact investment timing. Particularly, $x^*(\sigma)$ does not measure time, instead, it can only indirectly used to measure time, for example with the help of equation (7), (8) or (9). However, even in the canonical real option model the probability to invest in a given time τ as well as the expected time until investment do not only depend indirectly on uncertainty via the investment threshold x^* but also on uncertainty directly. Therefore, we cannot state if increasing uncertainty delays investment if we just know its impact on the investment threshold. Thus, the interpretation does not allow measuring the propensity to invest. Hence, it does not allow determining the sign of the investment-uncertainty relationship or calculating the partial derivative of the propensity to invest with respect to uncertainty, either. Furthermore, the investment threshold generally depends on time, i.e. $x^*(t)$. This would not be problematic if the sign of the investment-uncertainty relationship would be the same for all t. However, as has already be stated above, it has be shown by (Gryglewicz et al., 2008) that this is not always the case. However, as we can see in equation (5) the investment threshold does not depend on an additional parameter in the canonical real option case.

3.5 The influence of uncertainty on the expected time until investment or the probability to invest in finite time, respectively

If increasing uncertainty decreases the expected time until investment or if it increases the probability to invest in finite time, it can be concluded that uncertainty accelerates investment. Furthermore, the probability to invest in finite time always exists and is unique. The same applies for the expected time until investment. Thus, the interpretation allows measuring the propensity to invest for every given amount of uncertainty. Like the expected time until investment also the interpretation of (Gutiérrez, 2007) does not always have a numerical value. For example, in the canonical real options model the probability to invest equals one if $\sigma = \sqrt{2\mu}$ while simultaneously the expected time until investment is infinite. Though, the influence of uncertainty on the propensity to invest is still determinable in this case. In particular, if uncertainty is slightly lower than $\sqrt{2\mu}$ the expected time until investment is finite while if uncertainty is slightly higher than $\sqrt{2\mu}$ the probability to invest in finite time is a little bit lower than one. Thus, it becomes apparent that in this case increasing uncertainty lowers the propensity to invest. Thus, the interpretation allows determining the sign of the investment uncertainty relationship for every given amount of uncertainty. Thereby it does not depend on any additional parameter. However, the interpretation measures the influence of uncertainty on the propensity to invest in two different units: The expected time until investment is measured in time periods, while the probability to invest in finite time is dimensionless. Furthermore, the propensity to invest is the lower the higher the expected time until investment while it is the higher the higher the probability to invest in finite time. As a consequence, the partial derivation of the propensity to invest with respect to uncertainty is not well defined. Particularly, the partial derivative does not exist if the probability to invest in finite time equals one and the expected time until investment is infinite as it is the case in the canonical real option model for $\sigma = \sqrt{2\mu}$. Hence, using this interpretation the partial derivative of the propensity to invest with respect to uncertainty could at best be defined piecewise.

4. The non-monotonicity of the investment-uncertainty relationship in the canonical real options model

As has been shown by (Sarkar, 2000) with the help of a numerical example in the canonical real option model the probability to invest in a given time τ is not always monotonically decreasing with uncertainty σ. In the following we proof that this is not an exemption but the rule. Thereby, we always assume that the canonical real options model is non-trivial, i.e. $x_0 < x^*_{\sigma=0}$. Otherwise, under certainty instantaneous investment would always be optimal. First, we determine $\lim_{\sigma\downarrow 0} \mathbb{P}(t^*(\sigma) \leq t_0 + \tau)$. We have

$$\lim_{\sigma\downarrow 0} \Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right) + \left(\mu - \frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) = \lim_{\sigma\downarrow 0} \Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right) + \mu\tau}{\sigma\sqrt{\tau}}\right)$$
$$= \begin{cases} \Phi(\infty) & \tau > -\frac{1}{\mu} ln\left(\frac{x_0}{x^*(0)}\right) \\ \Phi(-\infty) & \tau < -\frac{1}{\mu} ln\left(\frac{x_0}{x^*(0)}\right) \end{cases} = \begin{cases} 1 & \tau > -\frac{1}{\mu} ln\left(\frac{x_0}{x^*(0)}\right) \\ 0 & \tau < -\frac{1}{\mu} ln\left(\frac{x_0}{x^*(0)}\right) \end{cases} \quad (12)$$

and for $\mu \leq 0$ it is obviously

$$\lim_{\sigma\downarrow 0}\left(\frac{x^*(\sigma)}{x_0}\right)^{\frac{\mu}{\sigma^2}-1}\Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)=0. \tag{13}$$

However, if $\mu > 0$ it is according to l'Hôspital's rule

$$\lim_{\sigma\downarrow 0}\left(\frac{x^*(\sigma)}{x_0}\right)^{\frac{\mu}{\sigma^2}-1}\Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)$$

$$=\lim_{\sigma\downarrow 0}\frac{\frac{\partial}{\partial\sigma}\Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)}{\frac{\partial}{\partial\sigma}\left(\frac{x^*(\sigma)}{x_0}\right)^{-\frac{\mu}{\sigma^2}+1}}=0. \tag{14}$$

Thus, finally we get

$$\lim_{\sigma\downarrow 0}\mathbb{P}(t^*(\sigma)\le t_0+\tau)=\begin{cases}1 & \tau>-\frac{1}{\mu}ln\left(\frac{x_0}{x^*(\sigma)}\right)\\ 0 & \tau<-\frac{1}{\mu}ln\left(\frac{x_0}{x^*(\sigma)}\right)\end{cases}. \tag{15}$$

Now, we determine $\lim_{\sigma\to\infty}\mathbb{P}(t^*(\sigma)\le t_0+\tau)$. We have

$$\lim_{\sigma\to\infty}\left(\frac{x^*(\sigma)}{x_0}\right)^{\frac{\mu}{\sigma^2}-1}=\lim_{\sigma\to\infty}\left(\frac{x^*(\sigma)}{x_0}\right)^{-1}=0 \tag{16}$$

and

$$\lim_{\sigma\to\infty}\Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right)=\Phi(-\infty)=0. \tag{17}$$

Thus, we get

$$\lim_{\sigma\to\infty}\mathbb{P}(t^*(\sigma)\le t_0+\tau)=0. \tag{18}$$

In equation (8) it is further easy to see that $\mathbb{P}(t^*(\sigma)\le t_0+\tau)>0\ \forall\tau,\sigma\in(0,\infty)$.

Especially, it is $\left(\frac{x^*(\sigma)}{x_0}\right)^{\frac{\mu}{\sigma^2}-1}>0$ and $\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}},\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\in\mathbb{R}$

implies $\Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)+\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right), \Phi\left(\frac{ln\left(\frac{x_0}{x^*(\sigma)}\right)-\left(\mu-\frac{1}{2}\sigma^2\right)\tau}{\sigma\sqrt{\tau}}\right) > 0$. Finally, we can deduce from equation (8) that $\mathbb{P}(t^*(\sigma) \leq t_0 + \tau)$ is continuous in σ. Hence, we can state the following proposition with the help of the intermediate value theorem:

Proposition 1: *In a non-trivial canonical real options model it is always possible to find a* $\tau > 0$ *and uncertainty values* $0 < \sigma_1 < \sigma_2 < \sigma_3$ *so that* $\mathbb{P}(t^*(\sigma_1) \leq t_0 + \tau) < \mathbb{P}(t^*(\sigma_2) \leq t_0 + \tau)$ *and* $\mathbb{P}(t^*(\sigma_2) \leq t_0 + \tau) > \mathbb{P}(t^*(\sigma_3) \leq t_0 + \tau)$.

Thus, we can in every non-trivial canonical real options model choose such a τ that the investment-uncertainty relationship is non-monotonic, if we interpret the investment-uncertainty relationship as the influence of uncertainty on the probability to invest in a given time τ. In particular, the non-monotonicity can also be obtained if $\lambda\rho = 0$.

So far, we have seen that the investment-uncertainty relationship crucially depends on τ. But it also strongly depends on the values of the other parameters. Notably, only slight variations can have a big impact on the shape of the curve. For example (Lund, 2005) shows that only a minor increase in the growth rate μ can make the difference between a strictly negative and a non-monotonic investment-uncertainty relationship. In regard of the initial value x_0 of the stochastic process, (Sarkar, 2000) contrarily states "that using a different value of x_0 will result in different Prob(Invest), but will make no difference to the relationship between σ and Prob(Invest), which is what we are interested in." However, our results depict that the influence of x_0 is of importance for the sign of the investment-uncertainty relationship. An illustration is given by Figure 1, where the exact values of (Sarkar, 2000) are used and the initial value x_0 is varied ceteris paribus. As can be seen the initial value dramatically affects the investment-uncertainty relationship.

INSERT FIGURE 1 HERE

---

5. The complexity of the investment-uncertainty relationship

In the following we will illustrate the complexity of the investment-uncertainty relationship by showing that all of the four interpretations discussed before are either not applicable or fail to show the full picture of the investment-uncertainty relationship. Therefore, we differ only slightly from the canonical real options model by two variations which are analogous to subsection 4.1 in (Gryglewicz et al., 2008). Particularly, the project-life is assumed to have a finite length $T_P > 0$, i.e. it will only generate profits for a specific period of time, and the option-life is assumed to have a finite length of $T_O > 0$, i.e. the company has the possibility to invest at every point of time in the time interval $[t_0, t_0 + T_O]$. If the company invests at a time $t \in [t_0, t_0 + T_O]$, therefore, it expects discounted cash flows of

$$\begin{aligned} V(x(t), t) &= \mathbb{E} \int_{t}^{t+T_P} x(s) e^{-(r-\mu+\lambda\rho\sigma)(s-t)}\, ds \\ &= \frac{x(t)}{r-\mu+\lambda\rho\sigma} \left(1 - e^{(r-\mu+\lambda\rho\sigma)T_P}\right) \end{aligned} \tag{19}$$

and will thus generate an expected profit of

$$\pi(x(t), t) = V(x(t), t) - I. \tag{20}$$

Hence, the company should wait with the investment until the cash flows reach the time-depending optimal investment threshold $x^*(t)$ which maximizes its expected profit. This is an optimal problem stopping problem for which the optimal investment time is determined by

$$t^* = inf\{t \geq t_0 | x(t) \geq x^*(t)\}. \tag{21}$$

Applying Ito's Lemma to equation (7) it can be proved that the option value of the investment opportunity $F(x(t), t)$ must follow the differential equation

$$\frac{1}{2}x^2\sigma^2\frac{\partial^2 F(x,t)}{\partial^2 x} + (\mu - \lambda\rho\sigma)x\frac{\partial F(x,t)}{\partial x} + \frac{\partial F(x,t)}{\partial t} - rF(x,t) = 0. \tag{22}$$

The differential equation can be solved using the following conditions: As zero is an absorbing boundary of the process $x(t)$ the boundary condition

$$F(0,t) = 0 \tag{23}$$

must hold. The value-matching condition

$$F(x^*(t),t) = \pi(x^*(t),t) = V(x^*(t),t) - I \tag{24}$$

ensures that upon optimal investment timing the value of the option is equal to its intrinsic value. The last condition for an optimal exercise is the smooth-pasting condition

$$\frac{\partial F(x^*(t),t)}{\partial x} = \frac{\partial \pi(x^*(t),t)}{\partial x} \tag{25}$$

as given by Dixit and Pindyck (1994). Additionally,

$$F(x(t),t) = 0 \; \forall t \geq t_0 + T_O \tag{26}$$

as the investment opportunity vanishes at time $t_0 + T_O$.

To discuss the investment-uncertainty relationship for this problem the model is solved numerically. In particular, the partial differential equations are solved by means of explicit finite differences. If not stated otherwise we assume the same values as (Gryglewicz et al., 2008), i.e. $\mu = 0.08$; $\sigma = 0.2$; $r = 0.1$; $\rho = 0.7$; $\lambda = 0.4$; $\tau = 5$; $T_P = 10$; $T_O = 10$; $I = 10$; $x_0 = 1$.

INSERT FIGURE 2 HERE

We begin the analysis of the investment-uncertainty relationship with the discussion of the optimal investment threshold $x^*(t)$ which is depicted in Figure 2 for varying σ. As expected and as in (Gryglewicz et al., 2008) the threshold is time-dependent. The threshold is monotonically decreasing in time, i.e. it is the lower the lower the remaining option life $T_O - t$. This behavior is well known from American options as the decreasing option life leads to a decreasing value of the option to wait. Although the characteristic sustains for all values of uncertainty the influence of uncertainty on the optimal investment threshold for a certain point of time is more complex. In the canonical real option model higher uncertainty does increase the threshold. However, in the given model the effect of uncertainty has a U-shaped pattern. Both low and high values can lead to higher investment thresholds compared to average values. In e.g. $t = 0$ the lowest threshold is observed for $\sigma = 0.2$ and both an increase and decrease of σ lead to higher optimal investment thresholds. (Gryglewicz et al., 2008) conclude that “the finite-life option assumption neither mitigates nor augments the positive relationship between investment and uncertainty due to the decreasing trigger.” However, as discussed above the optimal investment threshold does not measure timing and is only one step to discuss the investment decision. Essentially, we cannot make any inferences from the influence of uncertainty on the investment threshold on the influence of uncertainty on investment.

The investment-uncertainty relationship could be discussed from the point of view of the expected time until investment. However, under uncertainty the finite-project life might lead to trajectories of the stochastic cash flows which never end in the money despite of the decreasing investment threshold as the cash flow ends after T_p years and the option can only be exercised within T_O years. This implies a true probability that the company will never invest. Thus, in the given model the expected time until investment is not quantifiable and not a reliable source to examine the investment-uncertainty relation.

INSERT FIGURE 3 HERE

The probability to invest in a given time is the second interpretation which could be used to discuss the investment-uncertainty relationship. As can be seen in Figure 3 the influence of uncertainty on the probability to invest in a given time is non-monotonic and varying for different points of time. For low values of uncertainty the probability to invest is increasing except for $\tau = 10$ while it is decreasing for high values of uncertainty. We will discuss $\tau = 10$ as a special case later and focus first on the common characteristics of the other points in time. The relationship of uncertainty and the probability to invest could be explained by the characteristics of the investment threshold and the possible trajectories for varying uncertainty as discussed above. For low values of uncertainty the investment threshold is high and due to the low uncertainty only a low if any probability exists that the cash-flow will reach this threshold within the time τ. If uncertainty is increasing the investment threshold is decreasing. Enhanced by this effect increasing uncertainty thus is increasing the probability that the cash-flow will reach the investment threshold within the time τ. However, if the uncertainty is substantially high increasing uncertainty leads to an increasing investment threshold. Thus, for every given time τ there exists an uncertainty value from which on the effect of the increasing threshold outweighs the effect of the higher chance to reach the threshold due to the increasing uncertainty. Thus, the probability that the threshold is reached within the time τ is decreasing with uncertainty for higher values of uncertainty. This observed U-shape pattern is in line with the findings of (Sarkar, 2000) for special values of τ and x_0.

The characteristics of the influence of uncertainty on the probability to invest for $\tau = 10$ need to be discussed somewhat separately. As there will not be any cash flows after $\tau = 10$ the company obviously will not invest after $\tau = 10$ years. Hence, the probability that the investment takes place before $\tau = 10$ is equal to the probability that the investment takes place within finite time. As can be seen in Figure 3 the probability to invest before $\tau = 10$ is almost equal to one for low values of uncertainty, then starts to decline slightly before a more pronounced decline of the probability to invest can be observed.[7] The reason for the special characteristics might again be found considering the interaction of optimal investment threshold and the chance to reach the threshold for given uncertainty. For low value of sigma the investment threshold drops dramatically as the option to wait loses value before the end of the option life. (Gryglewicz et al., 2008) argue that the drop is caused by the low convenience yield for lower uncertainty which causes little profits when the option is exercised early. If the drop is pronounced enough as in our model the low uncertainty is enough for the cash flow to reach the new investment threshold before or at $\tau = 10$. For higher values of uncertainty the probability to invest follows the already discussed pattern. Overall, the model demonstrates that the effect of uncertainty on the probability to invest is non-monotonic and that it depends on the level of uncertainty σ and on

[7] It should be noted that this fast decline in probability is not a drop or jump but continuous.

the time τ.[8] And now the final question: Does increasing uncertainty accelerate investment in the given real option model? Obviously, there is no clear answer possible with any of the interpretations related to the investment-uncertainty relationship. The sign of the investment-uncertainty relationship stays ambiguous. In particular, we observe a difference between the influence of uncertainty on the propensity to invest in short term and the influence of uncertainty on the propensity to invest in long term. Though, not depending on an additional parameter the interpretation of the investment-uncertainty relationship proposed by (Gutiérrez, 2007) implicitly only takes the long-term perspective into account and thereby totally ignores the short-term perspective of the investment-uncertainty relationship.

6. Conclusion

The investment-uncertainty relationship, i.e. the influence of uncertainty on the propensity to invest is of great interest for policy makers. While real options literature mostly predicts a negative investment-uncertainty relationship recent literature has given various examples of a non-monotonic investment-uncertainty relationship. In this paper we have shown that the non-monotonicity is not an exemption but the rule even in the canonical real options model. However, this result does not doubt the main implication of real options theory. Under uncertainty the possibility to wait with an investment has a flexibility value and this flexibility value encourages delaying an investment especially if compared with the net present value decision rule. Though, in future research the

[8] As discussed above the probability to invest is also influenced by x_0.

formulation that increasing uncertainty generally delays investment should be better avoided if the investment-uncertainty relationship is not conscientiously analyzed in the considered situation. Particularly, this includes outlining explicitly how the propensity to invest is measured.

Furthermore, we analyzed the strengths and weaknesses of the four different interpretations of the investment-uncertainty relationship that have been used in the literature. Our findings reveal that the influence of uncertainty on the investment trigger is not a particularly suitable measure of the investment-uncertainty relationship as it does not measure investment timing directly. Though a reasonable measure, the second alternative, i.e. the influence of uncertainty on the expected time until investment is unusable in many cases. The third interpretation, i.e. the influence of uncertainty on the probability to invest always allows to determine the sign of the investment-uncertainty relationship and to calculate the partial derivative of the propensity to invest with respect to uncertainty. However, the interpretation depends on an additional time parameter whose influence on the sign of the investment-uncertainty relationship is crucial. The fourth interpretation, i.e. the influence of uncertainty on the expected time until investment if the probability to invest within finite time is equal to one or the influence of uncertainty on the probability to invest within finite time if this probability is lower than one, allows always determining the sign of the investment-uncertainty relationship. Though, the partial derivative of the propensity to invest with respect to uncertainty is not well defined. By the example of a certain real options model we furthermore demonstrated that sometimes it is not possible to simultaneously achieve clear results about the sign of the investment-uncertainty relationship and to get the full picture of this relationship with any of the four interpretations due to the critical role of time in the context. Thus, future research should avoid relying on only one of the four interpretations of the investment-uncertainty relationship but discussing this relationship from the view of several interpretations.

7. References

Dixit, A. K., & Pindyck, R. S. (1994). *Investment under Uncertainty*. Princeton: Princeton University Press.

Gryglewicz, S., Huisman, K. J. M., & Kort, P. M. (2008). Finite project life and uncertainty effects on investment. *Journal of Economic Dynamics and Control*, *32*(7), 2191–2213.

Gutiérrez, Ó. (2007). Devaluating projects and the investment–uncertainty relationship. *Journal of Economic Dynamics and Control*, *31*(12), 3881–3888.

Harrison, J. M. (1985). *Brownian Motion and Stochastic Flows*. New York: John Wiley & Sons.

Lambrecht, B. M. (2004). The timing and terms of mergers motivated by economies of scale. *Journal of Financial Economics*, *72*(1), 41–62.

Lukas, E., & Welling, A. (2013). On the Investment-Uncertainty Relationship: A Game Theoretic Real Option Approach. *Finance Research Letters*, *forthcoming*.

Lund, D. (2005). How to analyze the investment–uncertainty relationship in real option models? *Review of Financial Economics*, *14*(3-4), 311–322.

Mauer, D. C., & Ott, S. H. (1995). Investment under Uncertainty: The Case of Replacement Investment Decisions. *Journal of Financial and Quantitative Analysis*, *30*(4), 581–605.

McDonald, R. L., & Siegel, D. R. (1986). The value of waiting to invest. *Quarterly Journal of Economics*, *101*(4), 707–727.

Merton, R. C. (1973). An Intertemporal Capital Asset Pricing Model. *Econometrica*, *41*(5), 867–887.

Metcalf, G. E., & Hassett, K. A. (1995). Investment under alternative return assumptions. Comparing random walks and mean reversion. *Journal of Economic Dynamics and Control*, *19*, 1471–1488.

Sarkar, S. (2000). On the Investment-Uncertainty Relationship in a Real Options Model. *Journal of Economic Dynamics and Control*, *24*(2), 219–225.

Ting, S. H. M., Ewald, C.-O., & Wang, W.-K. (2013). On the investment–uncertainty relationship in a real option model with stochastic volatility. *Mathematical Social Sciences*, *66*(1), 22–32.

Whalley, E. A. (2011). Optimal R&D investment for a risk-averse entrepreneur. *Journal of Economic Dynamics and Control*, *35*(4), 413–429.

Wong, K. P. (2007). The effect of uncertainty on investment timing in a real options model. *Journal of Economic Dynamics and Control*, *31*(7), 2152–2167.

Table 1:

Criteria	Probability to invest within time τ	Expected time until investment	Investment threshold	Gutierrez (2007)
1: Allows measuring the propensity to invest	+	+	-	+
2: Allows determining the sign of the investment-uncertainty relationship	+	−	−	+
3: Allows calculating the partial derivative of the propensity to invest with respect to uncertainty	+	-	−	−
4: Does not depend on any additional parameter	-	+	⊙	+

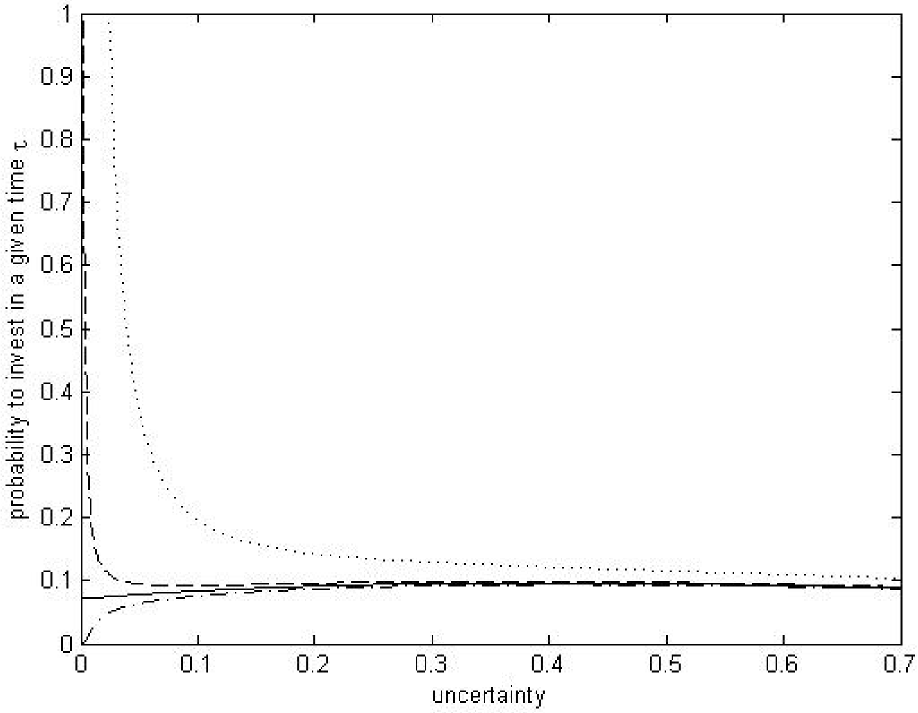

Figure 1: The probability to invest in a given time $\tau = 5$ in dependence of uncertainty for different x_0: dotted line: $x_0 = 0.11$; dash line: $x_0 = 0.101$; solid line: $x_0 = 0.1$; dash-dotted line: $x_0 = 0.099$.

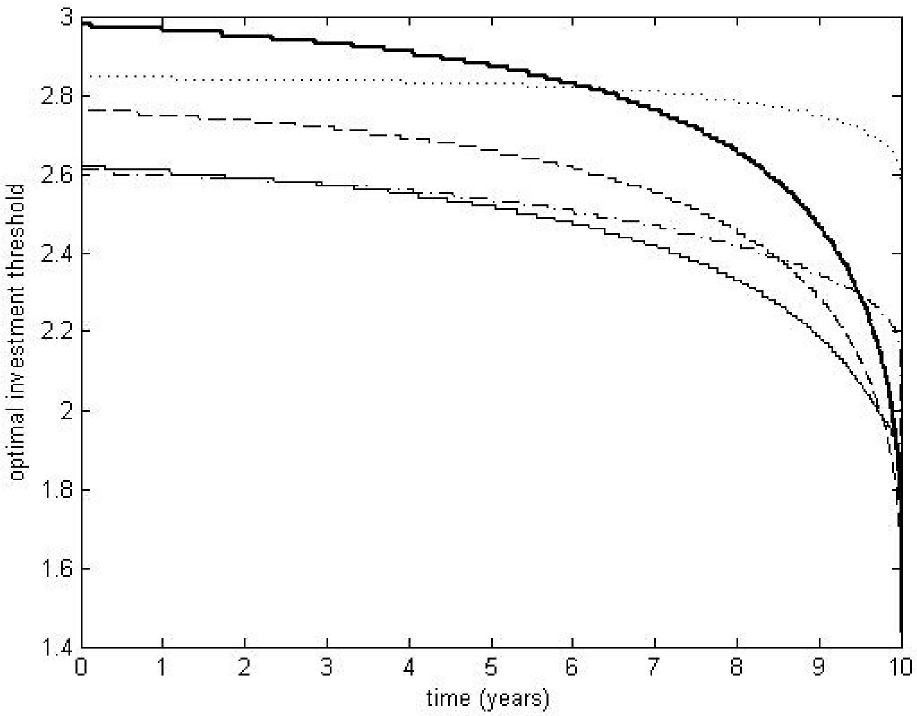

Figure 2: The optimal investment threshold in dependence of time for different degrees of uncertainty: fat solid line: $\sigma = 0.3$; dotted line: $\sigma = 0.1$; dashed line: $\sigma = 0.25$; solid line: $\sigma = 0.2$; dash-dotted line: $\sigma = 0.15$.

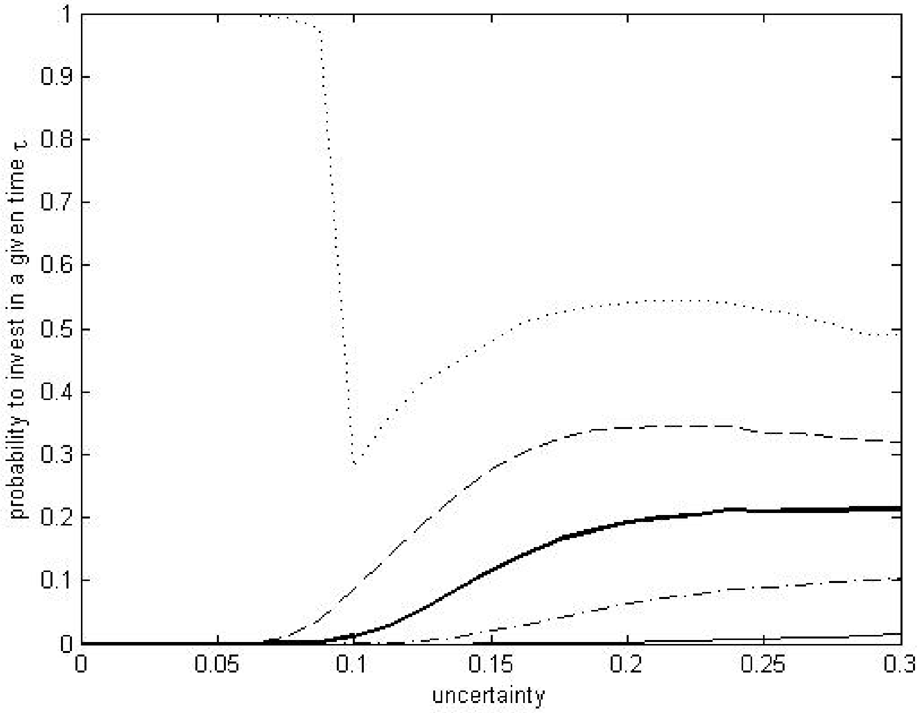

Figure 3: The probability to invest in a given time τ in dependence of uncertainty for different τ: dotted line: $\tau = 10$; dash line: $\tau = 8$; fat solid line: $\tau = 6$; dash-dotted line: $\tau = 4$; solid line: $\tau = 2$.