A mixed singular/switching control problem for a dividend policy with reversible technology investment

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June 8, 2006

Abstract

We consider a mixed stochastic control problem that arises in Mathematical Finance literature with the study of interactions between dividend policy and investment. This problem combines features of both optimal switching and singular control. We prove that our mixed problem can be decoupled in two pure optimal stopping and singular control problems. Furthermore, we describe the form of the optimal strategy by means of viscosity solution techniques and smooth-fit properties on the corresponding system of variational inequalities. Our results are of a quasi-explicit nature. From a financial viewpoint, we characterize situations where a firm manager decides optimally to postpone dividend distribution in order to invest in a reversible growth opportunity corresponding to a modern technology. In this paper, a reversible opportunity means that the firm may disinvest from the modern technology and return back to its old technology by receiving some gain compensation. The results of our analysis take qualitatively different forms depending on the parameters values.

Keywords: mixed singular / switching control problem, viscosity solution, smooth-fit property, system of variational inequalities.

MSC2000 subject classification: 60G40, 91B70, 93E20.

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1 Introduction

Stochastic optimization problems that involve both bounded variation control and/or optimal switching are becoming timely problems in the applied probability literature and more particulary in Mathematical Finance. On one hand, the study of singular stochastic control problems in corporate Finance originates with the research on optimal dividend policy for a firm whose cash reserve follows a diffusion model, see Jeanblanc and Shiryaev [11] and Choulli, Taksar and Zhou [3]. On the other hand, the combined singular / stopping control problems have emerged in target tracking models, see Davis and Zervos [6] and Karatzas, Ocone, Wang and Zervos [12] as well as in Mathematical Finance from firm investment theory. For instance, Guo and Pham [10] have studied the optimal time to activate production and to control it by buying or selling capital while Zervos [18] has applied this type of mixed problems in the field of real options theory. Finally, the theory of investment under uncertainty for a firm that can operate a production activity in different modes has led to optimal switching problems which have received a lot of attention in recent years from the applied mathematics community, see Brekke and Oksendal [2], Duckworth and Zervos [8], Ly Vath and Pham [13].

In this paper, we consider a combined stochastic control problem that has emerged in a recent paper by Décamps and Villeneuve [5] with the study of the interactions between dividend policy and investment under uncertainty. These authors have studied the interaction between dividend policy and irreversible investment decision in a growth opportunity. Our aim is to extend this work by relaxing the irreversible feature of the growth opportunity. In other words, we shall consider a firm with a technology in place that has the opportunity to invest in a new technology that increases its profitability. The firm self-finances the opportunity cost on its cash reserve. Once installed, the manager can decide to return back to the old technology by receiving some cash compensation. The mathematical formulation of this problem leads to a combined singular control/switching control for a one dimensional diffusion process. The diffusion process may take two regimes old or new that are switched at stopping times decisions. Within a regime, the manager has to choose a dividend policy that maximizes the expected value of all payouts until bankruptcy or regime transition. The transition from one regime to another incurs a cost or a benefit. The problem is to find the optimal mixed strategy that maximizes the expected returns.

Our analysis is rich enough to address several important questions that have arisen recently in the real option literature [1]. What is the effect of financing constraints on investment decision? When is it optimal to postpone dividends distribution in order to invest? Basically, two assumptions in the real option theory are that the investment decision is made independently of the financial structure of the investment firm and also that the cash process generated by the investment is independent of any managerial decision. In contrast, our model studies the investment under uncertainty with the following set of assumptions. The firm is cash constrained and must finance its investments on its cash benefits, and the cash process generated by the investment depends only on the managerial decision to pay or not dividends, to quit or not the project. Our major finding is to characterize the natural

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[1]See the book of Dixit and Pyndick [7] for an overview of this literature.

intuition that the manager will delay dividend payments if the investment is sufficiently valuable.

As usual in stochastic control theory, the problem developed in this paper leads via the dynamic programming principle to a Hamilton Jacobi Bellman equation which forms in this paper a system of coupled variational inequalities. Therefore, a classical approach based on a verification theorem fails since it is very difficult to guess the shape of both the value function and optimal strategy. To circumvent this difficulty, we use a viscosity solution approach and uniqueness result combined with smooth-fit properties for determining the solution to the HJB system. As by product, we also determine the shape of switching regions. Our findings take qualitatively different forms depending on both the profit rates of each technology and transition costs.

The paper is organized as follows. We formulate the combined stochastic control problem in Section 2. In Section 3, we characterize by means of viscosity solutions, the system of variational inequalities satisfied by the value function, and we also state some regularity properties. Section 4 is devoted to qualitative results concerning the switching regions and in Section 5 we give the quasi-explicit computation and description of the value function and the optimal strategies.

2 Model formulation : a mixed switching/singular control problem

We consider a firm whose activities generate cash process. The manager of the firm acts in the best interest of its shareholders and maximizes the expected present value of dividends up to bankruptcy when the cash reserve becomes negative. The firm has at any time the possibility to invest in a modern technology that increases the drift of the cash from μ_0 to μ_1 without affecting the volatility σ. This growth opportunity requires a fixed cost $g > 0$ self-financed by the cash reserve. Moreover, we consider a reversible investment opportunity for the firm : the manager can decide to return back to the old technology by receiving some fixed gain compensation $(1 - \lambda)g$, with $0 < \lambda < 1$.

The mathematical formulation of this mixed singular/switching control problem is as follows. Let W be a Brownian motion on a filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ satisfying the usual conditions.
- A strategy decision for the firm is a singular/switching control $\alpha = (Z, (\tau_n)_{n \geq 1}) \in \mathcal{A}$ where $Z \in \mathcal{Z}$, the set of $\mathbb{F}$-adapted cadlag nondecreasing processes, $Z_{0^-} = 0$, $(\tau_n)_n$ is an increasing sequence of stopping times, $\tau_n \to \infty$. Z represents the total amount of dividends paid until time t, (τ_n) the switching technology (regimes) time decisions. By convention regime $i = 0$ represents the old technology and $i = 1$ the modern technology.
- Starting from an initial state $(x, i) \in \mathbb{R} \times \{0, 1\}$ for the cash-regime value, and given a control $\alpha \in \mathcal{A}$, the dynamics of the cash reserve process of a firm is governed by :

$$dX_t = \mu_{I_t} dt + \sigma dW_t - dZ_t - dK_t, \quad X_{0^-} = x, \tag{2.1}$$

where :

$$I_t = \sum_{n\geq 0}\left(i1_{\tau_{2n}\leq t<\tau_{2n+1}} + (1-i)1_{\tau_{2n+1}\leq t<\tau_{2n+2}}\right), \quad I_{0^-} = i \tag{2.2}$$

$$K_t = \sum_{n\geq 0}\left(g_{i,1-i}1_{\tau_{2n+1}\leq t<\tau_{2n+2}} + g_{1-i,i}1_{\tau_{2n+2}\leq t<\tau_{2n+3}}\right),$$

with

$$0 \leq \mu_0 < \mu_1 \qquad \sigma > 0,$$

$$g_{01} = g > 0, \qquad g_{10} = -(1-\lambda)g < 0, \quad 0 < \lambda < 1.$$

(Here we used the convention $\tau_0 = 0$). We denote by $(X^{x,i}, I^i)$ the solution to (2.1)-(2.2) (as usual, we omit the dependance in the control α when there is no ambiguity). The time of strict bankruptcy is defined as

$$T = T^{x,i,\alpha} = \inf\left\{t \geq 0 : X_t^{x,i} < 0\right\},$$

and we set by convention $X_t^{x,i} = X_T^{x,i}$ for $t \geq T$. Thus, for $t \in [T \wedge \tau_{2n}, T \wedge \tau_{2n+1})$, the cash reserve $X^{x,i}$ is in technology i (its drift term is μ_i), while for $t \in [T \wedge \tau_{2n+1}, T \wedge \tau_{2n+2})$, $X^{x,i}$ is in technology $1-i$ (its drift term is μ_{1-i}). Moreover,

$$X_{T\wedge\tau_{2n+1}}^{x,i} = X_{(T\wedge\tau_{2n+1})^-}^{x,i} - g_{i,1-i} \quad \text{on} \quad \{\tau_{2n+1} < T\}$$

$$X_{T\wedge\tau_{2n+2}}^{x,i} = X_{(T\wedge\tau_{2n+2})^-}^{x,i} - g_{1-i,i} \quad \text{on} \quad \{\tau_{2n+2} < T\}.$$

The optimal firm value is

$$v_i(x) = \sup_{\alpha\in\mathcal{A}} \mathbb{E}\left[\int_0^{T^-} e^{-\rho t}dZ_t\right], \quad x \in \mathbb{R}, \ i = 0,1. \tag{2.3}$$

Notice that v_i is nonnegative, and $v_i(x) = 0$ for $x < 0$. Since $T = T^{x,i,\alpha}$ is obviously nondecreasing in x, the value functions v_i are clearly nondecreasing.

Remark 2.1 For any $x > 0$, $i = 0,1$, and given an arbitrary control $\alpha = (Z, (\tau_n)_{n\geq 1}) \in \mathcal{A}$, let us consider the control $\tilde{\alpha} = (\tilde{Z}, (\tau_n)_{n\geq 1}) \in \mathcal{A}$ with $\tilde{Z}_t = Z_t + \eta$, for $t \geq 0$, and $0 < \eta < x$. Then, by stressing the dependence of the state process on the control, we have $X_t^{x,i,\tilde{\alpha}} = X_t^{x-\eta,i,\alpha}$ for $0 \leq t < T^{x,i,\tilde{\alpha}} = T^{x-\eta,i,\alpha}$. We deduce

$$v_i(x) \geq \mathbb{E}\left[\int_0^{(T^{x,i,\tilde{\alpha}})^-} e^{-\rho t}d\tilde{Z}_t\right] = \mathbb{E}\left[\int_0^{(T^{x-\eta,i,\alpha})^-} e^{-\rho t}dZ_t\right],$$

which implies from the arbitrariness of α :

$$v_i(x) \geq v_i(x-\eta), \quad 0 < \eta < x.$$

This shows in particular that v is strictly increasing on $(0,\infty)$.

3 Dynamic programming and general properties on the value functions

We first introduce some notations. We denote by $R^{x,i}$ the cash reserve in absence of dividends distribution and in regime i, i.e. the solution to

$$dR_t = \mu_i dt + \sigma dW_t, \quad R_0 = x. \tag{3.1}$$

The associated second order differential operator is denoted $\mathcal{L}_i$:

$$\mathcal{L}_i \varphi(x) = \mu_i \varphi'(x) + \frac{1}{2}\sigma^2 \varphi''(x).$$

In view of the dynamic programming principle, recalled below (see (3.20)), we formally expect that the value functions v_i, $i = 0, 1$, satisfy the system of variational inequalities :

$$\min\left[\rho v_i(x) - \mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - v_{1-i}(x - g_{i,1-i})\right] = 0, \quad x > 0, \ i = 0, 1. \tag{3.2}$$

This statement will be proved rigorously later by means of viscosity solutions. For the moment, we first state a standard comparison principle for this system of PDE.

Proposition 3.1 *Suppose that φ_i, $i = 0, 1$, are two smooth functions on $(0, \infty)$ s.t. $\varphi_i(0^+)$:= $\lim_{x \downarrow 0} \varphi_i(x) \geq 0$, and*

$$\min\left[\rho \varphi_i(x) - \mathcal{L}_i \varphi_i(x), \varphi_i'(x) - 1, \varphi_i(x) - \varphi_{1-i}(x - g_{i,1-i})\right] \geq 0, \ x > 0, \ i = 0, 1, \tag{3.3}$$

where we set by convention $\varphi_i(x) = 0$ for $x < 0$. Then, we have $v_i \leq \varphi_i$, $i = 0, 1$.

Proof. Given an initial state-regime value $(x, i) \in \mathbb{R}_+ \times \{0, 1\}$, take an arbitrary control $\alpha = (Z, (\tau_n), n \geq 1) \in \mathcal{A}$, and set for $m > 0$, $\theta_{m,n} = \inf\{t \geq T \wedge \tau_{2n} : X_t^{x,i} \geq m\} \nearrow \infty$ a.s. when m goes to infinity. Apply then Itô's formula to $e^{-\rho t}\varphi_i(X_t^{x,i})$ between the stopping times $T \wedge \tau_{2n}$ and $\tau_{m,2n+1} := T \wedge \tau_{2n+1} \wedge \theta_{m,n}$. Notice that for $T \wedge \tau_{2n} \leq t < \tau_{m,2n+1}$, $X_t^{x,i}$ stays in regime i. Then, we have

$$\begin{aligned}
e^{-\rho \tau_{m,2n+1}}\varphi_i(X_{\tau_{m,2n}^-}^{x,i}) &= e^{-\rho(T \wedge \tau_{2n})}\varphi_i(X_{T \wedge \tau_{2n}}^{x,i}) + \int_{T \wedge \tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}(-\rho\varphi_i + \mathcal{L}_i\varphi_i)(X_t^{x,i})dt \\
&\quad + \int_{T \wedge \tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}\sigma\varphi_i'(X_t^{x,i})dW_t - \int_{T \wedge \tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}\varphi_i'(X_t^{x,i})dZ_t^c \\
&\quad + \sum_{T \wedge \tau_{2n} \leq t < \tau_{m,2n+1}} e^{-\rho t}\left[\varphi_i(X_t^{x,i}) - \varphi_i(X_{t^-}^{x,i})\right], \tag{3.4}
\end{aligned}$$

where Z^c is the continuous part of Z. We make the convention that when $T \leq \tau_n$, $(T \wedge \theta)^-$ = T for all stopping time $\theta > \tau_n$ a.s., so that (3.4) holds true a.s. for all n, m (recall that $\varphi_i(X_T^{x,i}) = 0$). Since $\varphi_i' \geq 1$, we have by the mean-value theorem $\varphi_i(X_t^{x,i}) - \varphi_i(X_{t^-}^{x,i})$ $\leq X_t^{x,i} - X_{t^-}^{x,i} = -(Z_t - Z_{t^-})$ for $T \wedge \tau_{2n} \leq t < \tau_{m,2n+1}$. By using also the supersolution

inequality of φ_i, taking expectation in the above Itô's formula, and noting that the integrand in the stochastic integral term is bounded by a constant (depending on m), we have

$$\mathbb{E}\left[e^{-\rho\tau_{m,2n+1}}\varphi_i(X^{x,i}_{\tau^-_{m,2n+1}})\right] \leq \mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] - \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{\tau_{m,2n+1}} e^{-\rho t}dZ^c_t\right]$$
$$- \mathbb{E}\left[\sum_{T\wedge\tau_{2n}\leq t<\tau_{m,2n+1}} e^{-\rho t}(Z_t - Z_{t-})\right],$$

and so

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{\tau^-_{m,2n+1}} e^{-\rho t}dZ_t + e^{-\rho\tau_{m,2n+1}}\varphi_i(X^{x,i}_{\tau^-_{m,2n+1}})\right]$$

By sending m to infinity, with Fatou's lemma, we obtain :

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right]$$
$$\geq \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{(T\wedge\tau_{2n+1})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+1})}\varphi_i(X^{x,i}_{(T\wedge\tau_{2n+1})-})\right]. \qquad (3.5)$$

Now, as $\varphi_i(x) \geq \varphi_{1-i}(x - g_{i,1-i})$ and recalling $X^{x,i}_{T\wedge\tau_{2n+1}} = X^{x,i}_{(T\wedge\tau_{2n+1})-} - g_{i,1-i}$ on $\{\tau_{2n+1} < T\}$, we have

$$\varphi_i(X^{x,i}_{(T\wedge\tau_{2n+1})-}) \geq \varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})-} - g_{i,1-i})$$
$$= \varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})}) \quad \text{on } \{\tau_{2n+1} < T\}. \qquad (3.6)$$

Moreover, notice that φ_i is nonnegative as $\varphi_i(0^+) \geq 0$ and $\varphi'_i \geq 1$. Hence, since $\varphi_{1-i}(X^{x,i}_{(T\wedge\tau_{2n+1})})$ $= \varphi_{i-1}(X^{x,i}_T) = 0$ on $\{T \leq \tau_{2n+1}\}$, we see that inequality (3.6) also holds on $\{T \leq \tau_{2n+1}\}$ and so a.s. Therefore, plugging into (3.5), we have

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n}}^{(T\wedge\tau_{2n+1})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+1})}\varphi_{1-i}(X^{x,i}_{T\wedge\tau_{2n+1}})\right].$$

Similarly, we have from the supersolution inequality of φ_{1-i} :

$$\mathbb{E}\left[e^{-\rho(T\wedge\tau_{2n+1})}\varphi_{1-i}(X^{x,i}_{T\wedge\tau_{2n+1}})\right] \geq \mathbb{E}\left[\int_{T\wedge\tau_{2n+1}}^{(T\wedge\tau_{2n+2})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n+2})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n+2}})\right].$$

By iterating these two previous inequalities for all n, we then obtain

$$\varphi_i(x) \geq \mathbb{E}\left[\int_0^{(T\wedge\tau_{2n})^-} e^{-\rho t}dZ_t + e^{-\rho(T\wedge\tau_{2n})}\varphi_i(X^{x,i}_{T\wedge\tau_{2n}})\right],$$
$$\geq \mathbb{E}\left[\int_0^{(T\wedge\tau_{2n})^-} e^{-\rho t}dZ_t\right], \quad \forall n \geq 0,$$

since φ_i is nonnegative. By sending n to infinity, we obtain the required result from the arbitrariness of the control α. $\qquad \square$

As a corollary, we show a linear growth condition on the value functions.

Corollary 3.1 *We have,*

$$v_0(x) \ \le \ x + \frac{\mu_1}{\rho}, \qquad v_1(x) \ \le \ x + \frac{\mu_1}{\rho} + (1-\lambda)g, \quad x > 0. \tag{3.7}$$

Proof. We set $\varphi_0(x) = x + \frac{\mu_1}{\rho}$, $\varphi_1(x) = x + \frac{\mu_1}{\rho} + (1-\lambda)g$, on $(0, \infty)$, and $\varphi_i(x) = 0$ for $x < 0$. A straightforward computation shows that we have the supersolution properties for φ_i, $i = 0, 1$:

$$\min\left[\rho\varphi_0(x) - \mathcal{L}_0\varphi_0(x), \varphi_0'(x) - 1, \varphi_0(x) - \varphi_1(x-g)\right] \ \ge \ 0, \quad x > 0,$$
$$\min\left[\rho\varphi_1(x) - \mathcal{L}_1\varphi_1(x), \varphi_1'(x) - 1, \varphi_1(x) - \varphi_0(x + (1-\lambda)g)\right] \ \ge \ 0, \quad x > 0.$$

We then conclude from Proposition 3.1. □

The next result states the initial-boundary data for the value functions.

Proposition 3.2 *1) The value function v_0 is continuous on $(0, \infty)$ and satisfies*

$$v_0(0^+) \ := \ \lim_{x \downarrow 0} v_0(x) \ = \ 0. \tag{3.8}$$

2) The value function v_1 satisfies

$$v_1(0^+) \ := \ \lim_{x \downarrow 0} v_1(x) \ = \ v_0((1-\lambda)g). \tag{3.9}$$

Proof. 1) a) We first state (3.8). For $x > 0$, let us consider the drifted Brownian $R^{x,1}$, defined in (3.1), and denote $\theta_0 = \inf\{t \ge 0 \ : R_t^{x,1} = 0\}$. It is well-known that :

$$\mathbb{E}\left[\sup_{0 \le t \le \theta_0} R_t^{x,1}\right] \ \rightarrow \ 0, \quad \text{as } x \downarrow 0. \tag{3.10}$$

We also have

$$\sup_{0 \le t \le \theta_0} R_t^{x,1} \ \downarrow \ 0, \quad a.s. \ \text{ as } x \downarrow 0. \tag{3.11}$$

Fix some $r > 0$, and denote $\theta_r = \inf\{t \ge 0 \ : R_t^{x,1} = r\}$. It is also well-known that

$$\mathbb{P}\left[\theta_0 > \theta_r\right] \ \rightarrow \ 0, \quad \text{as } x \downarrow 0. \tag{3.12}$$

Let $\alpha = (Z, (\tau_n)_{n \ge 1})$ be an arbitrary policy in $\mathcal{A}$, and denote $\eta = T \wedge \theta_r = T^{x,0,\alpha} \wedge \theta_r$. Since $\mu_0 < \mu_1$ and $g_{01} + g_{10} > 0$, we notice that $X_t^{x,0} \le R_t^{x,1} - Z_t \le R_t^{x,1}$ for all $t \ge 0$. Hence $T \le \theta_0$, $Z_t \le R_t^{x,1}$ for $t < T$, and in particular $Z_{\eta^-} \le R_\eta^{x,1}$. We then write :

$$\mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] \ = \ \mathbb{E}\left[\int_0^{\eta^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[1_{T > \eta} \int_\eta^{T^-} e^{-\rho t} dZ_t\right]$$
$$\le \ \mathbb{E}\left[Z_{\eta^-}\right] + \mathbb{E}\left[1_{T > \eta}\mathbb{E}\left[\int_\eta^{T^-} e^{-\rho t} dZ_t \,\middle|\, \mathcal{F}_\eta\right]\right]$$
$$\le \ \mathbb{E}\left[R_\eta^{x,1}\right] + \mathbb{E}\left[1_{T > \theta_r} e^{-\gamma\eta} v_0(X_{\eta^-}^{x,0})\right], \tag{3.13}$$

where we also used in the last inequality the definition of the value function v_0. Now, since $T \leq \theta_0$, we have $\eta \leq \theta_0$. Moreover, since v_0 is nondecreasing and $\eta \leq \theta_r$, we have $v_0(X_{\eta^-}^{x,0}) \leq v_0(r)$. Thus, inequality (3.13) yields

$$0 \leq v_0(x) \leq \mathbb{E}\left[\sup_{0 \leq t \leq \theta_0} R_t^{x,1}\right] + v_0(r)\mathbb{P}[\theta_0 > \theta_r] \longrightarrow 0, \quad \text{as } x \downarrow 0, \quad (3.14)$$

from (3.10)-(3.12). This proves $v_0(0^+) = 0$.

b) We next prove the continuity of v_0 at any $y > 0$. Let $\alpha = (Z, (\tau_n)_{n \geq 1}) \in \mathcal{A}$, $X^{y,0}$ be the corresponding process and $T = T^{y,0,\alpha}$ its bankruptcy time. According to (3.10) and (3.12), given a fixed $r > 0$, for any arbitrary small $\varepsilon > 0$, one can find $0 < \delta < y$ s.t. for $0 < x < \delta$,

$$\mathbb{E}\left[\sup_{0 \leq t \leq \theta_0} R_t^{x,1}\right] + v_0(r)\mathbb{P}[\theta_0 > \theta_r] \leq \varepsilon,$$

Then, following the same lines of proof as for (3.13)-(3.14), we show

$$\mathbb{E}\left[\int_\theta^{T^-} e^{-\rho t} dZ_t\right] \leq \varepsilon, \quad (3.15)$$

for any $0 < x < \delta$ and stopping time θ s.t. $X_\theta^{y,0} \leq x$. Given $0 < x < \delta$, consider the state process $X^{y-x,0}$ starting from $y - x$ in regime 0, and controlled by α. Denote θ its bankruptcy time, i.e. $\theta = T^{y-x,0,\alpha} = \inf\{t \geq 0 : X_t^{y-x,0} < 0\}$. Notice that $X_t^{y-x,0} = X_t^{y,0} - x$ for $t \leq \theta \leq T$, and so

$$X_\theta^{y,0} = X_\theta^{y-x,0} + x \leq x.$$

From (3.15), we then have

$$\mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] = \mathbb{E}\left[\int_0^{\theta^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[\int_\theta^{T^-} e^{-\rho t} dZ_t\right]$$
$$\leq v_0(y - x) + \varepsilon.$$

From the arbitrariness of α, and recalling that v_0 is nondecreasing, this implies

$$0 \leq v_0(y) - v_0(y - x) \leq \varepsilon,$$

which shows the continuity of v_0.

2) Given an arbitrary control $\alpha = (Z, (\tau_n)_{n \geq 1}) \in \mathcal{A}$, let us consider the control $\tilde{\alpha} = (\tilde{Z}, (\tilde{\tau}_n)_{n \geq 1}) \in \mathcal{A}$ defined by $\tilde{Z} = Z$, $\tilde{\tau}_1 = 0$, $\tilde{\tau}_n = \tau_{n-1}$, $n \geq 2$. Then, for all $x > 0$, and by stressing the dependence of the state process on the control, we have $X_t^{x,1,\tilde{\alpha}} = X_t^{x+(1-\lambda)g,0,\alpha}$ for $0 \leq t < T^{x,1,\tilde{\alpha}} = T^{x+(1-\lambda)g,0,\alpha}$. We deduce

$$v_1(x) \geq \mathbb{E}\left[\int_0^{(T^{x,1,\tilde{\alpha}})^-} e^{-\rho t} d\tilde{Z}_t\right] = \mathbb{E}\left[\int_0^{(T^{x+(1-\lambda)g,0,\alpha})^-} e^{-\rho t} dZ_t\right],$$

which implies from the arbitrariness of α :

$$v_1(x) \geq v_0(x + (1-\lambda)g), \quad x > 0. \quad (3.16)$$

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On the other hand, starting in the regime $i = 1$, for $x \geq 0$, let $\alpha = (Z, (\tau_n)_{n \geq 1})$ be an arbitrary control in $\mathcal{A}$. We denote $T_1 = T \wedge \tau_1 = T^{x,1,\alpha} \wedge \tau_1$, and we write :

$$\mathbb{E}\left[\int_0^{T^-} e^{-\rho t} dZ_t\right] = \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[1_{T > \tau_1} \int_{\tau_1}^{T^-} e^{-\rho t} dZ_t\right]. \qquad (3.17)$$

The first term in the r.h.s. of (3.17) is dealt similarly as in (3.13)-(3.14) : we set $\eta_1 = T_1 \wedge \theta_r$ with $\theta_r = \inf\{t \geq 0 : R_t^{x,1} = r\}$ for some fixed $r > 0$, and we notice that $X_t^{x,1} = R_t^{x,1} - Z_t \leq R_t^{x,1}$ for $t < \tau_1$. Hence $T_1 \leq \theta_0 = \inf\{t \geq 0 : R_t^{x,1} = 0\}$, and $Z_{\eta_1^-} \leq R_{\eta_1}^{x,1} \leq \sup_{0 \leq t \leq \theta_0} R_t^{x,1}$. Then, as in (3.13)-(3.14), we have :

$$\mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t\right] \leq \mathbb{E}\left[\sup_{0 \leq t \leq \theta_0} R_t^{x,1}\right] + v_1(r)\mathbb{P}[\theta_0 > \theta_r]. \qquad (3.18)$$

For the second term in the r.h.s. of (3.17), since there is a change of regime at τ_1 from $i = 1$ to $i = 0$, and by definition of the value function v_0, we have :

$$
\begin{aligned}
\mathbb{E}\left[1_{T > \tau_1} \int_{\tau_1}^{T^-} e^{-\rho t} dZ_t\right] &= \mathbb{E}\left[1_{T > \tau_1} \mathbb{E}\left[\int_{\tau_1}^{T^-} e^{-\rho t} dZ_t \,\Big|\, \mathcal{F}_{\tau_1}\right]\right] \\
&\leq \mathbb{E}\left[1_{T > \tau_1} e^{-\rho \tau_1} v_0(X_{\tau_1}^{x,1})\right] \\
&\leq \mathbb{E}\left[1_{T > \tau_1} v_0(X_{\tau_1^-}^{x,1} + (1 - \lambda)g)\right] \\
&\leq \mathbb{E}\left[v_0\left(\sup_{0 \leq t \leq \theta_0} R_t^{x,1} + (1 - \lambda)g\right)\right]. \qquad (3.19)
\end{aligned}
$$

Here, we used in the second inequality the fact that $X_{\tau_1}^{x,1} = X_{\tau_1^-}^{x,1} + (1 - \lambda)g$ on $\{\tau_1 < T\}$, and in the last one the observation that $X_t^{x,1} \leq R_t^{x,1}$ for $t < \tau_1$, and $\tau_1 = T_1 \leq \theta_0$ on $\{\tau_1 < T\}$. Hence, by combining (3.16)-(3.17)-(3.18)-(3.19), we obtain :

$$v_0(x + (1 - \lambda)g)$$
$$\leq v_1(x) \leq \mathbb{E}\left[\sup_{0 \leq t \leq \theta_0} R_t^{x,1}\right] + v_1(r)\mathbb{P}[\theta_0 > \theta_r] + \mathbb{E}\left[v_0\left(\sup_{0 \leq t \leq \theta_0} R_t^{x,1} + (1 - \lambda)g\right)\right].$$

Finally, by using the continuity of v_0, the limits (3.10)-(3.11)-(3.12), as well as the linear growth condition (3.7) of v_0, which allows to apply dominated convergence theorem, we conclude that $v_1(0^+) = v_0((1 - \lambda)g)$. $\qquad \square$

Remark 3.1 There is some asymmetry between the two value functions v_0 and v_1. Actually, v_0 is continuous at $0 : v_0(0^+) = v_0(0^-) = 0$, while it is not the case for v_1, since $v_1(0^+) = v_0((1 - \lambda)g) > 0 = v_1(0^-)$: When the reserve process in regime 0 approaches zero, we are ineluctably absorbed by this threshold. On the contrary, in regime 1, when the reserve process approaches zero, we have the possibility to change of regime, which pushes us above the bankruptcy threshold by receiving $(1 - \lambda)g$. In particular, at this stage, we do not know yet the continuity of v_1 on $(0, \infty)$. This will be proved in Theorem 3.1 as a consequence of the dynamic programming principle. In the sequel, we set by convention $v_i(0) = v_i(0^+)$ for $i = 0, 1$.

The following dynamic programming principle holds : for any $(x, i) \in \mathbb{R}_+ \times \{0, 1\}$, we have

$$
(\textbf{DP}) \quad v_i(x) = \sup_{\alpha \in \mathcal{A}} \mathbb{E}\left[\int_0^{(T \wedge \theta \wedge \tau_1)^-} e^{-\rho t} dZ_t \right.
$$
$$
\left. + \; e^{-\rho(T \wedge \theta \wedge \tau_1)} \left(v_i(X_{T \wedge \theta}^{x,i}) 1_{T \wedge \theta < \tau_1} + v_{1-i}(X_{\tau_1}^{x,i}) 1_{\tau_1 \leq T \wedge \theta} \right) \right] \quad (3.20)
$$

where θ is any stopping time, possibly depending on $\alpha \in \mathcal{A}$ in (3.20).

We then have the PDE characterization of the value functions v_i.

Theorem 3.1 *The value functions v_i, $i = 0, 1$, are continuous on $(0, \infty)$, and are the unique viscosity solutions with linear growth condition on $(0, \infty)$ and boundary data $v_0(0) = 0$, $v_1(0) = v_0((1 - \lambda)g)$ to the system of variational inequalities :*

$$
\min \left[\rho v_i(x) - \mathcal{L}_i v_i(x), v_i'(x) - 1, v_i(x) - v_{1-i}(x - g_{i,1-i}) \right] = 0, \quad x > 0, \; i = 0, 1. \quad (3.21)
$$

Actually, we prove some more regularity results on the value functions.

Proposition 3.3 *The value functions v_i, $i = 0, 1$, are C^1 on $(0, \infty)$. Moreover, if we set for $i = 0, 1$:*

$$
\begin{aligned}
\mathcal{S}_i &= \{x \geq 0 \; : v_i(x) = v_{1-i}(x - g_{i,1-i})\} \\
\mathcal{D}_i &= \{x > 0 \; : v_i'(x) = 1\}, \\
\mathcal{C}_i &= (0, \infty) \setminus (\mathcal{S}_i \cup \mathcal{D}_i),
\end{aligned}
$$

then v_i is C^2 on the open set $\mathcal{C}_i \cup \mathcal{D}_i$ of $(0, \infty)$, and we have in the classical sense

$$
\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i.
$$

The proofs of Theorem 3.1 and Proposition 3.3 follow and combine essentially arguments from [10] for singular control, and [14] for switching control, and are postponed to Appendix A and B.

$\mathcal{S}_i$ is the switching region from technology i to $1 - i$, $\mathcal{D}_i$ is the dividend region in technology i, and $\mathcal{C}_i$ is the continuation region in technology i. Notice from the boundary conditions on v_i that $\mathcal{S}_i$ contains 0. We denote $\mathcal{S}_i^* = \mathcal{S}_i \setminus \{0\}$.

4 Qualitative results on the switching regions

4.1 Benchmarks

We consider the firm value without investment/disinvestment in technology $i = 0$:

$$
\hat{V}_0(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_0^-} e^{-\rho t} dZ_t \right], \quad (4.1)
$$

where $T_0 = \inf\{t \geq 0 \;:\; X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 0 :

$$dX_t \;=\; \mu_0 dt + \sigma dW_t - dZ_t, \quad X_{0-} \;=\; x.$$

It is known that $\hat{V}_0$, as the value function of a pure singular control problem, is characterized as the unique continuous viscosity solution on $(0, \infty)$, with linear growth condition to the variational inequality :

$$\min \left[\rho \hat{V}_0 - \mathcal{L}_0 \hat{V}_0 \;,\; \hat{V}_0' - 1 \right] \;=\; 0, \quad x > 0, \tag{4.2}$$

and boundary data

$$\hat{V}_0(0) \;=\; 0.$$

Actually, $\hat{V}_0$ is C^2 on $(0, \infty)$ and explicit computations of this standard singular control problem are developed in Shreve, Lehoczky and Gaver [16], Jeanblanc and Shiryaev [11], or Radner and Shepp [15] :

$$\hat{V}_0(x) \;=\; \begin{cases} \dfrac{f_0(x)}{f_0'(\hat{x}_0)} & , 0 \leq x \leq \hat{x}_0 \\ x - \hat{x}_0 + \dfrac{\mu_0}{\rho}, & x \geq \hat{x}_0, \end{cases}$$

where

$$f_0(x) \;=\; e^{m_0^+ x} - e^{m_0^- x}, \quad \hat{x}_0 \;=\; \frac{1}{m_0^+ - m_0^-} \ln \left(\frac{(m_0^+)^2}{(m_0^-)^2} \right),$$

and $m_0^- < 0 < m_0^+$ are roots of the characteristic equation :

$$\rho - \mu_0 m - \frac{1}{2}\sigma^2 m^2 \;=\; 0.$$

In other words, this means that the optimal cash reserve process is given by the reflected diffusion process at the threshold $\hat{x}_0$ with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq \hat{x}_0$, the optimal dividend policy is to distribute immediately the amount $x - \hat{x}_0$ and then follows the dividend policy characterized by the local time.

As a second benchmark, we consider the firm value problem in technology $i = 1$ with nonnegative constant liquidation value L to be fixed later :

$$w_1^L(x) \;=\; \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} L \right], \tag{4.3}$$

$T_1 = \inf\{t \geq 0 \;:\; X_t \leq 0\}$ is the time bankruptcy of the cash reserve in regime 1 :

$$dX_t \;=\; \mu_1 dt + \sigma dW_t - dZ_t, \quad X_{0-} \;=\; x.$$

Again, as value function of a pure singular control problem, w_1^L is characterized as the unique continuous viscosity solution on $(0, \infty)$, with linear growth condition to the variational inequality :

$$\min \left[\rho w_1^L - \mathcal{L}_1 w_1^L \;,\; (w_1^L)' - 1 \right] \;=\; 0, \quad x > 0, \tag{4.4}$$

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and boundary data

$$w_1^L(0) \;\;=\;\; L. \tag{4.5}$$

Actually, w_1^L is C^2 on $(0,\infty)$ and explicit computations of this singular control problem are developed in Boguslavskaya [1] :

• If $L \geq \frac{\mu_1}{\rho}$, then :

$$w_1^L(x) \;\;=\;\; x + L, \quad x \geq 0.$$

The optimal strategy is to distribute the initial cash reserve immediately, and so to liquidate the firm at $X_t = 0$ by changing of technology to regime $i = 0$ and receiving L.

• If $L < \frac{\mu_1}{\rho}$, then

$$w_1^L(x) \;\;=\;\; \begin{cases} \frac{1 - L h_1'(\hat{x}_1)}{f_1'(\hat{x}_1)} f_1(x) + L h_1(x) & , 0 \leq x \leq x_1^L \\ x - x_1^L + \frac{\mu_1}{\rho}, & x \geq x_1^L, \end{cases}$$

with

$$f_1(x) \;\;=\;\; e^{m_1^+ x} - e^{m_1^- x}, \quad h_1(x) \;=\; e^{m_1^- x},$$

$m_1^- < 0 < m_1^+$, the roots of the characteristic equation :

$$\rho - \mu_1 m - \frac{1}{2}\sigma^2 m^2 \;\;=\;\; 0,$$

and x_1^L the solution to

$$L \frac{h_1(x) f_1'(x) - h_1'(x) f_1(x)}{f_1'(x)} + \frac{f_1(x)}{f_1'(x)} \;\;=\;\; \frac{\mu_1}{\rho}.$$

The optimal cash reserve process is given by the reflected diffusion process at the threshold x_1^L with an optimal dividend process given by the local time at this boundary. When the firm starts with a cash reserve $x \geq x_1^L$, the optimal dividend policy is to distribute immediately the amount $x - x_1^L$ and then follows the dividend policy characterized by the local time. In the sequel, we shall denote

$$\hat{V}_1 \;\;=\;\; w_1^L \quad \text{and} \quad \hat{x}_1 \;=\; x_1^L \quad \text{when} \quad L = \hat{V}_0((1-\lambda)g).$$

$L = \hat{V}_0((1-\lambda)g)$ is the minimal received liquidation value when one switches to regime 0 at $x = 0$ and do not switch anymore.

Remark 4.1 It is known (see e.g.[1]) that $\hat{V}_0$ and w_1^L are concave on $(0,\infty)$. As a consequence, $\hat{V}_0$ and w_1^L are globally Lipschitz since their first derivatives are bounded near zero. By convention, we set $\hat{V}_0(x) = w_1^L(x) = 0$ for $x < 0$.

Remark 4.2 We have $v_0 \geq \hat{V}_0$ and $v_1 \geq \hat{V}_1$ on $(0,\infty)$. This is rather clear since the class of controls over which maximization is taken in $\hat{V}_0$ and $\hat{V}_1$ is included in the class of controls of v_0 and v_1. This may be justified more rigorously by a maximum principle argument and by noting that v_0 and v_1 are (viscosity) supersolution to the variational inequality satisfied respectively by $\hat{v}_0$ and $\hat{V}_1$, with the same boundary data.

We first show the intuitive result that the value function for the dividend policy problem is nondecreasing in the rate of return of the cash reserve.

Lemma 4.1

$$\hat{V}_1(x) \geq \hat{V}_0(x + (1-\lambda)g), \quad \forall x \geq 0.$$

Proof. We set $w_1(x) = \hat{V}_1(x - (1-\lambda)g)$ for $x \geq (1-\lambda)g$. From (4.4), we see that $\hat{w}_1$ satisfies on $[(1-\lambda)g, \infty)$:

$$
\begin{aligned}
w_1'(x) &= \hat{V}_1'(x - (1-\lambda)g) \geq 1 \\
(\rho w_1 - \mathcal{L}_0 w_1)(x) &= (\rho - \mathcal{L}_1 \hat{V}_1 + (\mu_1 - \mu_0)\hat{V}_1')(x - (1-\lambda)g) > 0,
\end{aligned}
$$

since $\mu_1 > \mu_0$ and $\hat{V}_1$ is increasing. Moreover, $w_1((1-\lambda)g) = \hat{V}_1(0) = \hat{V}_0((1-\lambda)g)$. By standard maximum principle on the variational inequality (4.2), we deduce that $w_1 \geq \hat{V}_0$ on $[(1-\lambda)g, \infty)$, which implies the required result. $\qquad \square$

The next result precises conditions under which the value function in the old technology is larger than the value function in the modern technology after paying the switching cost from the old to the modern regimes.

Lemma 4.2 *Suppose that $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$. Then,*

$$\hat{V}_0(x) \geq \hat{V}_1(x - g), \quad \forall x \geq 0, \quad \text{if and only if} \quad \frac{\mu_1 - \mu_0}{\rho} \leq \hat{x}_1 + g - \hat{x}_0.$$

Proof. Similar arguments as in Lemma 2.1 in Decamps and Villeneuve [5]. $\qquad \square$

Remark 4.3 Recalling that $\hat{V}_0$ and $\hat{V}_1$ are increasing and concave, the above Lemma shows also that if $\frac{\mu_1 - \mu_0}{\rho} > \hat{x}_1 + g - \hat{x}_0$, then there exists $\hat{x}_{01} \geq g$ s.t.

$$\max\left(\hat{V}_0(x), \hat{V}_1(x-g)\right) = \begin{cases} \hat{V}_0(x), & x \leq \hat{x}_{01} \\ \hat{V}_1(x-g), & x > \hat{x}_{01} \end{cases}$$

4.2 Preliminary results on the switching regions

In this section, we shall state some preliminary qualitative results concerning the switching regions.

Lemma 4.3 *If $x \in \mathcal{S}_i$ then $x - g_{i,1-i} \notin \mathcal{S}_{1-i}$.*

Proof. Since $v_i(x) > v_i(x - \lambda g)$ for every $x > 0$ and $i \in \{0, 1\}$, we have for $x \in \mathcal{S}_i$,

$$v_{1-i}(x - g_{i,1-i}) = v_i(x) > v_i(x - \lambda g) = v_i(x - g_{i,1-i} - g_{1-i,i}).$$

Therefore, $x - g_{i,1-i} \notin \mathcal{S}_{1-i}$ for $x \in \mathcal{S}_i$. $\qquad \square$

Let us recall the notation $\mathcal{S}_i^* = \mathcal{S}_i \setminus \{0\}$. We have the following inclusion :

Lemma 4.4 $\mathcal{S}_1^* \subset \mathcal{D}_1$.

Proof. We make a proof by contradiction by assuming that there exists some $x \in \mathcal{S}_1^* \setminus \mathcal{D}_1$. According to Proposition 3.3, we have $v_0'(x + (1-\lambda)g) = v_1'(x) > 1$, and so $x + (1-\lambda)g \notin \mathcal{D}_0$. Applying Lemma 4.3 with $i = 1$ implies $x + (1-\lambda)g \in \mathcal{C}_0$. Therefore,

$$
\begin{aligned}
\rho v_1(x) - \mathcal{L}_1 v_1(x) &= \rho v_1(x) - \mathcal{L}_0 v_1(x) + (\mu_0 - \mu_1) v_1'(x) \\
&= \rho v_0(x + (1-\lambda)g) - \mathcal{L}_0 v_0(x + (1-\lambda)g) + (\mu_0 - \mu_1) v_1'(x) \\
&= (\mu_0 - \mu_1) v_1'(x) \quad \text{since} \quad x + (1-\lambda)g \in \mathcal{C}_0 \\
&< 0,
\end{aligned}
$$

which contradicts Theorem 3.1. $\qquad\square$

We now introduce the following definition.

Definition 4.1 y *is a left boundary of the closed set* $\mathcal{D}_i$ *if there is some* $\delta > 0$ *such that* $y - \varepsilon$ *does not belong to* $\mathcal{D}_i$ *for every* $0 < \varepsilon < \delta$.

Lemma 4.5 *Let* $y > 0$ *be a left boundary of* $\mathcal{D}_i$.

- *If there is some* $\varepsilon > 0$ *such that* $(y - \varepsilon, y) \subset \mathcal{C}_i$, *then* $v_i(y) = \dfrac{\mu_i}{\rho}$.

- *If not,* $v_i(y) = \dfrac{\mu_{1-i}}{\rho}$.

Proof. Since y is a left boundary of $\mathcal{D}_i$, there is some $\varepsilon > 0$ such that $(y - \varepsilon, y) \subset \mathcal{C}_i \cup \mathcal{S}_i$. Therefore, two cases have to be considered.

$\star$ *Case 1*: If $(y - \varepsilon, y) \subset \mathcal{C}_i$. Then, according to Proposition 3.3, v_i is twice differentiable at x, for $y - \varepsilon < x < y$ and satisfies $v_i'(y) = 1$ and $v_i''(y) = 0$. Therefore, we have

$$
0 = \rho v_i(x) - \mathcal{L}_i v_i(x) = \rho v_i(x) - \mu_i v_i'(x) - \frac{\sigma^2}{2} v_i''(x).
$$

By sending x to y, we obtain that $v_i(y) = \dfrac{\mu_i}{\rho}$.

$\star$ *Case 2*: If not, there is an increasing sequence $(y_n)_n$ valued in $\mathcal{S}_i$, and converging to y which therefore belongs to $\mathcal{S}_i$. We then have $v_i(y_n) = v_{1-i}(y_n - g_{i,1-i})$ and also $v_i'(y_n) > 1$ for n great enough since y is a left boundary of $\mathcal{D}_i$. Thus, $y_n - g_{i,1-i} \notin \mathcal{D}_{1-i}$. Moreover, according to Lemma 4.3, we also have $y_n - g_{i,1-i} \notin \mathcal{S}_{1-i}$ and therefore, $y_n - g_{i,1-i} \in \mathcal{C}_{1-i}$ or equivalently

$$
\rho v_{1-i}(y_n - g_{i,1-i}) - \mathcal{L}_{1-i} v_{1-i}(y_n - g_{i,1-i}) = 0.
$$

By letting n tends to ∞, we obtain $v_{1-i}(y - g_{i,1-i}) = \dfrac{\mu_{1-i}}{\rho}$. Since $y \in \mathcal{S}_i$, this implies $v_i(y) = v_{1-i}(y - g_{i,1-i}) = \dfrac{\mu_{1-i}}{\rho}$.

$\qquad\square$

The next result shows that the switching region from modern technology $i = 1$ to the old technology $i = 0$ is either reduced to the zero threshold or to the entire state reserve domain $\mathbb{R}_+$, depending on the gain $(1-\lambda)g$ for switching from regime 1 to regime 0.

Proposition 4.1 *The two following cases arise :*

(i) *If* $v_0((1 - \lambda)g) < \dfrac{\mu_1}{\rho}$ *then* $\mathcal{S}_1 = \{0\}$.

(ii) *If* $v_0((1 - \lambda)g) \geq \dfrac{\mu_1}{\rho}$ *then* $\mathcal{S}_1 = \mathcal{D}_1 = \mathbb{R}_+$.

Proof. (i) Assume $v_0((1 - \lambda)g) < \dfrac{\mu_1}{\rho}$.

We shall make a proof by contradiction by considering the existence of some $x_0 \in S_1^*$. By Lemma 4.4, one can introduce the finite nonnegative number

$$\underline{x} = \inf\{y > 0 \ : [y, x_0] \subset \mathcal{D}_1\}.$$

Hence, $\underline{x}$ is a left boundary of $\mathcal{D}_1$. Moreover, Lemma 4.5 gives $v_1(\underline{x}) = \dfrac{\mu_1}{\rho}$ or $\dfrac{\mu_0}{\rho}$.

1. We first check that $\underline{x} > 0$. If not, we would have: $v_1(y) = y + v_0((1 - \lambda)g)$ for any $0 < y < x_0$. But, in this case, we have for $0 < y < x_0$,

$$\rho v_1(y) - \mathcal{L}_1 v_1(y) = \rho(y + v_0((1 - \lambda)g)) - \mu_1.$$

   Therefore, under the assumption (i), $\rho v_1(y) - \mathcal{L}_1 v_1(y) < 0$ for y small enough which is a contradiction.

2. We now prove that $v_1(\underline{x}) = \dfrac{\mu_1}{\rho}$. To see this, we shall show that the closed set $\mathcal{D}_1$ is an interval of $\mathbb{R}_+$. Let $a, b \in \mathcal{D}_1$ with $a < b$, we want to show that $(a, b) \subset \mathcal{D}_1$. If not, from Lemma 4.4, we can find a subinterval (c, d) with $c, d \in \mathcal{D}_1$ and $(c, d) \subset \mathcal{C}_1$. But, for $c < x < d$, we have

$$0 \quad = \quad \rho v_1(x) - \mathcal{L}_1 v_1(x) \quad = \quad \rho v_1(x) - \mu_1 v_1'(x) - \frac{\sigma^2}{2} v_1''(x).$$

   By sending x to c and d, we obtain that $v_1(c) = v_1(d) = \dfrac{\mu_1}{\rho}$ which contradicts the fact that v_1 is strictly increasing. Since $\mathcal{D}_1$ is an interval of $\mathbb{R}_+$, we have $\underline{x} = \inf \mathcal{D}_1$. Thus, recalling that $\underline{x} > 0$, we can find from Lemma 4.4, some $\varepsilon > 0$ such that $(\underline{x} - \varepsilon, \underline{x}) \subset \mathcal{C}_1$, and deduce from Lemma 4.5 that $v_1(\underline{x}) = \dfrac{\mu_1}{\rho}$.

3. We now introduce
$$\bar{x} = \inf\{y \geq \underline{x} \,|\, y \in \mathcal{S}_1\}.$$

   Observe that $\bar{x} + (1 - \lambda)g \in \mathcal{D}_0$. Moreover, according to Lemma 4.3, $\bar{x} + (1 - \lambda)g \notin \mathcal{S}_0$ and thus a left neighbourhood of $\bar{x} + (1 - \lambda)g$ belongs to $\mathcal{C}_0$. We first prove that $\bar{x} + (1 - \lambda)g$ cannot be a left boundary of $\mathcal{D}_0$. On the contrary, we would have from Lemma 4.5,

$$v_1(\bar{x}) \quad = \quad v_0(\bar{x} + (1 - \lambda)g) \quad = \quad \frac{\mu_0}{\rho} \quad < \quad \frac{\mu_1}{\rho} \quad = \quad v_1(\underline{x}),$$

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which contradicts the fact that v_1 is increasing. Therefore, $\bar{x} + (1-\lambda)g \in \overset{o}{\mathcal{D}}_0$, and we can find $y < \bar{x}$ such that $y + (1-\lambda)g$ is a left boundary of $\mathcal{D}_0$. Hence,

$$v_1(\bar{x}) \;=\; v_0(\bar{x} + (1-\lambda)g) \;=\; \bar{x} - y + v_0(y + (1-\lambda)g) \;\leq\; \bar{x} - y + v_1(y).$$

Since the reverse inequality is always true, we obtain that $y \in \mathcal{S}_1$ which contradicts the definition of $\bar{x}$. We conclude that $\bar{x}$ cannot be strictly positive, which is a contradiction with the first step. This proves finally that $x_0 \in \mathcal{S}_1^*$.

(ii) Assume that $v_0((1-\lambda)g) \geq \dfrac{\mu_1}{\rho}$. Let y be a left boundary of $\mathcal{D}_1$. We shall prove that y necessarily equals zero. If not, according to Lemma 4.5, $v_1(y) \leq \dfrac{\mu_1}{\rho} \leq v_1(0)$ where the second inequality comes from the hypothesis and (3.9). Since the function v_1 is strictly increasing, we get the desired contradiction. Therefore, $\mathcal{D}_1 = [0, a]$. It remains to prove that a is infinite. From Lemma 4.4, the open set (a, ∞) belongs to $\mathcal{C}_1$ if $a < \infty$. Using the regularity of v_1 on $\mathcal{C}_1$, we get by the same reasoning as in the proof of Lemma 4.5 that $v_1(a) = \dfrac{\mu_1}{\rho}$, which gives the same contradiction as before. Hence, $\mathcal{D}_1 = [0, \infty)$. We then have for any $x > 0$,

$$v_1(x) \;=\; x + v_0((1-\lambda)g) \;\leq\; v_0(x + (1-\lambda)g).$$

Since the reverse inequality is always true by definition, we conclude that $\mathcal{S}_1 = [0, \infty)$. $\quad\square$

The next proposition describes the structure of the switching region from technology $i = 0$ to $i = 1$, in the case where the growth rate μ_1 in the modern technology $i = 1$, is large enough.

Proposition 4.2 *Suppose that*

$$\frac{\mu_1 - \mu_0}{\rho} \;>\; \hat{x}_1 + g - \hat{x}_0, \quad and \quad \hat{V}_0((1-\lambda)g) \;<\; \frac{\mu_1}{\rho}.$$

Then, there exists $x_{01}^* \in [g, \infty)$ s.t.

$$\mathcal{S}_0^* \;=\; [x_{01}^*, \infty).$$

Proof. We first notice that $\mathcal{S}_0^* \neq \emptyset$. On the contrary, we would have $v_0 = \hat{V}_0$, and so $\hat{V}_0(x) \geq v_1(x-g) \geq \hat{V}_1(x-g)$ for all x, which is in contradiction with Lemma 4.2. Moreover, since $v_1(x-g) = v_0(x) > 0$ for all $x \in \mathcal{S}_0^*$, we deduce that $\mathcal{S}_0^* \subset [g, \infty)$ and so

$$x_{01}^* \;:=\; \inf \mathcal{S}_0^* \;\in\; [g, \infty).$$

Let us now consider the function

$$w_0(x) \;=\; \begin{cases} v_0(x), & x < x_{01}^* \\ v_1(x-g), & x \geq x_{01}^*. \end{cases}$$

We claim that w_0 is a viscosity solution, with linear growth condition and boundary data $w_0(0^+) = 0$, to

$$\min\left[\rho w_0(x) - \mathcal{L}_0 w_0(x), w_0'(x) - 1, w_0(x) - v_1(x-g) \right] \;=\; 0, \;\; x > 0.$$

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For $x < x_{01}^*$, this is clear since $w_0 = v_0$ on $(0, x_{01}^*)$. For $x > x_{01}^*$, we see that $w_0' \geq 1$ and

$$\begin{aligned}
\rho w_0 - \mathcal{L}_0 w_0 &= (\rho v_1 - \mathcal{L}_1 v_1 + (\mu_1 - \mu_0) v_1')(x - g) \\
&\geq (\mu_1 - \mu_0) v_1'(x - g) \geq 0.
\end{aligned}$$

Hence, the viscosity property is also satisfied for $x > x_{01}^*$. It remains to check the viscosity property for $x = x_{01}^*$. The viscosity subsolution property at x_{01}^* is trivial since $w_0(x_{01}^*) = v_1(x_{01}^* - g)$. For the viscosity supersolution property, take some C^2 test function φ s.t. x_{01}^* is a local minimum of $w_0 - \varphi$. From the smooth-fit condition of the value function v_0 at the switching boundary, it follows that w_0 is C^1 at x_{01}^*. Hence $w_0'(x_{01}^*) = \varphi'(x_{01}^*)$. Moreover, since $w_0 = v_0$ is C^2 for $x < x_{01}^*$, we also have $\varphi''(x_{01}^*) \leq w_0''(x_{01}^{*-}) := \lim_{x \nearrow x_{01}^*} w''(x)$. Since $\rho w_0(x) - \mathcal{L}_0 w_0(x) \geq 0$ for $x < x_{01}^*$, we deduce by sending x to x_{01}^* :

$$\rho w_0(x_{01}^*) - \mathcal{L}_0 \varphi(x_{01}^*) \geq 0.$$

This implies the required viscosity supersolution inequality at $x = x_{01}^*$. By uniqueness, we conclude that $w_0 = v_0$, which proves that $\mathcal{S}_0^* = [x_{01}^*, \infty)$. $\square$

5 Main result and description of the solution

We give an explicit description of the structure of the solution to our control problem, which depends crucially on parameter values.

5.1 The case : $\hat{V}_0((1 - \lambda)g) \geq \frac{\mu_1}{\rho}$

Theorem 5.1 *Suppose that $\hat{V}_0((1 - \lambda)g) \geq \frac{\mu_1}{\rho}$. Then, we have $v_0(x) = \hat{V}_0(x)$ and $v_1(x) = \hat{V}_0(x + (1 - \lambda)g) = x + (1 - \lambda)g - x_0 + \frac{\mu_0}{\rho}$. It is optimal to never switch from regime 0 to regime 1. In regime 1, it is optimal to distribute all the surplus as dividends and to switch to regime 0.*

Proof. Under the condition of the theorem, and since $v_0 \geq \hat{V}_0$, we have $v_0((1 - \lambda)g) \geq \frac{\mu_1}{\rho}$. By Proposition 4.1, this implies $\mathcal{S}_1 = \mathcal{D}_1 = \mathbb{R}_+$. Recalling also the boundary data $v_1(0) = v_0((1 - \lambda)g)$, we get $v_1(x) = x + v_0((1 - \lambda)g)$ for $x \geq 0$. We next prove that the region $\mathcal{S}_0^*$ is empty. To see this, we have to prove that for $x \geq g$, $v_0(x) \geq v_1(x - g)$. Let us consider for $x \geq g$ the function $\theta(x) = v_0(x) - (x - g + v_0((1 - \lambda)g))$. Since $\lambda > 0$, we have $\theta(g) > 0$. Moreover, $\theta'(x) = v_0'(x) - 1 \geq 0$. Thus, $\theta(x) > 0$ for $x \geq g$ which is equivalent to $\mathcal{S}_0^* = \emptyset$. As a consequence, v_0 is a smooth solution of the variational inequality

$$\min \left[\rho v(x) - \mathcal{L}_0 v(x), v'(x) - 1 \right] = 0,$$

with initial condition $v(0) = 0$. By uniqueness, we deduce that $v_0 = \hat{V}_0$. To close the proof, it suffices to note that $\hat{V}_0((1 - \lambda)g) \geq \frac{\mu_1}{\rho}$ implies that $(1 - \lambda)g \geq \hat{x}_0$. Therefore, $v_0((1 - \lambda)g) = (1 - \lambda)g - x_0 + \frac{\mu_0}{\rho}$. $\square$

5.2 The case : $\hat{V}_0((1-\lambda)g) < \frac{\mu_1}{\rho}$

First observe that in this case, we have

$$v_0((1-\lambda)g) \quad < \quad \frac{\mu_1}{\rho}.$$

Indeed, on the contrary, from Theorem 5.1, we would get $v_0 = \hat{V}_0$, and so an obvious contradiction $\hat{V}_0((1-\lambda)g) \geq \frac{\mu_1}{\rho}$ with the considered case. From Proposition 4.1, we then have $\mathcal{S}_1 = \{0\}$ so that v_1 is the unique viscosity solution to

$$\min\left[\rho v_1 - \mathcal{L}_1 v_1, v_1' - 1\right] \quad = \quad 0, \quad x > 0,$$

with the boundary data $v_1(0) = v_0((1-\lambda)g)$. Therefore, v_1 is the firm value problem in technology $i = 1$ with liquidation value $v_0((1-\lambda)g)$:

$$v_1(x) \quad = \quad \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} v_0((1-\lambda)g)\right], \tag{5.1}$$

The form of v_1 is described in (4.3) with liquidation value $L = v_0((1-\lambda)g)$: we denote $x_1 = x_1^L$ the corresponding threshold.

Remark 5.1 Since v_1 and $\hat{V}_1$ are increasing with $v_1(x_1) = \hat{V}_1(\hat{x}_1) = \frac{\mu_1}{\rho}$, we have $x_1 \leq \hat{x}_1$.

Notice that the expression of v_1 is not completely explicit since we do not know at this stage the liquidation value $v_0((1-\lambda)g)$. The next result give an explicit solution when

$$\frac{\mu_1 - \mu_0}{\rho} \quad \leq \quad \hat{x}_1 + g - \hat{x}_0.$$

Theorem 5.2 *Suppose that*

$$\hat{V}_0((1-\lambda)g) \quad < \quad \frac{\mu_1}{\rho} \quad \leq \quad \frac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0. \tag{5.2}$$

Then $v_0 = \hat{V}_0$ and $v_1 = \hat{V}_1$. It is never optimal, once in regime $i = 0$, to switch to regime $i = 1$. In regime 1, it is optimal to switch to regime 0 at the threshold $x = 0$.

Proof. From Lemma 4.1 and Lemma 4.2, and recalling the variational inequalities (4.2) and (4.4), we see that $\hat{V}_0$ and $\hat{V}_1$ are viscosity solutions to

$$\min\left[\rho\hat{V}_0(x) - \mathcal{L}_0\hat{V}_0(x) , \ \hat{V}_0'(x) - 1 , \ \hat{V}_0(x) - \hat{V}_1(x-g)\right] \quad = \quad 0, \quad x > 0,$$

$$\min\left[\rho\hat{V}_1(x) - \mathcal{L}_1\hat{V}_1(x) , \ \hat{V}_1'(x) - 1 , \ \hat{V}_1(x) - \hat{V}_0(x+(1-\lambda)g)\right] \quad = \quad 0, \quad x > 0,$$

together with the boundary data $V_0(0^+) = 0$ and $\hat{V}_1(0^+) = \hat{V}_0((1-\lambda)g)$. By uniqueness to this system of variational inequalities, we conclude that $(v_0, v_1) = (\hat{V}_0, \hat{V}_1)$. □

In the sequel, we suppose that

$$\frac{\mu_1 - \mu_0}{\rho} \quad > \quad \hat{x}_1 + g - \hat{x}_0. \tag{5.3}$$

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From Proposition 4.2, the switching region from regime 0 to regime 1 has the form :

$$\mathcal{S}_0^* \;=\; \{x > 0 \;:\; v_0(x) \;=\; v_1(x-g)\} \;=\; [x_{01}^*, \infty),$$

for some $x_{01}^* \in [g, \infty)$. Moreover, since $x_1 \leq \hat{x}_1$ (see Remark 5.1), the above condition (5.3) implies $\frac{\mu_1 - \mu_0}{\rho} > x_1 + g - \hat{x}_0$. By same arguments as in Remark 4.3, there exists some $\bar{x}_{01} \geq g$ s.t.

$$\max\left(\hat{V}_0(x), v_1(x-g)\right) \;=\; \begin{cases} \hat{V}_0(x), & x \leq \bar{x}_{01} \\ v_1(x-g), & x > \bar{x}_{01} \end{cases}$$

Following [5], we introduce the pure stopping time problem

$$\bar{v}_0(x) \;=\; \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R_{\tau \wedge T_0}^{x,0}), v_1(R_{\tau \wedge T_0}^{x,0} - g)\right)\right], \tag{5.4}$$

where $\mathcal{T}$ denotes the set of stopping times valued in $[0, \infty]$. We also denote $\mathcal{E}_0$ the exercice region for $\bar{v}_0$:

$$\mathcal{E}_0 \;=\; \left\{x \geq 0 \;:\; \bar{v}_0(x) = \max\left(\hat{V}_0(x), v_1(x-g)\right)\right\}.$$

The next result shows that the original mixed singular/switching control problems may be reformulated as a coupled pure optimal stopping time and pure singular problem.

Theorem 5.3 *Suppose that*

$$\hat{V}_0((1-\lambda)g) \;<\; \frac{\mu_1}{\rho} \quad and \quad \frac{\mu_1 - \mu_0}{\rho} \;>\; \hat{x}_1 + g - \hat{x}_0. \tag{5.5}$$

Then, we have

$$v_0 \;=\; \bar{v}_0$$

and v_1 given by (5.1). Moreover,

$$\mathcal{E}_0 \;=\; \left\{0 \leq x < \bar{x}_{01} \;:\; v_0(x) = \hat{V}_0(x)\right\} \cup [x_{01}^*, \infty).$$

Proof. The proof follows along the lines of those of Theorem 3.1 in [5]. We will give only the road map of it in our context and omit the details.

Let us first note that the process $(e^{-\rho(t \wedge T_0)} v_0(R_{t \wedge T_0}^{x,0}))_{t \geq 0}$ is a supermartingale that dominates the function $\max(\hat{V}_0, v_1(. - g))$. Therefore, according to Snell envelope theory, we have $v_0 \geq \bar{v}_0$.

To prove the reverse inequality, it is enough to show that $\bar{v}_0' \geq 1$ (see Proposition 3.4 in [5]) and to use the uniqueness result of Theorem 3.1. To this end, we will precise the shape of the exercise region $\mathcal{E}_0$. According to Lemma 4.3 by Villeneuve [17], $\bar{x}_{01}$ does not belong to $\mathcal{E}_0$. Thus, the exercise region can be decomposed into two subregions

$$\mathcal{E}_{00} = \left\{x < \bar{x}_{01} : v_0(x) = \hat{V}_0(x)\right\}$$

and
$$\mathcal{E}_{01} = \{x > \bar{x}_{01} : v_0(x) = v_1(x - g)\}.$$

Two cases have to be considered :

Case (i). If the subregion $\mathcal{E}_{00}$ is empty, the optimal stopping problem defined by $\bar{v}_0$ can be solved explicitly, and we have, see [5] Lemma 3.3,

$$\bar{v}_0 = \begin{cases} \frac{e^{m_0^+ x} - e^{m_0^- x}}{e^{m_0^+ x_{01}^*} - e^{m_0^- x_{01}^*}} v_1(x_{01}^* - g) & x < x_{01}^* \\ v_1(x - g) & x \geq x_{01}^*. \end{cases}$$

The smooth-fit principle allows us to conclude that $\bar{v}_0' \geq 1$ since $v_1' \geq 1$.

Case (ii). If the subregion $\mathcal{E}_{00}$ is non empty, we can prove using the arguments of Proposition 3.5 and Lemma 3.4 in [5] that

$$\mathcal{E}_0 = [0, a] \cup [x_{01}^*, \infty),$$

with $a \geq \hat{x}_0$ and the value function $\bar{v}_0$ satisfies

$$\bar{v}_0(x) = A e^{m_0^+ x} + B e^{m_0^- x} \text{ for } x \in (a, x_{01}^*).$$

The smooth fit principle gives $\bar{v}_0'(a) = \hat{V}_0'(a) \geq 1$ and $\bar{v}_0'(x_{01}^*) = v_1'(x_{01}^* - g) \geq 1$. Clearly, $\bar{v}_0$ is convex in a right neighbourhood of a since $\hat{V}_0$ is linear at a. Therefore, if $\bar{v}_0$ remains convex on (a, x_{01}^*), the proof is over. If not, the second derivative of $\bar{v}_0$ given by $A(m_0^+)^2 e^{m_0^+ x} + B(m_0^-)^2 e^{m_0^- x}$ vanishes at most one time on (a, x_{01}^*), say in d. Hence,

$$1 = \bar{v}_0'(a) \leq (\bar{v}_0)'(x) \leq \bar{v}_0'(d) \text{ for } x \in (a, d),$$

and

$$1 \leq \bar{v}_0'(x_{01}^*) \leq \bar{v}_0'(x) \leq \bar{v}_0'(d) \text{ for } x \in (d, x_{01}^*),$$

which completes the proof. $\qquad\qquad\square$

Notice that the representation (5.1)-(5.4) of pure optimal singular and stopping problems for v_1 and v_0 is coupled, and so not easily computable. We decouple this representation by considering the sequence of pure optimal stopping and singular control problems, starting from $\hat{V}_1^{(0)} = \hat{V}_1$ and $\hat{V}_0^{(0)} = \hat{V}_0$:

$$\hat{V}_0^{(k)}(x) = \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \max\left(\hat{V}_0(R_{\tau \wedge T_0}^{x,0}), \hat{V}_1^{(k-1)}(R_{\tau \wedge T_0}^{x,0} - g) \right) \right], \quad k \geq 1,$$

$$\hat{V}_1^{(k)}(x) = \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{V}_0^{(k)}((1 - \lambda)g) \right], \quad k \geq 1.$$

The next result shows the convergence of this procedure.

Proposition 5.1 *Under the conditions (5.5) of Theorem 5.3, we have for all $x > 0$:*

$$\lim_{k \to \infty} \hat{V}_0^{(k)}(x) = v_0(x), \qquad \lim_{k \to \infty} \hat{V}_1^{(k)}(x) = v_1(x).$$

Proof. We will first prove that the increasing sequence $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ converges uniformly on every compact subsets of $\mathbb{R}_+$. To see this, we will apply Arzela-Ascoli Theorem by first proving the equi-continuity of the functions $\hat{V}_i^{(k)}$. Let us first remark that the functions $\hat{V}_1^{(k)}$ are Lipschitz continuous uniformly in k since they are concave with bounded first derivative (see Remark 4.2) independently of k. Let us also check that the functions $\hat{V}_0^{(k)}$ are Lipschitz continuous uniformly in k. Using the inequality $\max(a, b) - \max(c, d) \leq \max(a - c, b - d)$, and by setting

$$\Delta(x, y) = \max\left(\hat{V}_0(R_{\tau \wedge T_0}^{x,0}) - \hat{V}_0(R_{\tau \wedge T_0}^{y,0}), \hat{V}_1^{(k-1)}(R_{\tau \wedge T_0}^{x,0} - g) - \hat{V}_1^{(k-1)}(R_{\tau \wedge T_0}^{y,0} - g)\right),$$

we get by recalling also that $\hat{V}_0$ is Lipschitz (see Remark 4.2) :

$$
\begin{aligned}
|\hat{V}_0^{(k)}(x) - \hat{V}_1^{(k)}(y)| &\leq \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)}|\Delta(x, y)|\right] \\
&\leq K_0 \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)}|R_{\tau \wedge T_0}^{x,0} - R_{\tau \wedge T_0}^{y,0}|\right] \\
&\leq K_0|x - y| \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)}|\mu_0 \tau \wedge T_0 + \sigma W_{\tau \wedge T_0}|\right] \\
&\leq K_1|x - y|.
\end{aligned}
$$

According to Corollary 3.7, the set $\{(\hat{V}_0^{(k)}(x), \hat{V}_1^{(k)}(x)), k \in \mathbb{N}\}$ is bounded for every $x > 0$. Therefore, Arzela-Ascoli Theorem gives that the increasing sequence $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ converges uniformly on every compact subset of $\mathbb{R}_+$ to some $(\hat{V}_0^{(\infty)}, \hat{V}_1^{(\infty)})$.

On the other hand, for a fixed k, $(\hat{V}_0^{(k)}, \hat{V}_1^{(k)})$ is the unique viscosity solution with linear growth to the system of variational inequalities

$$F_0^{(k)}(u_0, u_0', u_0'') = \min\left(\rho u_0 - \mathcal{L}_0 u_0, u_0 - \max(\hat{V}_0, \hat{V}_1^{(k-1)}(. - g))\right) = 0,$$

$$F_1(u_1, u_1', u_1'') = \min\left(\rho u_1 - \mathcal{L}_1 u_1, u_1' - 1\right) = 0,$$

with initial condition $u_0(0) = 0$, $u_1(0) = \hat{V}_0^{(k)}((1 - \lambda)g)$.

Since $\hat{V}_1^{(k-1)}$ converges uniformly on every compact subset of $\mathbb{R}_+$, the Hamiltonian $F_0^{(k)}$ converges to F_0 on every compact subset of $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, with

$$F_0(u, u', u'') = \min\left(\rho u - \mathcal{L}_0 u, u - \max(\hat{V}_0, \hat{V}_1^\infty(. - g))\right) = 0.$$

According to standard stability results for viscosity solution, see for instance Lemma 6.2 page 73 in Fleming and Soner [9], the couple $(\hat{V}_0^{(\infty)}, \hat{V}_1^{(\infty)})$ is a viscosity solution of the system of variational inequalities

$$\min\left(\rho \hat{V}_0^\infty - \mathcal{L}_0 \hat{V}_0^\infty, \hat{V}_0^\infty - \max(\hat{V}_0, \hat{V}_1^\infty(. - g))\right) = 0, \tag{5.6}$$

$$\min\left(\rho \hat{V}_1^\infty - \mathcal{L}_1 \hat{V}_1^\infty, (\hat{V}_1^\infty)' - 1\right) = 0, \tag{5.7}$$

with initial conditions $\hat{V}_1^\infty(0) = \hat{V}_0^\infty((1-\lambda)g)$ and $\hat{V}_0^\infty(0) = 0$. By uniqueness to the system (5.6)-(5.7), we conclude that $\hat{V}_0^\infty = \bar{v}_0 = v_0$ and $\hat{V}_1^\infty = v_1$. $\qquad \square$

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We will close this section by describing the optimal strategy. According to Proposition 5.1, the value functions can be constructed recursively starting from $(\hat{V}_0, \hat{V}_1)$. Two cases have to be considered :

Case A : $\hat{V}_0^{(1)}((1-\lambda)g) = \hat{V}_0((1-\lambda)g)$. Then we have

$$
\begin{aligned}
\hat{V}_1^{(1)}(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{V}_0^{(1)}((1-\lambda)g)\right] \\
&= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{V}_0((1-\lambda)g)\right] \\
&= \hat{V}_1(x).
\end{aligned}
$$

Therefore, we deduce by a straightforward induction that the sequence $(\hat{V}_0^{(k)})_k$ is constant for $k \geq 1$ and the sequence $(\hat{V}_1^{(k)})_k$ is constant for $k \geq 0$. Therefore, we deduce from Proposition 5.1 that $v_0 = \hat{V}_0^{(1)}$ and $v_1 = \hat{V}_1$.

In regime 0, the optimal strategy consists in computing the optimal thresholds a and x_{01}^* associated to the optimal stopping problem $\hat{V}_0^{(1)}$. It is optimal to switch from regime 0 to regime 1 if the state process R^0 crosses the threshold x_{01}^* while it is optimal to pay dividends and therefore abandon the growth opportunity forever if R^0 falls below the threshold a. At the level a, it is too costly to wait reaching the threshold x_{01}^* even if the growth option is valuable. The shareholders prefer to receive today dividends than waiting a more profitable payment in the future.

In regime 1, the optimal strategy consists in paying dividends above $\hat{x}_1$ and switching to regime 0 only when the manager is being forced by its cash constraints.

Case B : $\hat{V}_0^{(1)}((1-\lambda)g) > \hat{V}_0((1-\lambda)g)$. Let us introduce the sequence

$$
\begin{aligned}
\hat{\theta}_0^{(k)}(x) &= \sup_{\tau \in \mathcal{T}} \mathbb{E}\left[e^{-\rho(\tau \wedge T_0)} \hat{\theta}_1^{(k-1)}(R_{\tau \wedge T_0}^{x,0} - g)\right], \quad k \geq 1, \\
\hat{\theta}_1^{(k)}(x) &= \sup_{Z \in \mathcal{Z}} \mathbb{E}\left[\int_0^{T_1^-} e^{-\rho t} dZ_t + e^{-\rho T_1} \hat{\theta}_0^{(k)}((1-\lambda)g)\right], \quad k \geq 1.
\end{aligned}
$$

starting from $\hat{\theta}_1^{(0)} = \hat{V}_1$ and $\hat{\theta}_0^{(0)} = \hat{V}_0$. Proceeding analogously as in the proof of Proposition 5.1, we can prove that the sequence $(\theta_0^{(k)}, \theta_1^{(k)})$ converges to $(\theta_0^{(\infty)}, \theta_1^{(\infty)})$ solution of the system of variational inequalities :

$$
\min\left(\rho \hat{\theta}_0^\infty - \mathcal{L}_0 \hat{\theta}_0^\infty, \hat{\theta}_0^\infty - \hat{\theta}_1^\infty(. - g)\right) = 0,
$$

$$
\min\left(\rho \hat{\theta}_1^\infty - \mathcal{L}_1 \hat{\theta}_1^\infty, (\hat{\theta}_1^\infty)' - 1\right) = 0,
$$

with initial conditions $\hat{\theta}_1^\infty(0) = \hat{\theta}_0^\infty((1-\lambda)g)$ and $\hat{\theta}_0^\infty(0) = 0$.

Note that the function $\hat{\theta}_0^\infty$ corresponds to the managerial decision to accumulate cash reserve at the expense of shareholder's dividend payment in order to invest in the modern technology.

The key feature of our model in **case B**, which has to be viewed as the analogue of Proposition 3.5 in [5], can be summarized as follows :

$\star$ If the net expected value evaluated at the threshold $\hat{x}_0$ dominates the firm value running under the old technology that is $\hat{\theta}_0^\infty(\hat{x}_0) > \hat{V}_0(\hat{x}_0)$ then the manager postpones dividend distribution in order to invest in the modern technology and thus $v_0 = \hat{\theta}_0^\infty$. Moreover, in regime 1, the manager always prefers to run under the modern technology until the cash process X_t^1 reaches zero forcing the manager to return back in regime 0 with the value $\hat{\theta}_0^\infty((1-\lambda)g)$, that is $v_1 = \hat{\theta}_1^\infty$.
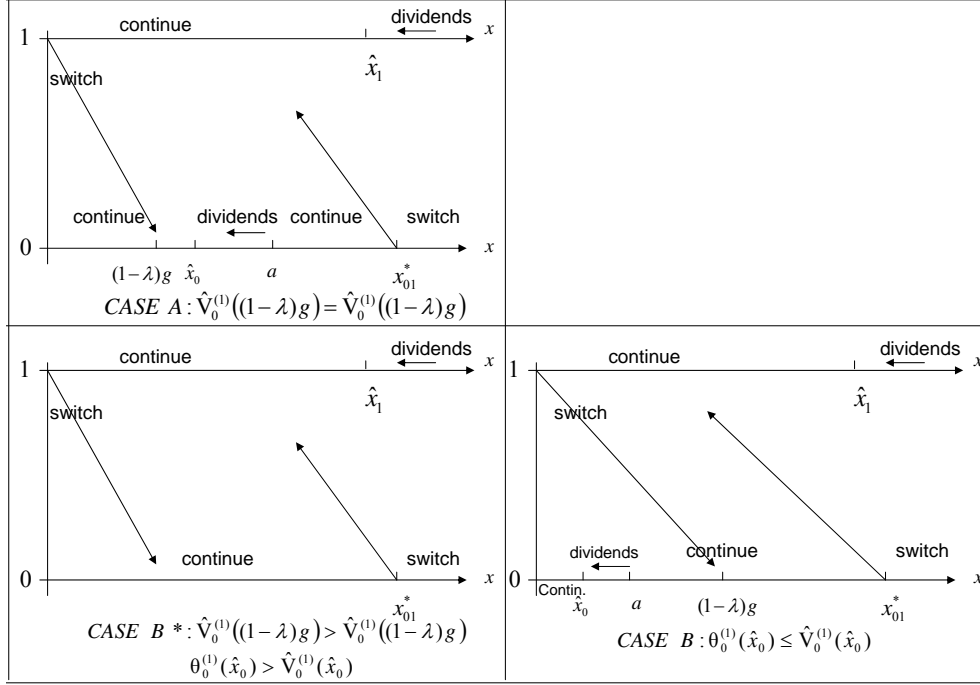
$\star$ If, on the contrary $\hat{\theta}_0^\infty(\hat{x}_0) \leq \hat{V}_0(\hat{x}_0)$ then the manager optimally ignores the strategy $\hat{\theta}_0^\infty$. Several situations can occur. For small values of the cash process $(X_t^0 \leq a)$, the manager optimally runs the firm under the old technology and pays out any surplus above $\hat{x}_0$ as dividends. For high values of the cash process $(X_t^0 \geq x_{01}^*)$, the manager switches optimally to regime one. For intermediary values of the cash process $(a \leq X_t^0 \leq x_{01}^*)$, there is an inaction region where the manager has not enough information to decide whether or not the investment is valuable.

We summarize all the results in Synthetic Table 1 and Figure 1.

<u>Synthetic table 1</u>

$\dfrac{\mu_1}{\rho} \leq \hat{V}_0((1-\lambda)g)$	$\hat{V}_0((1-\lambda)g) < \dfrac{\mu_1}{\rho} \leq \dfrac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0$	$\dfrac{\mu_1}{\rho} > \max\left(\hat{V}_0((1-\lambda)g), \dfrac{\mu_0}{\rho} + \hat{x}_1 + g - \hat{x}_0\right)$
$v_0(x) = \hat{V}_0(x)$ $v_1(x) = x + (1-\lambda)g - \hat{x}_0 + \dfrac{\mu_0}{\rho}$	$v_0(x) = \hat{V}_0(x)$ $v_1(x) = \hat{V}_1(x)$	$v_0(x) = \hat{V}_0^\infty(x)$ $v_1(x) = \hat{V}_1^\infty(x)$
		See figure 1

Figure 1

CASE A : $\hat{V}_0^{(1)}\big((1-\lambda)g\big) = \hat{V}_0^{(1)}\big((1-\lambda)g\big)$

CASE B * : $\hat{V}_0^{(1)}\big((1-\lambda)g\big) > \hat{V}_0^{(1)}\big((1-\lambda)g\big)$
$\theta_0^{(1)}(\hat{x}_0) > \hat{V}_0^{(1)}(\hat{x}_0)$

CASE B : $\theta_0^{(1)}(\hat{x}_0) \le \hat{V}_0^{(1)}(\hat{x}_0)$

Appendix A : Proof of Theorem 3.1

We divide the proof in several steps.

Proof of the continuity of v_1 on $(0,\infty)$.
We prove that v_1 is continuous at any $y > 0$. We fix an arbitrary small $\varepsilon > 0$. Applying the dynamic programming principle **(DP)** to v_1, there exists a control $\alpha = (Z,(\tau_n)_{n\geq1}) \in \mathcal{A}$ s.t.

$$
\begin{aligned}
v_1(y) - \frac{\varepsilon}{3} &\leq \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t}dZ_t + e^{-\rho(\tau_1\wedge T)}\left(v_1(X_T^{y,1})1_{T<\tau_1} + v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right)\right], \\
&= \mathbb{E}\left[\int_0^{(\tau_1\wedge T)^-} e^{-\rho t}dZ_t + e^{-\rho(\tau_1\wedge T)}v_0(X_{\tau_1}^{y,1})1_{\tau_1\leq T}\right], \quad\quad\quad\text{(A.1)}
\end{aligned}
$$

with $T = T^{y,1,\alpha}$ the bankruptcy time of the process $X^{y,1,\alpha}$, and since $v_1(X_T^{y,1}) = 0$ for $X_T^{y,1} < 0$.

For any $0 < x < y$, let $\theta = T^{y-x,1,\alpha}$ be the bankruptcy time of the process $X^{y-x,1,\alpha}$. We notice that $\theta \leq T$ and $X^{y-x,1,\alpha} = X^{y,1,\alpha} - x$ for all $0 < t < \theta \leq T$. Applying the dynamic programming principle **(DP)**, we then have

$$
\begin{aligned}
v_1(y-x) &\geq \mathbb{E}\left[\int_0^{(\theta\wedge\tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho(\theta\wedge\tau_1)}\left(v_1(X_\theta^{y-x,1})1_{\theta<\tau_1} + v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta}\right)\right] \\
&\geq \mathbb{E}\left[\int_0^{(\theta\wedge\tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho(\theta\wedge\tau_1)}v_0(X_{\tau_1}^{y-x,1})1_{\tau_1\leq\theta}\right]
\end{aligned}
$$

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$$\geq \mathbb{E}\left[\int_0^{(\tau_1 \wedge T)^-} e^{-\rho t}dZ_t + e^{-\rho(\tau_1 \wedge T)}v_0(X^{y,1}_{\tau_1})1_{\tau_1 \leq T}\right] - \mathbb{E}\left[\int_{\theta \wedge \tau_1}^{(T \wedge \tau_1)^-} e^{-\rho t}dZ_t\right]$$

$$+ \mathbb{E}\left[e^{-\rho(\theta \wedge \tau_1)}v_0(X^{y-x,1}_{\tau_1})1_{\tau_1 \leq \theta} - e^{-\rho(T \wedge \tau_1)}v_0(X^{y,1}_{\tau_1})1_{\tau_1 \leq T}\right] \tag{A.2}$$

Notice that $\theta \to T$ as x goes to zero. Hence, by the continuity of v_0 and the dominated convergence theorem, one can find $0 < \delta_1 < y$ s.t. for $0 < x < \delta_1$:

$$\mathbb{E}\left[e^{-\rho(\theta \wedge \tau_1)}v_0(X^{y-x,1}_{\tau_1})1_{\tau_1 \leq \theta} - e^{-\rho(T \wedge \tau_1)}v_0(X^{y,1}_{\tau_1})1_{\tau_1 \leq T}\right] \geq -\frac{\varepsilon}{3}. \tag{A.3}$$

We also have

$$-\mathbb{E}\left[\int_{\theta \wedge \tau_1}^{(T \wedge \tau_1)^-} e^{-\rho t}dZ_t\right] \geq -\mathbb{E}\left[Z_{(T \wedge \tau_1)^-} - Z_{\theta \wedge \tau_1}\right]$$

From the dominated convergence theorem, one can find $0 < \delta_2 < y$ s.t. for $0 < x < \delta_2$:

$$-\mathbb{E}\left[\int_{\theta \wedge \tau_1}^{(T \wedge \tau_1)^-} e^{-\rho t}dZ_t\right] \geq -\frac{\varepsilon}{3} \tag{A.4}$$

Plugging inequalities (A.3) and (A.4) into (A.2), we obtain for $0 < x < \min\{\delta_1, \delta_2\}$

$$v_1(y-x) \geq \mathbb{E}\left[\int_0^{(\tau_1 \wedge T)^-} e^{-\rho t}dZ_t + e^{-\rho(\tau_1 \wedge T)}v_0(X^{y,1}_{\tau_1})1_{\tau_1 \leq T}\right] - \frac{2\varepsilon}{3}$$

Using the inequality (A.1), and recalling that v_1 is nondecreasing, this implies

$$0 \leq v_1(y) - v_1(y-x) \leq \varepsilon,$$

which shows the left-continuity of v_1. By proceeding exactly in the same manner, we may obtain for a given $y > 0$ and any arbitrary $\varepsilon > 0$, the existence of $0 < \delta < y$ such that for all $0 < x < \delta$,

$$0 \leq v_1(y+x) - v_1(y) \leq \varepsilon$$

which shows the right-continuity of v_1. $\qquad\square$

Proof of supersolution property.
Fix $i \in \{0,1\}$. Consider any $\overline{x} \in (0, \infty)$ and $\varphi \in C^2(0, \infty)$ s.t. $\overline{x}$ is a minimum of $v_i - \varphi$ in a neighbourhood $B_\varepsilon(\overline{x}) = (\overline{x} - \varepsilon, \overline{x} + \varepsilon)$ of $\overline{x}$, $\overline{x} > \varepsilon > 0$, and $v_i(\overline{x}) = \varphi(\overline{x})$. First, by considering the admissible control $\overline{\alpha} = (\overline{Z}, \overline{\tau}_n, n \geq 1)$ where we decide to take immediate switching control, i.e. $\overline{\tau}_1 = 0$, while deciding not to distribute any dividend $\overline{Z} = 0$, we obtain

$$v_i(\overline{x}) \geq v_{i-1}(\overline{x} - g_{i,1-i}). \tag{A.5}$$

On the other hand, let us consider the admissible control $\hat{\alpha} = (\hat{Z}, \hat{\tau}_n, n \geq 1)$ where we decide to never switch regime, while the dividend policy is defined by $\hat{Z}_t = \eta$ for $t \geq 0$,

with $0 \leq \eta \leq \varepsilon$. Define the exit time $\tau_\varepsilon = \inf\{t \geq 0, X_t^{\bar{x},i} \notin \overline{B}_\varepsilon(\bar{x})\}$. We notice that $\tau_\varepsilon < T$. From the dynamic programming principle **(DP)**, we have

$$
\begin{aligned}
\varphi(\bar{x}) = v(\bar{x}) &\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t} d\hat{Z}_t + e^{-\rho(\tau_\varepsilon \wedge h)} v_i(X_{\tau_\varepsilon \wedge h}^{\bar{x},i})\right] \\
&\geq \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t} d\hat{Z}_t + e^{-\rho(\tau_\varepsilon \wedge h)} \varphi(X_{\tau_\varepsilon \wedge h}^{\bar{x},i})\right].
\end{aligned}
\tag{A.6}
$$

Applying Itô's formula to the process $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $\tau_\varepsilon \wedge h$, and taking the expectation, we obtain

$$
\begin{aligned}
\mathbb{E}\left[e^{-\rho(\tau_\varepsilon \wedge h)}\varphi(X_{\tau_\varepsilon \wedge h}^{\bar{x},i})\right] &= \varphi(\bar{x}) + \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t}(-\rho\varphi + \mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt\right] \\
&+ \mathbb{E}\left[\sum_{0 \leq t \leq \tau_\varepsilon \wedge h} e^{-\rho t}[\varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i})]\right].
\end{aligned}
\tag{A.7}
$$

Combining relations (A.6) and (A.7), we have

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t}(\rho\varphi - \mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt\right] &- \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t} d\hat{Z}_t\right] \\
-\mathbb{E}\left[\sum_{0 \leq t \leq \tau_\varepsilon \wedge h} e^{-\rho t}[\varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i})]\right] &\geq 0.
\end{aligned}
\tag{A.8}
$$

$\star$ Take first $\eta = 0$. We then observe that X is continuous on $[0, \tau_\varepsilon \wedge h]$ and only the first term of the relation (A.8) is non zero. By dividing the above inequality by h with $h \to 0$, we conclude that

$$
(\rho\varphi - \mathcal{L}_i\varphi)(\bar{x}) \geq 0.
\tag{A.9}
$$

$\star$ Take now $\eta > 0$ in (A.8). We see that $\hat{Z}$ jumps only at $t = 0$ with size η, so that

$$
\mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge h} e^{-\rho t}(\rho\varphi - \mathcal{L}_i\varphi)(X_t^{\bar{x},i})dt\right] - \eta - (\varphi(\bar{x} - \eta) - \varphi(\bar{x})) \geq 0.
\tag{A.10}
$$

By sending $h \to 0$, and then dividing by η and letting $\eta \to 0$, we obtain

$$
\varphi'(\bar{x}) - 1 \geq 0.
\tag{A.11}
$$

This proves the required supersolution property

$$
\min\left[(\rho\varphi - \mathcal{L}_i\varphi)(\bar{x}), \ \varphi'(\bar{x}) - 1, v_i(\bar{x}) - v_{1-i}(\bar{x} - g_{i,1-i})\right] \geq 0.
\tag{A.12}
$$

Proof of the subsolution property.

We prove the subsolution property by contradiction. Suppose that the claim is not true. Then, there exists $\bar{x} > 0$ and a neighbourhood $B_\varepsilon(\bar{x}) = (\bar{x} - \varepsilon, \bar{x} + \varepsilon)$ of $\bar{x}$, $\bar{x} > \varepsilon > 0$, a C^2 function φ with $(\varphi - v_*)(\bar{x}) = 0$ and $\varphi \geq v_i$ on $\overline{B}_\varepsilon(\bar{x})$, and $\eta > 0$, s.t. for all $x \in \overline{B}_\varepsilon(\bar{x})$:

$$
\begin{aligned}
\rho\varphi(x) - \mathcal{L}_i\varphi(x) &> \eta, \tag{A.13} \\
\varphi'(x) - 1 &> \eta, \tag{A.14} \\
v_i(x) - v_{i-1}(x - g_{i,1-i}) &> \eta. \tag{A.15}
\end{aligned}
$$

For any admissible control $\alpha = (Z, \tau_n, n \geq 1)$, consider the exit time $\tau_\varepsilon = \inf\{t \geq 0, X_t^{\bar{x},i} \notin \overline{B}_\varepsilon(\bar{x})\}$. We notice that $\tau_\varepsilon < T$. Applying Itô's formula to the process $e^{-\rho t}\varphi(X_t^{\bar{x},i})$ between 0 and $(\tau_\varepsilon \wedge \tau_1)^-$, and by noting that before $(\tau_\varepsilon \wedge \tau_1)^-$, $X^{x,i}$ stays in regime i and in the ball $\overline{B}_\varepsilon(\bar{x})$, we obtain

$$
\begin{aligned}
\mathbb{E}\left[e^{-\rho(\tau_\varepsilon \wedge \tau_1)^-}\varphi(X_{(\tau_\varepsilon \wedge \tau_1)^-}^{\bar{x},i})\right] &= \varphi(\bar{x}) + \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}(-\rho\varphi(X_t^{\bar{x},i}) + \mathcal{L}_i\varphi(X_t^{\bar{x},i}))dt\right] \\
&\quad - \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}\varphi'(X_t^{\bar{x},i})dZ_t^c\right] \\
&\quad + \mathbb{E}\left[\sum_{0 \leq t < \tau_\varepsilon \wedge \tau_1} e^{-\rho t}[\varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i})]\right]. \quad\quad (A.16)
\end{aligned}
$$

From Taylor's formula and (A.14), and noting that $\Delta X_t^{\bar{x},i} = -\Delta Z_t$ for all $0 \leq t < \tau_\varepsilon \wedge \tau_1$, we have

$$
\begin{aligned}
\varphi(X_t^{\bar{x},i}) - \varphi(X_{t^-}^{\bar{x},i}) &= \Delta X_t^{\bar{x},i} \, \varphi'(X_t^{\bar{x},i} + z\Delta X_t^{\bar{x},i}) \\
&\leq -(1+\eta)\Delta Z_t \quad\quad (A.17)
\end{aligned}
$$

Plugging the relations (A.13), (A.14), and (A.17) into (A.16), we obtain

$$
\begin{aligned}
v_i(\bar{x}) = \varphi(\bar{x}) &\geq \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho(\tau_\varepsilon \wedge \tau_1)^-}\varphi(X_{(\tau_\varepsilon \wedge \tau_1)^-}^{\bar{x},i})\right] \\
&\quad + \eta\left(\mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dZ_t\right]\right). \\
&\geq \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dZ_t + e^{-\rho\tau_\varepsilon^-}\varphi(X_{\tau_\varepsilon^-}^{\bar{x},i})\mathbb{1}_{\tau_\varepsilon < \tau_1} + e^{-\rho\tau_1^-}\varphi(X_{\tau_1^-}^{\bar{x},i})\mathbb{1}_{\tau_1 \leq \tau_\varepsilon}\right] \\
&\quad + \eta\left(\mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t}dZ_t\right]\right). \quad\quad (A.18)
\end{aligned}
$$

Notice that while $X_{\tau_\varepsilon^-}^{\bar{x},i} \in \overline{B}_\varepsilon(\bar{x})$, $X_{\tau_\varepsilon}^{\bar{x},i}$ is either on the boundary $\partial\overline{B}_\varepsilon(\bar{x})$ or out of $\overline{B}_\varepsilon(\bar{x})$. However, there is some random variable γ valued in $[0,1]$ s.t.

$$
\begin{aligned}
X^{(\gamma)} &= X_{\tau_\varepsilon^-}^{\bar{x},i} + \gamma\Delta X_{\tau_\varepsilon}^{\bar{x},i} \\
&= X_{\tau_\varepsilon^-}^{\bar{x},i} - \gamma\Delta Z_{\tau_\varepsilon} \in \partial\overline{B}_\varepsilon(\bar{x}).
\end{aligned}
$$

Then similarly as in (A.17), we have

$$
\varphi(X^{(\gamma)}) - \varphi(X_{\tau_\varepsilon^-}^{\bar{x},i}) \leq -\gamma(1+\eta)\Delta Z_{\tau_\varepsilon}. \quad\quad (A.19)
$$

Noting that $X^{(\gamma)} = X_{\tau_\varepsilon}^{\bar{x},i} + (1-\gamma)\Delta Z_{\tau_\varepsilon}$, we have

$$
v_i(X^{(\gamma)}) \geq v_i(X_{\tau_\varepsilon}^{\bar{x},i}) + (1-\gamma)\Delta Z_{\tau_\varepsilon}. \quad\quad (A.20)
$$

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Recalling that $\varphi(X^{(\gamma)}) \geq v_i(X^{(\gamma)})$, inequalities (A.19) and (A.20) imply

$$\varphi(X_{\tau_\varepsilon^-}) \geq v_i(X_{\tau_\varepsilon}^{\bar{x},i}) + (1 + \gamma\eta)\Delta Z_{\tau_\varepsilon}$$

Plugging into (A.18) and using (A.15), we have

$$
\begin{aligned}
v_i(\bar{x}) \quad \geq \quad & \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho\tau_\varepsilon} v_i(X_{\tau_\varepsilon}^{\bar{x},i}) 1_{\tau_\varepsilon < \tau_1} + e^{-\rho\tau_1} v_{1-i}(X_{\tau_1}^{\bar{x},i}) 1_{\tau_1 \leq \tau_\varepsilon}\right] \\
& + \eta\, \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge \tau_1} e^{-\rho t} dt + \int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho\tau_1} 1_{\tau_1 \leq \tau_\varepsilon} + \gamma e^{-\rho\tau_\varepsilon \wedge \tau_1} \Delta Z_{\tau_\varepsilon} 1_{\tau_\varepsilon < \tau_1}\right] \\
& + \mathbb{E}\left[e^{-\rho\tau_\varepsilon} \Delta Z_{\tau_\varepsilon} 1_{\tau_\varepsilon < \tau_1}\right]. \tag{A.21}
\end{aligned}
$$

We now claim that there exists a constant $c_0 > 0$ such that for any admissible control

$$\mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge \tau_1} e^{-\rho t} dt + \int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t} dZ_t + e^{-\rho\tau_1} 1_{\tau_1 \leq \tau_\varepsilon} + \gamma e^{-\rho\tau_\varepsilon \wedge \tau_1} \Delta Z_{\tau_\varepsilon} 1_{\tau_\varepsilon < \tau_1}\right] \geq c_0. \tag{A.22}$$

The C^2 function $\psi(x) = c_0\left[1 - \frac{(x-\bar{x})^2}{\varepsilon^2}\right]$, with

$$0 < c_0 \leq \min\left\{\left(\rho + \frac{2}{\varepsilon}\mu_i + \frac{1}{\varepsilon^2}\sigma^2\right)^{-1}, \frac{\varepsilon}{2}\right\}.$$

satisfies

$$\left\{\begin{array}{rcll} \min\left\{-\rho\psi + \mathcal{L}_i\psi + 1, 1 - \psi', -\psi + 1\right\} & \geq & 0, & \text{on } \overline{B}_\varepsilon(\bar{x}), \\ \psi & = & 0, & \text{on } \partial\overline{B}_\varepsilon(\bar{x}), . \end{array}\right. \tag{A.23}$$

Applying Itô's formula, we then obtain

$$
\begin{aligned}
0 < c_0 = \psi(\bar{x}) \quad \leq \quad & \mathbb{E}\left[e^{-\rho(\tau_\varepsilon \wedge \tau_1)} \psi(X_{(\tau_\varepsilon \wedge \tau_1)^-}^{\bar{x},i})\right] \\
& + \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge \tau_1} e^{-\rho t} dt\right] + \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t} dZ_t\right]. \tag{A.24}
\end{aligned}
$$

Noting that $\psi'(x) \leq 1$, we have

$$\psi(X_{\tau_\varepsilon^-}^{\bar{x},i}) - \psi(X^{(\gamma)}) \quad \leq \quad (X_{\tau_\varepsilon^-}^{\bar{x},i} - X^{(\gamma)}) = \gamma\Delta Z_{\tau_\varepsilon}$$

Plugging into (A.24), we obtain

$$
\begin{aligned}
0 < c_0 \quad \leq \quad & \mathbb{E}\left[e^{-\rho\tau_1} \psi(X_{\tau_1^-}^{\bar{x},i}) 1_{\tau_1 \leq \tau_\varepsilon}\right] + \mathbb{E}\left[\int_0^{\tau_\varepsilon \wedge \tau_1} e^{-\rho t} dt\right] \\
& + \mathbb{E}\left[\int_0^{(\tau_\varepsilon \wedge \tau_1)^-} e^{-\rho t} dZ_t\right] + \mathbb{E}\left[\gamma e^{-\rho\tau_\varepsilon} \Delta Z_{\tau_\varepsilon} 1_{\tau_\varepsilon < \tau_1}\right]. \tag{A.25}
\end{aligned}
$$

Since $\psi(x) \leq 1$ for all $x \in B_\varepsilon(\bar{x})$, this proves the claim (A.22).

Finally, by taking the supremum over all admissible control α, and using the dynamic programming principle **(DP)**, (A.21) implies $v_i(\overline{x}) \geq v_i(\overline{x}) + \eta c_0$, which is a contradiction. Thus we obtain the required viscosity subsolution property :

$$\min\left[(\rho\varphi - \mathcal{L}_i\varphi)(\overline{x}), \ \varphi'(\overline{x}) - 1, v_i(\overline{x}) - v_{i-1}(\overline{x} - g_{i,i-1})\right] \leq 0. \tag{A.26}$$

$\square$

Proof of the uniqueness property.

Suppose u_i, $i = 0, 1$, are continuous viscosity subsolutions to the system of variational inequalities on $(0, \infty)$, and w_i, $i = 0, 1$, continuous viscosity supersolutions to the system of variational inequalities on $(0, \infty)$, satisfying the boundary conditions $u_i(0^+) \leq w_i(0^+)$, $i = 0, 1$, and the linear growth condition :

$$|u_i(x)| + |w_i(x)| \leq C_1 + C_2 x, \quad \forall x \in (0, \infty), \ i = 1, 2, \tag{A.27}$$

for some positive constants C_1 and C_2. We want to prove that

$$u_i \leq w_i, \quad \text{on} \quad (0, \infty), \ i = 0, 1,$$

Step 1. We first construct strict supersolutions to the system with suitable perturbations of w_i, $i = 0, 1$. We set

$$h_i(x) = A_i + B_i x + C x^2, \quad x > 0,$$

where

$$
\begin{aligned}
A_0 &= \frac{\mu_1 B_1 + C\sigma^2 + 1}{\rho} + \frac{C}{4}\left(\frac{B_1}{C} - 2\frac{\mu_1}{\rho}\right)^2 + \frac{C}{4}\left(\frac{B_0}{C} - 2\frac{\mu_0}{\rho}\right)^2 + w_0(0^+) + w_1(0^+), \\
A_1 &= A_0 + \frac{3}{2}g + \frac{g}{\lambda}, \\
B_0 &= 3, \ B_1 = 2 + \frac{2}{\lambda}, \\
C &= \frac{1}{\lambda g}.
\end{aligned}
$$

We then define for all $\gamma \in (0, 1)$, the continuous functions on $(0, \infty)$ by :

$$w_i^\gamma = (1 - \gamma)w_i + \gamma h_i, \quad i = 0, 1.$$

We then see that for all $\gamma \in (0, 1), i = 0, 1$:

$$
\begin{aligned}
w_i^\gamma(x) - w_{1-i}^\gamma(x - g_{i,1-i}) &= (1 - \gamma)\left[w_i(x) - w_{1-i}(x - g_{i,1-i})\right] + \gamma\left[h_i(x) - h_{1-i}(x - g_{i,1-i})\right], \\
&\geq \gamma\left[(2Cg_{i,1-i} + B_i - B_{1-i})x + A_i - A_{1-i} - Cg_{i,1-i}^2 + B_{1-i}g_{i,1-i}\right], \\
&\geq \gamma\frac{g}{2}, \quad i = 0, 1. \tag{A.28}
\end{aligned}
$$

Furthermore, we also easily obtain

$$h_i'(x) - 1 = B_i + 2Cx - 1 \geq 1. \tag{A.29}$$

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A straight calculation will also provide us with the last required inequality, i.e.

$$\rho h_i(x) - \mathcal{L}_i h_i(x) \geq 1. \tag{A.30}$$

Combining (A.28), (A.29), and (A.30), this shows that w_i^γ is a strict supersolution of the system : for $i = 0, 1$, we have on $(0, \infty)$

$$\min\left[\rho w_i^\gamma(x) - \mathcal{L}_i w_i^\gamma(x), \, w_i^{\gamma\prime}(x) - 1, \, w_i^\gamma(x) - w_{i-1}^\gamma(x - g_{i,1-i})\right] \geq \gamma \min\{1, \frac{g}{2}\} = \delta. \tag{A.31}$$

Step 2. In order to prove the comparison principle, it suffices to show that for all $\gamma \in (0, 1)$:

$$\max_{i \in \{0,1\}} \sup_{(0,+\infty)} (u_i - w_i^\gamma) \leq 0,$$

since the required result is obtained by letting γ to 0. We argue by contradiction and suppose that there exist some $\gamma \in (0, 1)$ and $i \in \{0, 1\}$, s.t.

$$\theta := \max_{j \in \{0,1\}} \sup_{(0,+\infty)} (u_j - w_j^\gamma) = \sup_{(0,+\infty)} (u_i - w_i^\gamma) > 0. \tag{A.32}$$

Notice that $u_i(x) - w_i^\gamma(x)$ goes to $-\infty$ when x goes to infinity. We also have $\lim_{x \to 0^+} u_i(x) - \lim_{x \to 0^+} w_i^\gamma(x) \leq \gamma(\lim_{x \to 0^+} w_i(x) - A_i) \leq 0$. Hence, by continuity of the functions u_i and w_i^γ, there exists $x_0 \in (0, \infty)$ s.t.

$$\theta = u_i(x_0) - w_i^\gamma(x_0).$$

For any $\varepsilon > 0$, we consider the functions

$$\begin{aligned}
\Phi_\varepsilon(x, y) &= u_i(x) - w_i^\gamma(y) - \phi_\varepsilon(x, y), \\
\phi_\varepsilon(x, y) &= \frac{1}{4}|x - x_0|^4 + \frac{1}{2\varepsilon}|x - y|^2,
\end{aligned}$$

for all $x, y \in (0, \infty)$. By standard arguments in comparison principle, the function Φ_ε attains a maximum in $(x_\varepsilon, y_\varepsilon) \in (0, \infty)^2$, which converges (up to a subsequence) to (x_0, x_0) when ε goes to zero. Moreover,

$$\lim_{\varepsilon \to 0} \frac{|x_\varepsilon - y_\varepsilon|^2}{\varepsilon} = 0. \tag{A.33}$$

Applying Theorem 3.2 in [4], we get the existence of $M_\varepsilon, N_\varepsilon \in \mathbb{R}$ such that:

$$\begin{aligned}
(p_\varepsilon, M_\varepsilon) &\in J^{2,+} u_i(x_\varepsilon), \\
(q_\varepsilon, N_\varepsilon) &\in J^{2,-} w_i^\gamma(y_\varepsilon),
\end{aligned}$$

and

$$\begin{pmatrix} M_\varepsilon & 0 \\ 0 & N_\varepsilon \end{pmatrix} \leq D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) + \varepsilon(D^2\phi(x_\varepsilon, y_\varepsilon))^2, \tag{A.34}$$

where

$$p_\varepsilon = D_x\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3,$$

$$q_\varepsilon = -D_y\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon),$$

$$D^2\phi_\varepsilon(x_\varepsilon, y_\varepsilon) = \begin{pmatrix} 3(x_\varepsilon - x_0)^2 + \frac{1}{\varepsilon} & -\frac{1}{\varepsilon} \\ -\frac{1}{\varepsilon} & \frac{1}{\varepsilon} \end{pmatrix}.$$

By writing the viscosity subsolution property of u_i and the viscosity supersolution property (A.31) of w_i^γ, we have the following inequalities:

$$\min\left\{\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right)\mu_i - \frac{1}{2}\sigma^2 M_\varepsilon,\right.$$

$$\left.\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1, \ u_i((x_\varepsilon) - u_{1-i}(x_\varepsilon - g_{i,1-i})\right\} \leq 0, \qquad (A.35)$$

$$\min\left\{\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\mu_i - \frac{1}{2}\sigma^2 N_\varepsilon, \ \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1,\right.$$

$$\left. w_i^\gamma(y_\varepsilon) - w_{i-1}^\gamma(x_\varepsilon - g_{i,1-i})\right\} \geq \delta. \qquad (A.36)$$

We then distinguish the following three cases :

$\star$ *Case 1* : $u_i(x_\varepsilon) - u_{1-i}(x_\varepsilon - g_{i,1-i}) \leq 0$ in (A.35).
From the continuity of u_i and by sending $\varepsilon \to 0$, this implies

$$u_i(x_0) \leq u_{1-i}(x_0 - g_{i,1-i}). \qquad (A.37)$$

On the other hand, from (A.36), we also have

$$w_i^\gamma(y_\varepsilon) - w_{i-1}^\gamma(x_\varepsilon - g_{i,1-i}) \geq \delta,$$

which implies, by sending $\varepsilon \to 0$ and using the continuity of w_i :

$$w_i^\gamma(x_0) \geq w_{i-1}^\gamma(x_0 - g_{i,1-i}) + \delta. \qquad (A.38)$$

Combining (A.37) and (A.38), we obtain

$$\theta = u_i(x_0) - w_i^\gamma(x_0) \leq u_{1-i}(x_0 - g_{i,1-i}) - w_{i-1}^\gamma(x_0 - g_{i,1-i}) - \delta,$$
$$\leq \theta - \delta,$$

which is a contradiction.

$\star$ *Case 2* : $\left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3\right) - 1 \leq 0$ in (A.35)
Notice that by (A.36), we have

$$\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) - 1 \geq \delta,$$

which implies in this case

$$(x_\varepsilon - x_0)^3 \leq -\delta.$$

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By sending ε to zero, we obtain again a contradiction.

$\star$ *Case 3* : $\rho u_i(x_\varepsilon) - \left(\frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon) + (x_\varepsilon - x_0)^3 \right) \mu_i - \frac{1}{2}\sigma^2 M_\varepsilon \leq 0$ in (A.35)
From (A.36), we have

$$\rho w_i^\gamma(y_\varepsilon) - \frac{1}{\varepsilon}(x_\varepsilon - y_\varepsilon)\mu_i - \frac{1}{2}\sigma^2 N_\varepsilon \geq \delta,$$

which implies in this case

$$\rho \left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon) \right) - \mu_i(x_\varepsilon - x_0)^3 - \frac{1}{2}\sigma^2(M_\varepsilon - N_\varepsilon) \leq -\delta, \tag{A.39}$$

From (A.34), we have

$$\frac{1}{2}\sigma^2(M_\varepsilon - N_\varepsilon) \leq \frac{3}{2}\sigma^2(x_\varepsilon - x_0)^2 \left[1 + 3\varepsilon(x_\varepsilon - x_0) \right].$$

Plugging it into (A.39) yields

$$\rho \left(u_i(x_\varepsilon) - w_i^\gamma(y_\varepsilon) \right) \leq \mu_i(x_\varepsilon - x_0)^3 + \frac{3}{2}\sigma^2(x_\varepsilon - x_0)^2 \left[1 + 3\varepsilon(x_\varepsilon - x_0) \right] - \delta.$$

By sending ε to zero and using the continuity of u_i and w_i^γ, we obtain the required contradiction : $\rho\theta \leq -\delta < 0$. This ends the proof. $\qquad\square$

Appendix B : Proof of Proposition 3.3

C^1 property

We prove in three steps that for a given $i \in 0, 1$, v_i is a C^1 function on $(0, \infty)$. Notice first that since v_i is a strictly nondecreasing continuous function on $(0, \infty)$, it admits a nonnegative left and right derivative $v_i'^-(x)$ and $v_i'^+(x)$ for all $x > 0$.

Step 1. We start by proving that $v_i'^-(x) \geq v_i'^+(x)$ for all $x \in (0, \infty)$.
Suppose on the contrary that there exists some x_0 such that $v_i'^-(x_0) < v_i'^+(x_0)$. Take then some $q \in (v_i'^-(x), v_i'^+(x))$, and consider the function

$$\varphi(x) = v_i(x_0) + q(x - x_0) + \frac{1}{2\varepsilon}(x - x_0)^2,$$

with $\varepsilon > 0$. Then x_0 is a local minimum of $v_i - \varphi_i$, with $\varphi'(x_0) = q$ and $\varphi''(x_0) = \frac{1}{\varepsilon}$. Therefore, we get the required contradiction by writing the supersolution inequality :

$$0 \leq \rho v_i(x_0) - \mu_i\varphi'(x_0) - \frac{\sigma^2}{2}\varphi''(x_0) = \rho v_i(x_0) - \mu_i q - \frac{\sigma^2}{2\varepsilon},$$

and choosing ε small enough.

Step 2. We now prove that for $i = 0, 1$, v_i is C^1 on $(0, \infty) \backslash \mathcal{S}_i$.
Suppose there exists some $x_0 \notin \mathcal{S}_i$ s.t. $v_i'^-(x_0) > v_i'^+(x_0)$. We then fix some $q \in (v_i'^+(x_0), v_i'^-(x_0))$ and consider the function

$$\varphi(x) = v_i(x_0) + q(x - x_0) - \frac{1}{2\varepsilon}(x - x_0)^2,$$

with $\varepsilon > 0$. Then x_0 is a local maximum of $v_i - \varphi$, with $\varphi'(x_0) = q > 1$, $\varphi''(x_0) = -\frac{1}{\varepsilon}$. Since $x_0 \notin \mathcal{S}_i$, the subsolution inequality property implies :

$$\rho v_i(x_0) - \mu_i q + \frac{\sigma^2}{2\varepsilon} \leq 0,$$

which leads to a contradiction, by choosing ε sufficiently small. By combining the results from step 1 and step 2, we obtain that v_i is C^1 on the open set $(0, \infty) \backslash \mathcal{S}_i$.

Step 3. We now prove that v_i is C^1 on $(0, \infty)$.
From Step 2, we have to prove the C^1 property of v_i on $\mathcal{S}_i^*$. Fix then some $x_0 \in \mathcal{S}_i^*$ so that $v_i(x_0) = v_{1-i}(x_0 - g_{i,1-i})$. Hence, x_0 is a minimum of $v_i - v_{1-i}(. - g_{i,1-i})$, and so

$$v_i'^-(x_0) - v_{1-i}'^-(x_0 - g_{i,1-i}) \leq v_i'^+(x_0) - v_{1-i}'^+(x_0 - g_{i,1-i}). \tag{B.1}$$

Now, from Lemma 4.3, $x_0 - g_{i,1-i}$ belongs to the open set $(0, \infty) \backslash \mathcal{S}_{1-i}$. From step 2, v_{1-i} is C^1 on $(0, \infty) \backslash \mathcal{S}_{1-i}$, and so $v_{1-i}'^+(x_0 - g_{i,1-i}) = v_{1-i}'^-(x_0 - g_{i,1-i})$. From (B.1), we thus obtain

$$v_i'^-(x_0) \leq v_i'^+(x_0),$$

which is the required result, since the reverse inequality is already satisfied from Step 1.

C^2 **property**
We now turn to the proof of the C^2 property of v_i on the open set $\mathcal{C}_i \cup \mathcal{D}_i$ of $(0, \infty)$.
Step 1. First, we prove that v_i is C^2 on $\mathcal{C}_i$. By standard arguments, we check that v_i is a viscosity solution to

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad x \in \mathcal{C}_i. \tag{B.2}$$

Indeed, let $\overline{x} \in \mathcal{C}_i$ and φ a C^2 function on $\mathcal{C}_i$ s.t. $\overline{x}$ is a local maximum of $v_i - \varphi$, with $v_i(\overline{x}) = \varphi(\overline{x})$. Then, $\varphi'(\overline{x}) = v_i'(\overline{x}) > 1$. By definition of $\mathcal{C}_i$, we also have $v_i(\overline{x}) > v_{1-i}(x - g_{i,1-i})$ and so from the subsolution viscosity property (A.26) of v_i, we have

$$\rho \varphi(\overline{x}) - \mathcal{L}_i \varphi(\overline{x}) \leq 0.$$

The supersolution inequality for (B.2) is immediate from (A.12).
Now, for any arbitrary bounded interval $(x_1, x_2) \subset \mathcal{C}_i$, consider the Dirichlet boundary linear problem:

$$\rho w(x) - \mathcal{L}_i w(x) = 0, \quad \text{on } (x_1, x_2) \tag{B.3}$$
$$w(x_1) = v_i(x_1), \quad w(x_2) = v_i(x_2). \tag{B.4}$$

Classical results provide the existence and uniqueness of a smooth C^2 function w solution on (x_1, x_2) to (B.3)-(B.4). In particular, this smooth function w is a viscosity solution to (B.2) on (x_1, x_2). From standard uniqueness results for (B.3)-(B.4), we get $v_i = w$ on (x_1, x_2). From the arbitrariness of $(x_1, x_2) \subset \mathcal{C}_i$, this proves that v_i is smooth C^2 on $\mathcal{C}_i$.

Step 2. It is clear that v_i is C^2 on $\mathcal{D}_i$. We now prove that v_i is C^2 on $\mathcal{C}_i \cup \mathcal{D}_i$, i.e. v_i is also C^2 on any point $x_0 \in \overline{\mathcal{C}_i} \cap \overline{\mathcal{D}_i}$. We need to prove that $\lim_{x \uparrow x_0^-} v''_i(x) = \lim_{x \downarrow x_0^+} v''_i(x) = 0$. We set

$$x_a = \inf \{x \leq x_0, (x, x_0) \subset \mathcal{C}_i \cup \mathcal{D}_i\}, \quad x_b = \sup \{x \geq x_0, (x_0, x) \subset \mathcal{C}_i \cup \mathcal{D}_i\},$$

and we distinguish the three following cases :

$\star$ *Case 1* : $(x_a, x_0) \subset \mathcal{C}_i$ and $[x_0, x_b) \subset \mathcal{D}_i$.

By definition of $\mathcal{D}_i$, we then have for all $x \in [x_0, x_b)$, $v_i(x) = x - x_0 + v_i(x_0)$. From the viscosity supersolution property of v_i, this implies

$$\rho(x - x_0) + \rho v_i(x_0) - \mu_i \geq 0, \quad \forall x \in (x_0, x_b).$$

Hence, by sending $x \downarrow x_0^+$, we obtain

$$\rho v_i(x_0) - \mu_i \geq 0. \tag{B.5}$$

On the other hand, from the above step 1, we also know that v_i is a classical C^2 solution to the equation (B.2), and so

$$\rho v_i(x) - \mathcal{L}_i v_i(x) = 0, \quad \forall x \in (x_a, x_0). \tag{B.6}$$

By sending $x \uparrow x_0^-$, and using the fact that v_i is C^1 on $(0, \infty)$, in particular, $v_i'(x_0^-) = v_i'(x_0^+) = 1$, we obtain the existence of $v_i''(x_0^-)$ s.t.

$$\rho v_i(x_0) - \mu_i - \frac{\sigma^2}{2} v''_i(x_0^-) = 0. \tag{B.7}$$

Plugging it into (B.5), we obtain $v_i''(x_0^-) \geq 0 = v_i''(x_0^+)$. Suppose now that $v_i''(x_0^-) > 0$. This leads to a contradiction since it would mean that v_i' is strictly non decreasing in a left neighbourhood of x_0, i.e. $v_i' < v_i'(x_0) = 1$, which is impossible given the viscosity supersolution property. Therefore, $v''(x_0)$ exists and v'' is continuous on x_0.

$\star$ *Case 2* : $(x_a, x_0] \subset \mathcal{D}_i \cap \mathcal{S}_i$ and $(x_0, x_b) \subset \mathcal{C}_i$.

We show actually that this case is impossible. Indeed, we would have

$$v_i(x) = v_{1-i}(x - g_{i,1-i}) = x - x_0 + v_i(x_0), \quad \forall x \in (x_a, x_0],$$

and so $(x_a - g_{i,1-i}, x_0 - g_{i,1-i}] \subset \mathcal{D}_{1-i}$. Hence, from the viscosity supersolution property of v_i and v_{1-i}, this would imply

$$\rho v_{1-i}(x_0 - g_{i,1-i}) - \mu_{1-i} \geq 0, \tag{B.8}$$

$$\rho v_i(x_0) - \mu_i \geq 0. \tag{B.9}$$

We then consider the functions w_i and w_{1-i} :

$$w_i(x) = \begin{cases} x - x_0 + v_i(x_0), & x > x_0 \\ v_i(x), & x \leq x_0, \end{cases}$$

$$w_{1-i}(x) = \begin{cases} x - (x_0 - g_{i,1-i}) + v_{1-i}(x_0 - g_{i,1-i}), & x > x_0 - g_{i,1-i}, \\ v_{1-i}(x), & x \leq x_0 - g_{i,1-i}. \end{cases}$$

We now claim that $(w_j)_{j \in \{i, 1-i\}}$ are viscosity solutions on $(0, \infty)$ to

$$\min \left[\rho w_j(x) - \mathcal{L}_j w_j(x), w_j'(x) - 1, w_j(x) - w_{1-j}(x - g_{j,1-j}) \right] = 0, \quad x > 0, \tag{B.10}$$

Let us check the viscosity solution property of w_i. Take some $\overline{x} \in (0, \infty)$ and φ a C^2 function s.t $\overline{x}$ is local minimum of $w_i - \varphi$ with $w_i(\overline{x}) = \varphi(\overline{x})$. If $\overline{x} \leq x_0$, given the definition of w_i and noting that $w_i \leq v_i$, we obtain that $\overline{x}$ is also a local minimum of $v_i - \varphi$ with $v_i(\overline{x}) = \varphi(\overline{x})$. From the viscosity supersolution of v_i and observing that $v_{1-i} \geq w_{1-i}$, this yields

$$\min\left[\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}), \varphi'(\overline{x}) - 1, \varphi(\overline{x}) - w_{1-i}(\overline{x} - g_{i,1-i})\right] \geq 0.$$

If $\overline{x} > x_0$, we have

$$\varphi'(\overline{x}) = w_i'(\overline{x}) = 1, \tag{B.11}$$
$$\varphi''(\overline{x}) \leq w_i''(\overline{x}) = 0, \tag{B.12}$$
$$w_i(\overline{x}) = w_{1-i}(\overline{x}). \tag{B.13}$$

From (B.11)-(B.12), we have after straightforward calculation :

$$\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}) \geq \rho\varphi(\overline{x}) - \mu_i = \rho v_i(\overline{x}) - \mu_i \geq 0, \tag{B.14}$$

by (B.9). Hence, from (B.11), (B.13), and (B.14), we obtain

$$\min\left[\rho\varphi(\overline{x}) - \mathcal{L}_i\varphi(\overline{x}), \varphi'(\overline{x}) - 1, \varphi(\overline{x}) - w_{1-i}(\overline{x} - g_{i,1-i})\right] \geq 0,$$

which proves the viscosity supersolution property of w_i to (B.10).

We now turn to the viscosity subsolution property of w_i. Let $\overline{x} \in (0, \infty)$ and φ a C^2 s.t $\overline{x}$ is a local maximum of $w_i - \varphi$. If $\overline{x} < x_0$ and using the definition of w_i, we obtain the required desired subsolution property. If $\overline{x} \geq x_0$, we have $\varphi'(\overline{x}) = w_i'(\overline{x}) = 1$, and so the desired subsolution property.

By proceeding in the same manner, we also obtain that w_{1-i} is a viscosity solution to (B.10). Hence, w_j, $j = 0, 1$, satisfy the same boundary constraints and linear growth as v_j, $j = 0, 1$, which proves, by uniqueness property, that $w_j = v_j$, $j = 0, 1$. This is a contradiction given the definition of w_j.

$\star$ *Case 3 :* $(x_a, x_0] \subset \mathcal{D}_i$ with $(x_a, x_0] \cap \mathcal{S}_i = \emptyset$, and $(x_0, x_b) \subset \mathcal{C}_i$.

In this case, we distinguish two separate possibilities:

- **(i)** $x_0 + g_{1-i,i} \in \mathcal{S}_{1-i}$

Let $x \in (x_a, x_0]$, we have

$$v_i(x) = x - x_0 + v_i(x_0),$$
$$v_{1-i}(x_0 + g_{1-i,i}) \geq x_0 - x + v_{1-i}(x + g_{1-i,i}), \tag{B.15}$$
$$\geq x_0 - x + v_i(x) \tag{B.16}$$

But we also have $v_i(x_0) = v_{1-i}(x_0 + g_{1-i,i})$. As such, the inequalities in (B.15)-(B.16) become equalities, and

$$v_{1+i}(x + g_{1-i,i}) = x - x_0 + v_{1-i}(x_0 + g_{1-i,i})$$
$$\text{i.e. } (x_a + g_{1-i,i}, x_0 + g_{1-i,i}] \subset \mathcal{D}_{1-i}$$

We now consider the following functions:

$$
w_i(x) = \begin{cases} x - x_0 + v_i(x_0), & x > x_0 \\ v_i(x), & x \le x_0, \end{cases}
$$

$$
w_{1-i}(x) = \begin{cases} x - (x_0 + g_{1-i,i}) + v_{1-i}(x_0 + g_{1-i,i}), & x > x_0 + g_{1-i,i}, \\ v_{1-i}(x), & x \le x_0 + g_{1-i,i}. \end{cases}
$$

By the same arguments as above, we obtain that $(w_j)_{j=0,1}$ are viscosity solutions to

$$
\min \left[\rho u_j(x) - \mathcal{L}_j u_j(x), u_j'(x) - 1, u_j(x) - u_{1-j}(x - g_{j,1-j}) \right] = 0, \quad \forall x > 0,
$$

and due to the uniqueness property, $w_i = v_i$ for $i = 0, 1$, which is a contradiction given the definition of w_i.

- **(ii)** $x_0 + g_{1-i,i} \notin \mathcal{S}_{1-i}$

Let $x_1 = \inf\{x \ge x_0 + g_{1-i,i}, x \notin \mathcal{S}_{1-i}\}$. By the continuity of $(v_i)_{i=0,1}$, we have $x_1 > x_0 + g_{1-i,i}$. Recall that $v_i(x) = x - x_0 + v_i(x_0)$ for all $x \in (x_a, x_0] \subset \mathcal{D}_i$. Moreover, since $x_0 \notin \mathcal{S}_i$, we have $v_i(x_0) > v_{1-i}(x_0 - g_{i,1-i})$. Consider the function $h(x) = x - x_0 + v_i(x_0) - v_{1-i}(x - g_{i,1-i})$ so that $h(x_0) > 0$. Let $x_2 = \sup\{x \ge x_0, h(x) \ge 0\}$. Given the continuity of h, we have $x_2 > x_0$. We fix $x_c = \min\{x_1 - g_{1-i,i}, x_2\}$. We now consider the following functions:

$$
w_i(x) = \begin{cases} x - x_0 + v_i(x_0), & x \in (x_0, x_c), \\ v_i(x), & x \notin (x_0, x_c), \end{cases}
$$

$$
w_{1-i} = v_{1-i}, \quad \text{on } (0, \infty)
$$

We may easily prove that $(w_j)_{j=0,1}$ are viscosity solutions to (3.21), which implies from uniqueness property that $w_j = v_j$, $j = 0, 1$, a contradiction.

We therefore conclude that v_i is C^2 on $\mathcal{C}_i \cup \mathcal{D}_i$, which ends the proof. $\qquad \square$

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